RESOURCE MANAGEMENT TECHNIQUES

UNIT I - LINEAR PROGRAMMING

Operation Research is coined as a result of conducting research on military operations during world war II

INTRODUCTION TO OPERATIONS RESEARCH

Operations Research is a scientific approach to problem solving for executive decision making which requires the formulation of mathematical, economic and satistical models for decision & control problems to deal with situations arising out of risk and uncertainity.

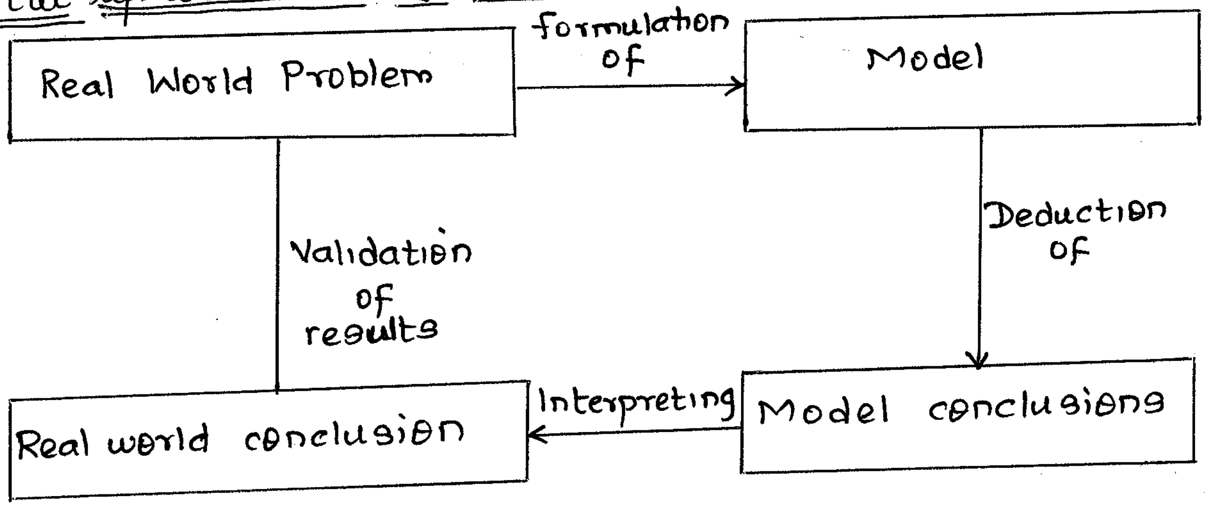
SCOPE OF OR

1. Military operations
2. Production Management
3. Marketing
4. Finance
5. Research & development
6. Personnel Management

PHASES OF OR

1. Definition of the problem / Formulation of the problem
2. Construction of the model
3. Solution of the model
4. Validation of the model
5. Implementation of the solution

Pictorial representation of Phases of OR



Advantages of OR

1. Structure approach to problems
2. Critical approach to problem solving

Limitations of OR

1. Mathematical models are applicable to only specific categories of problems as they do not take the qualitative factors into account.
2. It requires huge calculations which cannot be handled manually and require computers, resulting in heavy costs.

ESSENTIAL CHARACTERISTICS OF DESIGN PROBLEM

The essential characteristics of all decisions are

1. Objectives 2. Alternatives 3. Influencing factors (constraints)

These characteristics are expressed in terms of scientific quantifications or mathematical equations. It provides the management with the most needed tools for improving their decisions.

PRINCIPLE COMPONENTS OF DECISION PROBLEM

There are certain basic components which are required in every decision problem model

1. Controllable (Decision) variable

They are the issues or factors in the problem whose values are to be determined by solving the model. The possible values assigned to these variables are called decision alternatives or strategies or coures of action.

2. Uncontrollable variables:

They are the factors whose numerical value depends upon the external environment prevailing in the organization.

3. Objective function:

It is a representation of the criterion that express the decision maker's manner of the evaluating the desirability of alternative values of the decision variables; and how the criterion is to be optimized (minimized or maximized).

4. Constraints or limitations

These are the restrictions on the values of the decision variables. These restrictions can arise due to limited resources such as space, money, manpower, material etc. The constraints may be in the form of equations or inequalities.

5. Functional Relationships

In a decision problem, the decision variables in the objective function and in the constraints are connected by a specific functional relationship.

A general decision problem can be written as

Optimize (Max. or Min.) $z = f(x)$

Subject to the constraints

$g_i(x) \; \{ \leq, =, \geq \} \; b_i \, , \quad i = 1, 2, \cdots, m$

x = a vector of decision variables $(x_1, x_2, \cdots, x_m)$

$f(x)$ = Criterion or objective function to be optimized.

$g_i(x)$ = the i^{th} constraint

b_i = fixed amount of the i^{th} resource

6. Parameters :

These are constants in the functional relationships. Parameters can be deterministic or probablistic in nature. A deterministic parameter is one whose value is assumed to occur with certainity.

MODELING PHASES OF OPERATIONS RESEARCH

1. PROBLEM DEFINITION

It involves defining the scope of the problem under investigation. This function should be carried out by entire OR team.

The aim is to identify three principal elements of the decision problem

1) Description of the decision problem
2) Determination of the objective of the study.
3) Specifications of the limitations under which the modeled system operates.

2. Model Construction

It entails* an attempt to translate the problem definition into mathematical relationships. If the resulting model fits one of the standard mathematical models, such as linear programming, the solution is obtained by using available algorithms. If the mathematical relationships are too complex to obtain a solution, heuristic approach may be considered.

*entails - involves

3. Model Solution

It entails the use of well-defined optimization algorithms. An important aspect of the world solution phase is sensitivity analysis. It deals with obtaining additional information about the behaviour of the optimum solution. Sensitivity analysis is needed. when the parameters of the model cannot be estimated accurately.

4. Model Validity

It predict adequately the behaviour of the system. A common method for checking the validity of a model is to compare its output with historical output data. The model is valid, if under similar output conditions, it reasonably duplicates past performance.

5. Implementation:

Implementation of the solution of a validated model involves the translation of the results into understandable operating instructions to be issued to the people who will administer the recommended system.

LINEAR PROGRAMMING

Linear programming is the general technique to optimum allocation of resources to several activities on the basis of a given criterion of optimality.

Methods to solve LPP :

1. Graphical Method - Applicable when two variables are involved.

2. Simplex Method - Applicable for any number of variables.

Assumptions of Linear programming

1. Certainity : It is assumed that all modern parameters such as availability of resources, profit contribution of a unit and consumption of resources must be known. LPP is therefore assumed to be deterministic in nature.

2. Divisibility : The solution values of decision variables and resources are assumed to have either whole numbers (integers) or mixed numbers.

3. Additivity : This means that the total output for a given combination of activity levels is the algebraic sum of the output of each individual process.

4. Linearity : For any decision variable, the amount of particular resource say i used and its contribution to the cost in objective function must be proportional to its amount.

Properties of Linear Programming Solution:

1. Feasible solution:

If all the constraints of the given linear programming model are satisfied by the solution of the model, then that solution is known as a feasible solution.

2. Optimal Solution:

If there is no other superior solution to the solution obtained for a given linear programming model, then the solution is optimal solution.

3. Alternate optimum solution:

For some linear programming model, there may be more than one combination of values of the decision variables yielding the best objective function value. Such combinations of the values of the decision variables are known as alternate optimum solutions.

4. Unbounded Solution:

For some LP model, the objective function value can be increased/decreased infinitely without any limitation, Such solution is known as unbounded solution.

5. Infeasible Solution:

If there is no combination of the values of the decision variables satisfying all the constraints of the LP model, then the model is said to have infeasible solution. There is no solution for the given model which can be implemented.

6. Degenerate Solution:

Intersection of two constraints will define a corner point of the feasible region.

FORMULATION OF LP PROBLEM

The procedure for mathematical formulation of LPP consists of 5 steps.

Step 1: Define decision variables.

Step 2: formulate the constraints

Express them as linear equality or inequality in terms of the decision variables.

Step 3: formulate the objective function:

It determines whether the objective function is to be maximized or minimized.

Step 4: Add the non-negativity constraints.

MATHEMATICAL MODEL OF LP

The general formulation of LPP can be stated as to find the values of "n" decision variables $x_1, x_2, \cdots x_n$, optimize (max or minimize) the objective function.

$$Z = C_1x_1 + C_2x_2 + \cdots + C_nx_n$$

Subject to linear constraints

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq, =, \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq, =, \geq)\ b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq, =, \geq)\ b_m \end{array}}$$

and $x_1, x_2, \cdots, x_n \geq 0$

Matrix Model of LPP:

optimize $z = cx$ (objective function)

Subject to $Ax \ (\leq, =, \geq) \ b$ (constraint)

$b > 0, \ x \geq 0$ (Non-negativity restrictions)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

PROBLEMS ON LPP

1. A company manufactures two types of products P_1 & P_2. Each product uses Lathe & milling machine. The processing time per unit of P_1 on the lathe is 5 hours and on the milling machine is 4 hours. The processing time per unit of P_2 on the lathe is 10 hours & on the milling machine is 4 hours. The maximum numbers of hours available per week on the lathe & the milling machine are 60 hours & 40 hours. Also the profit per unit of selling P_1 & P_2 are Rs. 6 & Rs. 8. Formulate a linear programming model to determine the production volume of each of the products such that the total profit is maximized.

Machine	Product P_1	Product P_2	Limit on Machine
Lathe	5	10	60
Milling Machine	4	4	40
Profit (Rs.)	6	8	

Let x_1, x_2 are the production volumes of P_1 & P_2

LPP Model: Max. $z = 6x_1 + 8x_2$

Subject to $5x_1 + 10x_2 \leq 60$

$4x_1 + 4x_2 \leq 40$

$x_1, x_2 \geq 0$

2. A manufacturer produces two types of models M_1 & M_2. Each model of the type M_1 requires 4 hours of grinding & 2 hours of polishing whereas each model of the type M_2 requires 2hours of grinding & 5 hours of polishing. The manufacturer has 2 grinders & 3 polishers. Each grinders work 40 hours a week & each polisher works for 60 hours a week. Profit on M_1 Model is Rs. 3 and on model M_2 is Rs. 4. whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week?

	Model M_1	Model M_2	Limit On working
Grinding	4	2	40×2 = 80
Polishing	2	5	60×3 = 180
Profit (Rs.)	3	4	

LPP Model:

$$\text{Max. } Z = 3x_1 + 4x_2$$

$$\text{Subject to } \quad 4x_1 + 2x_2 \leq 80$$

$$2x_1 + 5x_2 \leq 180$$

$$x_1, x_2 \geq 0$$

Let x_1, x_2 be the number of units of M_1, M_2 Models.

3. A person requires 10, 12 & 12 units of chemicals A, B & C for his garden. A liquid product contains 5, 2 & 1 units of A, B & C per jar. A dry product contains 1, 2, 4 units of A, B & C per carton. If the liquid product is sold for Rs. 3 per jar & the dry product is sold for Rs. 2 per carton, how many units of each product should be purchased in order to minimize the cost & meet the requirements?

	Liquid	Product Dry	chemicals
A	5	1	10
B	2	2	12
C	1	4	12
cost	3	2	

LPP Model :

$$\text{Min. } z = 3x_1 + 2x_2$$

Subject to

$$5x_1 + x_2 \geq 10$$

$$2x_1 + 2x_2 \geq 12$$

$$x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

4. A manufacturing company is engaged in producing three types of products A, B & C. The production department daily produces components sufficient to make 50 units of A, 25 units of B, 30 units of C. The management is confronted with the problem of optimizing the daily production of products in assembly department where only 100 men-hours are available daily to assemble the products. The following additional information is available.

The company has a daily order commitment for 20 units of product A & a total of 15 units of B & C. Formulate this problem as an LP model so as to maximize the total profit

Type of product	Profit contribution per unit of product (Rs.)	Assembly time per product (hrs)
A	12	0.8
B	20	1.7
C	45	2.5

	Product			
	A	B	C	
Production	50	25	30	
Assembly	0.8	1.7	2.5	100
Order	20	15	B & C 15	
	12	20	45	

LPP Model: $\text{Max } z = 12x_1 + 20x_2 + 45x_3$

Subject to $0.8x_1 + 1.7x_2 + 2.5x_3 \leq 100$

$x_1 \leq 50$, $x_2 \leq 25$, $x_3 \leq 30$

$x_1 \geq 20$, $x_2 + x_3 \geq 15$

$x_1, x_2, x_3 \geq 0$

5. A television company operates two assembly sections A and B. Each section is used to assemble the components of these types of televisions: colour, Standard, Economy. The expected daily production on each section is as follows:

TV MODEL	SECTION A	SECTION B
Colour	3	1
Standard	1	1
Economy	2	6

The daily running costs for two sections average Rs-6000 for section A and Rs-4000 for section B. It is given that the company must produce atleast 24 colours, 16 Standard & 14 economy. TV. sets for which an order is pending. Formulate this as L.P.P so as to minimize the total cost.

Model	A	B	Minimum product
Colour	3	1	24
Standard	1	1	16
Economy	2	6	14
Average (Rs.)	6000	4000	

LP Model : Minimize $z = 6000x_1 + 4000x_2$

Subject to

$$3x_1 + x_2 \geq 24$$
$$x_1 + x_2 \geq 16$$
$$2x_1 + 6x_2 \geq 14$$
$$x_1, x_2 \geq 0$$

GRAPHICAL LP SOLUTION

LPP involving only two variables can be effectively solved by a graphical method. It provides a pictorial representation of the problem & its solution.

Redundant constraints are automatically eliminated from the system. Graphical solution is not a powerful tool of linear programming as most of the practical situations do involve more than two variables.

PROCEDURE FOR GRAPHICAL LP SOLUTION

1. Determination of the solution space that defines all feasible solutions of the model.
2. Determination of the optimum solution from among all the feasible points in the solution space.

PROBLEMS:

1. Solve the following LPP using graphical method.

Maximize $z = 6x_1 + 8x_2$

Subject to $5x_1 + 10x_2 \leq 60$

$4x_1 + 4x_2 \leq 40$

$x_1 + x_2 \geq 0$

1. First consider the inequality constraints as equality.

$$5x_1 + 10x_2 = 60$$
$$4x_1 + 4x_2 = 40$$
$$x_1 = 0$$
$$x_2 = 0$$

2. when $x_1 = 0$, $5x_1 + 10x_2 = 60$

$x_2 = 6$

$(0, 6)$

$x_2 = 0, \quad 5x_1 + 10x_2 = 60$

$5x_1 = 60$

$x_1 = 12$

(12,0) First constraint (0,6), (12,0)

3. when $x_1 = 0, \quad 4x_1 + 4x_2 = 40$

$4x_2 = 40$

$x_2 = 10$

(0,10)

$x_2 = 0, \quad 4x_1 + 4x_2 = 40$

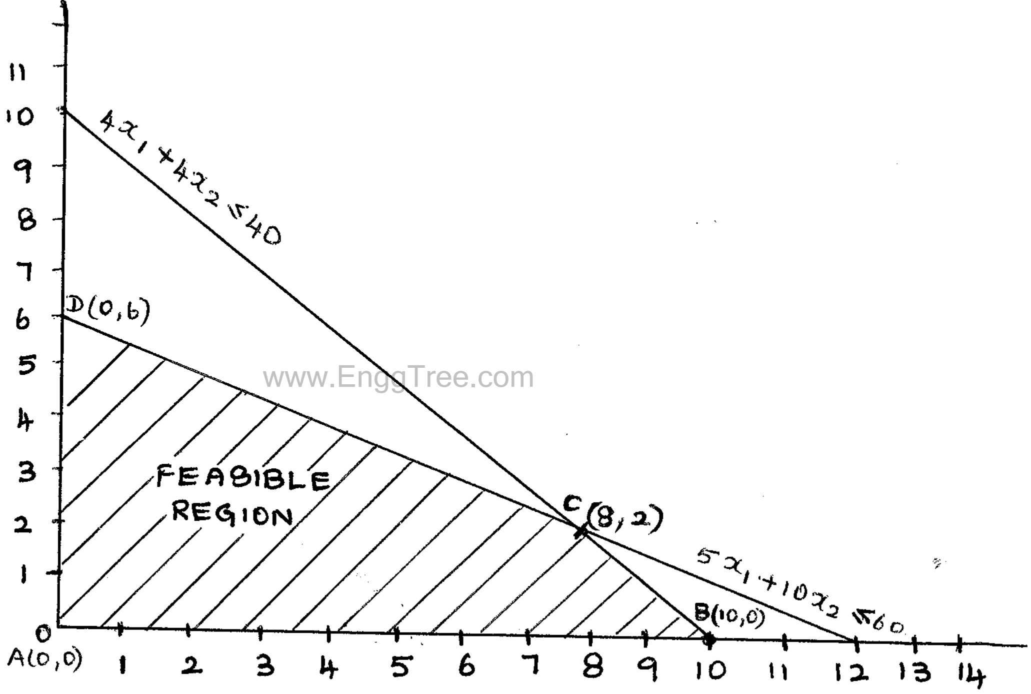
$4x_1 = 40$

$x_1 = 10$

(10,0)

Second constraint (0,10),(10,0)

4.



5. The closed polygon A.B.C.D is the feasible region.

6. The objective function at each corner point is calculated

$Z = 6x_1 + 8x_2$

A(0,0)	$Z = 6(0) + 8(0) = 0$	C(8,2)	$Z = 6(8) + 8(2) = 64$
B(10,0)	$Z = 6(10) + 8(0) = 60$	D(0,6)	$Z = 6(0) + 8(6) = 48$

Type of the objective function is maximization, the solution corresponding to the maximum Z value is to be selected as the optimum solution. $x_1 = 8, x_2 = 2$

Optimum, $Z = 64$

2. Reddy mikks produces both interiors & exterior paints from two raw materials, M_1 and M_2.

	Tons of raw materials per ton of		Maximum daily availability (tons)
	Exterior point	Interior point	
Raw Material M1	6	4	24
Raw Material M2	1	2	6
profit per ton ($1000)	5	4	

A Market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also the maximum daily demand of interior paint is 2 tons.

Reddy Mikks want to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

LP Model: Maximize $z = 5x_1 + 4x_2$

Subject to $6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

First demand restriction says that the difference between the daily production of interior & exterior paints, $x_2 - x_1$ does not exceed 1 ton.

$$\therefore \quad x_2 - x_1 \leq 1$$

Second demand restriction is that the maximum daily demand of interior paint is limited to 2 tons

$$\therefore \quad x_2 \leq 2$$

The complete Reddy Mikks model is

$$\boxed{\begin{array}{ll} \text{Max} & z = 5x_1 + 4x_2 \\ \text{Subject to} & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}}$$

1. First consider the inequality constraint as equalities

$$6x_1 + 4x_2 = 24$$
$$x_1 + 2x_2 = 6$$
$$-x_1 + x_2 = 1$$
$$x_2 = 2$$
$$x_1 = 0$$
$$x_2 = 0$$

2. when $x_1 = 0$ $6x_1 + 4x_2 = 24$

(0,6) $x_2 = 6$

$x_2 = 0$ $6x_1 + 4x_2 = 24$

(4,0) $x_1 = 4$

First constraint (0,6), (4,0)

3. when $x_1 = 0$ $x_1 + 2x_2 = 6$

$2x_2 = 6$

(0,3) $x_2 = 3$

$x_2 = 0$ $x_1 + 2x_2 = 6$

(6,0) $x_1 = 6$

Second constraint (0,3), (6,0)

4. when $x_1 = 0$ $-x_1 + x_2 = 1$

(0,1) $x_2 = 1$

$x_2 = 0$ $-x_1 = 1$

(-1,0) $x_1 = -1$

Negative values are omitted.

Third constraint (0,1).

Remaining points to plot are calculated,

when $x_1 = 1$ $-x_1 + x_2 = 1$

(1,2) $x_2 = 2$

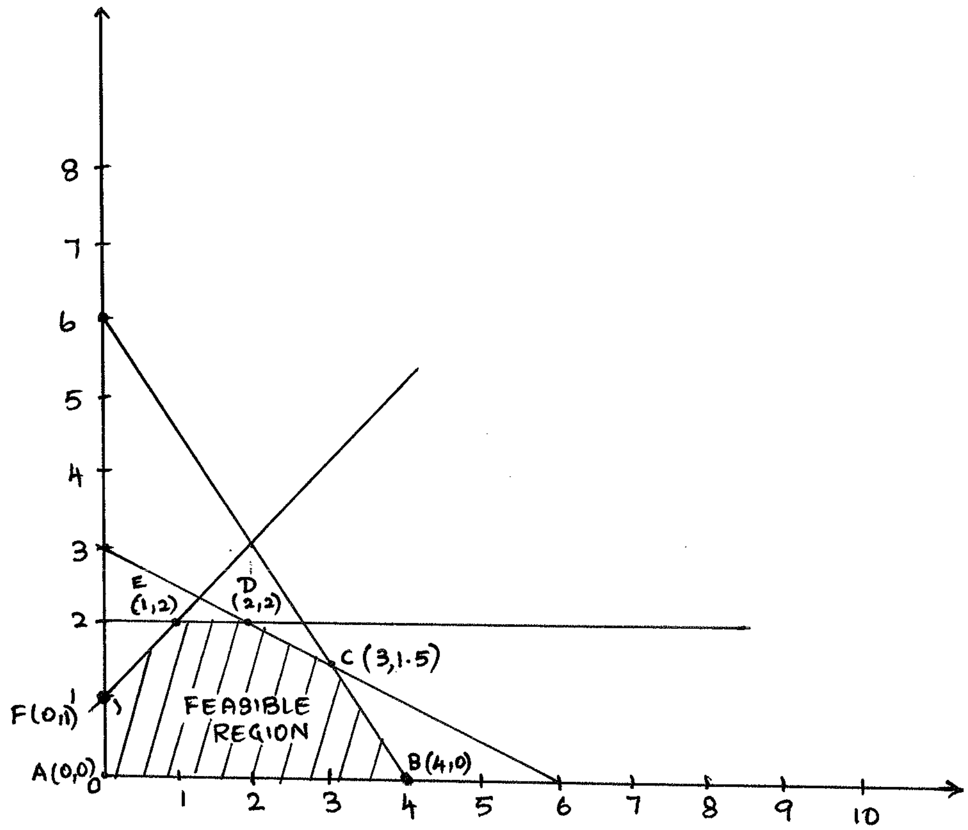
when $x_1 = 2$ $-x_1 + x_2 = 1$

(2,3) $x_2 = 3$

∴ Third constraint (0,1) (1,2) (2,3)

Fouth constraint $x_2 = 2$ for all values of x_1

(ie) (0,2) (1,2) (2,2)



6. The closed polygon A-B-C-D-E-F is the feasible region

7. The objective function at each corner points is calculated.

$z = 5x_1 + 4x_2$

A (0,0)	$z = 0$
B (4,0)	$z = 5(4) + 4(0) = 20$
C (3,1.5)	$z = 5(3) + 4(1.5) = 21$
D (2,2)	$z = 5(2) + 4(2) = 18$
E (1,2)	$z = 5(1) + 4(2) = 13$
F (0,1)	$z = 5(0) + 4(1) = 4$

The optimum solution is when $x_1 = 3$, $x_2 = 1.5$

(ie) $z = 21$ [thousand dollars]

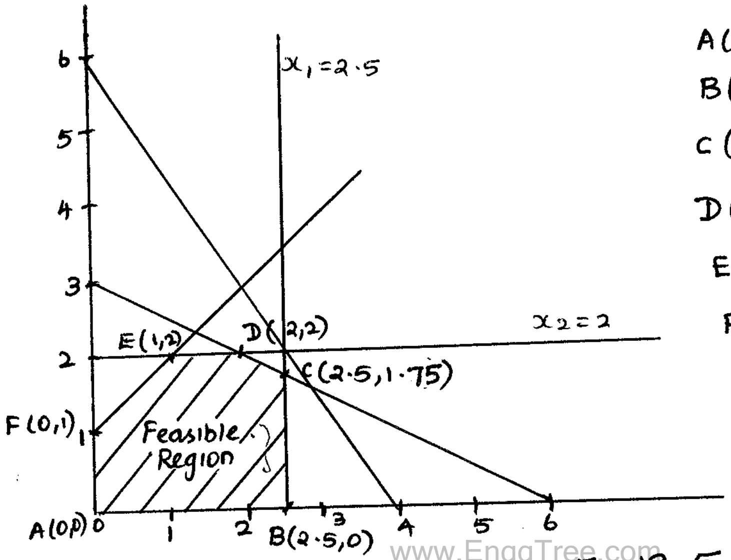
3. Determine the optimum solution & solution space of the Reddy Mikks model for each of the following independent changes.

a) The maximum daily demand for exterior paint is atmost 2.5 tons.

$x_1 \leq 2.5$

$\therefore x_1 = 2.5$ for all values of x_2

(ie) constraint (2.5, 0), (2.5, 1) (2.5, 2) ...



$Z = 5x_1 + 4x_2$

A(0,0) $Z = 5(0)+4(0) = 0$

B(2.5,0) $Z = 5(2.5)+4(0) = 12.5$

C(2.5,1.75) $Z = 5(2.5)+4(1.75) = 19.5$

D(2,2) $Z = 5(2)+4(2) = 18$

E(1,2) $Z = 5(1)+4(2) = 13$

F(0,1) $Z = 5(0)+4(1) = 4$

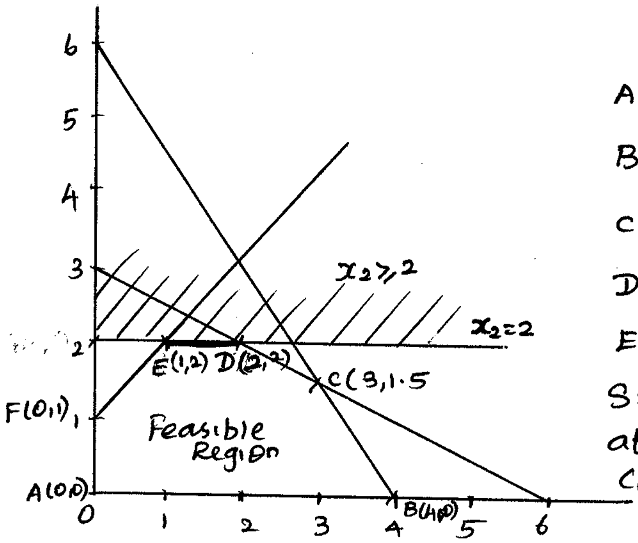
The optimum solution is $Z = 19.5$

$x_1 = 2.5$, $x_2 = 1.75$

b) The daily demand for interior paint is atleast 2 tons

$x_2 \geq 2$

$x_2 = 2$, for all values of x_1. $\therefore$ constraint is (0,2) (1,2) (2,2) ...



$Z = 5x_1 + 4x_2$

A(0,0) $Z = 5(0) + 4(0) = 0$

B(4,0) $Z = 5(4) + 4(0) = 20$

C(3,1.5) $Z = 5(3) + 4(1.5) = 21$

D(2,2) $Z = 5(2)+4(2) = 18$

E(1,2) $Z = 5(1) + 4(2) = 13$

Since interior paint should b atleast 2 tons (ie, $x_2 = 2$) consider D, E out of which D produces maximum. ie, (2,2

The optimum solution is $Z = 18$. $x_1 = 2$ & $x_2 = 2$

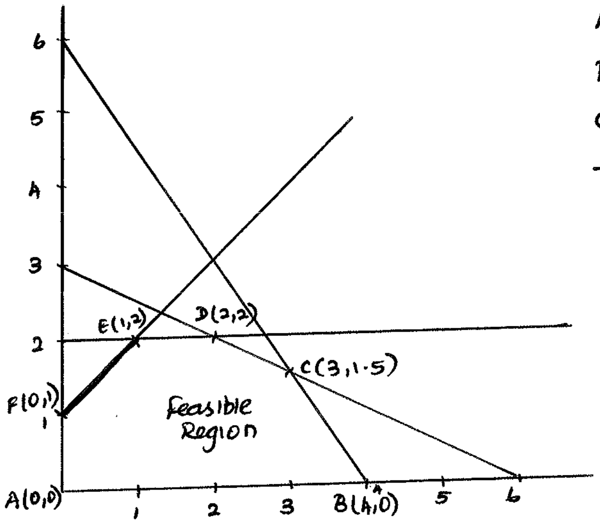
c)The daily demand for interior paint is exactly 1 ton higher than that for exterior paint.

∴ $x_1+1 = x_2$

$-x_1+x_2 = 1$ ∴ x_2 always 1 ton more than x_1

constraint

(0,1), (1,2) (2,3)···



$z = 5x_1+4x_2$

A(0,0) $z=5(0)+4(0)=0$

B(4,0) $z=5(4)+4(0)=20$

C(3,1.5) $z=5(3)+4(1.5)=21$

D(2,2) $z=5(2)+4(2)=18$

E(1,2) $z=5(1)+4(2)=13$

F(0,1) $z=5(0)+4(1)=4$

Since interior paint demand should be exactly 1ton higher than that for exterior paint.

Among the 6 points, the points E, F has x_2 one more than x_1 Out of which the optimum pt is (1,2) (ie. E)

The optimum solution is

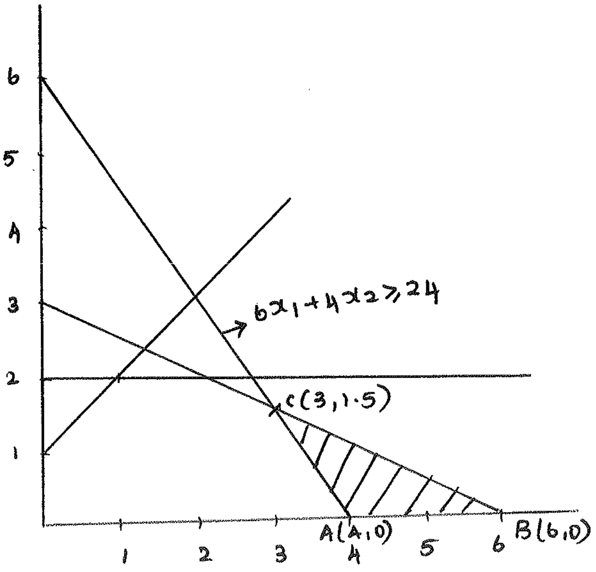
$Z=13$. $x_1=1$, $x_2=2$

d)The daily availability of raw material M_1 is atleast 24 tons

$6x_1+4x_2 \geq 24$

∴ $6x_1+4x_2 = 24$

constraint (4,0), (2,3) (0,6)



$z = 5x_1+4x_2$

A(4,0) $z=5(4)+4(0)=20$

B(6,0) $z=5(6)+4(0)=30$

C(3,1.5) $z=5(3)+4(1.5)=21$

The optimum solution is

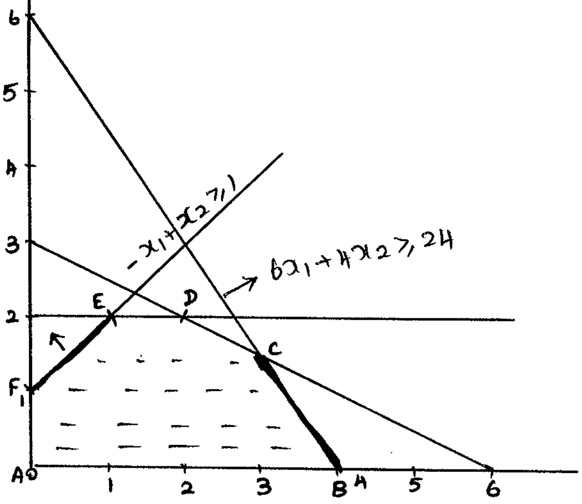
$Z=30$

$x_1=6$ $x_2=0$

e) The daily availability of raw material M1 is atleast 24 tons and the daily demand for the interior paint exceeds that of exterior paint by atleast 1 ton.

$6x_1 + 4x_2 \geq 24$ constraint (4,0) (0,6) (2,3)

$-x_1 + x_2 \geq 1$ constraint (0,1) (1,2) (2,3)



The two constraints does not have a common point within the feasible region

∴ No feasible space.

4. A pineapple form produces two products, canned pineapple & canned juice. The specific amounts of materials, labour & equipment required to produce each product & the availability of each of these resources as shown in the table.

	Canned Juice	Canned pineapple	Available Resources
Labour	3	2	12
Equipment	1	2.3	6.9
Material	1	1.4	4.9

Assuming one unit of canned juice & canned pineapple has profit margins Rs. 2 & Rs. 1. Formulate this as LPP & solve it graphically also.

LPP Model: Let x_1 be number of units of canned juice
x_2 be number of units of canned pineapple

Maximize $Z = 2x_1 + x_2$

Subject to $3x_1 + 2x_2 \leq 12$

$x_1 + 2.3x_2 \leq 6.9$

$x_1 + 1.42x_2 \leq 4.9$

$x_1, x_2 \geq 0$

1. First consider the inequality constraints as equalities

$$3x_1 + 2x_2 = 12$$
$$x_1 + 2.3x_2 = 6.9$$
$$x_1 + 1.4x_2 = 4.9$$
$$x_1 = 0$$
$$x_2 = 0$$

2. when $x_1 = 0$, $3x_1 + 2x_2 = 12$

$2x_2 = 12$

(0,6) $x_2 = 6$

3. $x_2 = 0$, $3x_1 + 2x_2 = 12$

$3x_1 = 12$

(4,0) $x_1 = 4$

3. when $x_1 = 0$, $x_1 + 2.3x_2 = 6.9$

$x_2 = 3$

(0,3)

$x_2 = 0$, $x_1 + 2.3x_2 = 6.9$

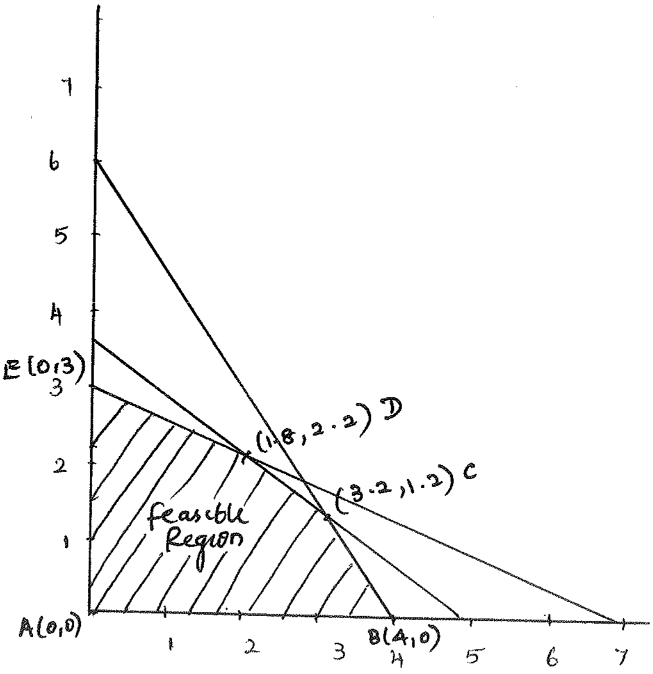
(6.9,0) $x_1 = 6.9$

4. when $x_1 = 0$, $x_1 + 1.4x_2 = 4.9$

(0,3.5) $x_2 = 3.5$

$x_2 = 0$, $x_1 + 1.4x_2 = 4.9$

(4.9,0) $x_1 = 4.9$



5. The closed polygon A-B-C-D-E is the feasible region

6. $Z = 2x_1 + x_2$

A(0,0) $Z = 2(0)+0 = 0$

B(4,0) $Z = 2(4)+0 = 8$

C(3.2,1.2) $Z = 2(3.2)+1.2 = 7.6$

D(1.8,2.2) $Z = 2(1.8)+2.2 = 5.8$

E(0,3) $Z = 2(0)+3 = 3$

7. The optimum solution

$Z = 8$

$x_1 = 4$, $x_2 = 0$

5. A company manufactures 2 types of printed circuits. The requirements of transistors, resistors & capacitors for each type of printed circuits along with other data.

	CIRCUIT A	CIRCUIT B	Stock Available
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
profit (Rs.)	5	8	

How many circuits of each type should the company produce from the stock to earn maximum profit.

LPP Model:

x_1 be the no. of type A circuits

x_2 be the no. of type B circuits

Max. $z = 5x_1 + 8x_2$

Subject to
$$15x_1 + 10x_2 \leq 180$$
$$10x_1 + 20x_2 \leq 200$$
$$15x_1 + 20x_2 \leq 210$$
$$x_1, x_2 \geq 0$$

Graphical Method:

1. First consider the inequality constraint as equalities
$$15x_1 + 10x_2 = 180$$
$$10x_1 + 20x_2 = 200$$
$$15x_1 + 20x_2 = 210$$
$$x_1, x_2 = 0$$

2. when $x_1 = 0$, $15x_1 + 10x_2 = 180$
$x_2 = 18$

(0,18)

$x_2 = 0$, $15x_1 + 10x_2 = 180$
$x_1 = 12$

(12,0)

3. When $x_1 = 0$, $10x_1 + 20x_2 = 200$

$x_2 = 10$

(0,10)

$x_2 = 0$, $10x_1 + 20x_2 = 200$

$x_1 = 20$

(20,0)

4. when $x_1 = 0$, $15x_1 + 20x_2 = 210$

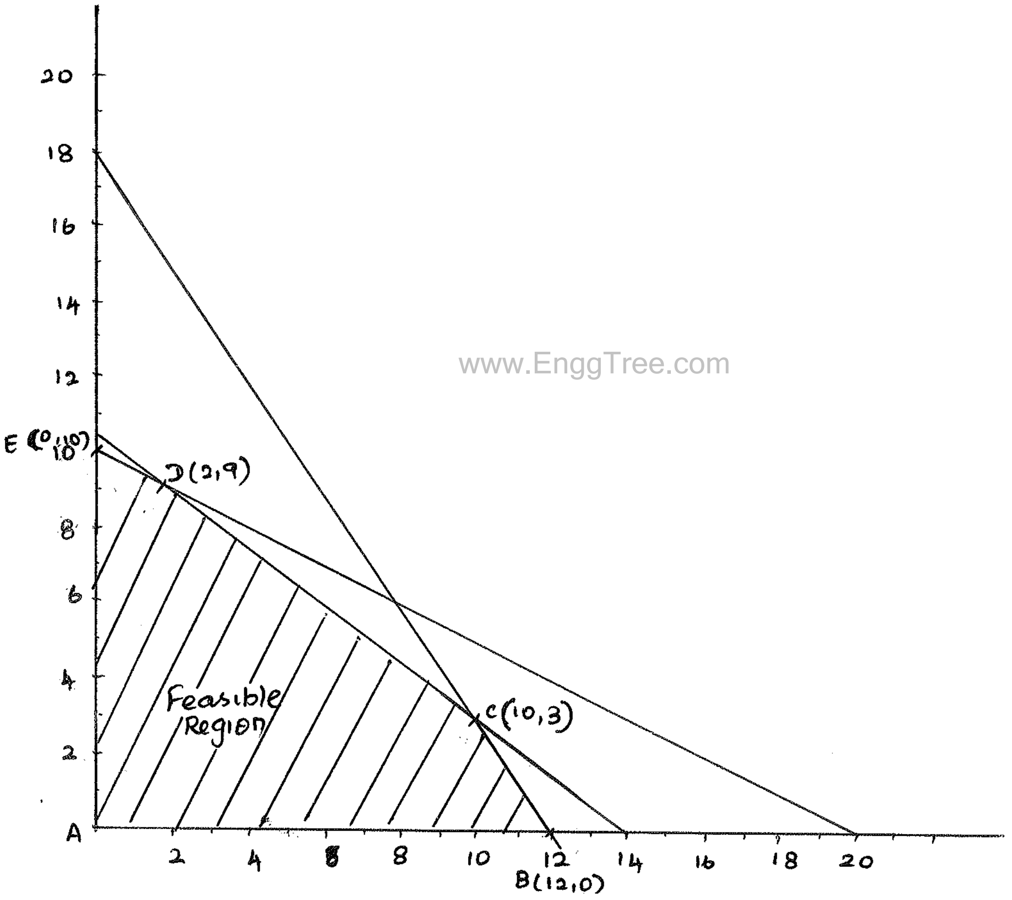
$x_2 = 10.5$

(0,10.5)

$x_2 = 0$, $15x_1 + 20x_2 = 210$

$x_1 = 14$

(14,0)



5. Feasible region is A-B-C-D-E

6. $Z = 5x_1 + 8x_2$

A(0,0) $Z = 5(0) + 8(0) = 0$

B(12,0) $Z = 5(12) + 8(0) = 60$

C(10,3) $Z = 5(10) + 8(3) = 74$

D(2,9) $Z = 5(2) + 8(9) = 82$

E(0,10) $Z = 5(0) + 8(10) = 80$

(7) The optimal solution is

$Z = 82$

$x_1 = 2$, $x_2 = 9$

Examples of Minimization Model:

① Ozark farm uses atleast 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions.

Feedstuff	Protein	Fiber	Cost
Corn	.09	.02	.30
Soyabean Meal	.6	.06	.90

The dietary requirements of the special feed are atleast 30% protein & atmost 5% fiber. Ozark farms wishes to determine the daily minimum-cost feed mix.

Let x_1 = corn in daily mix
x_2 = Soyabean in daily mix.

The objective function is

$$\text{Min. } z = .3x_1 + .9x_2$$

Subject to

1. Farms needs atleast 800 lb of feed a day.

$$x_1+x_2 \geq 800$$

(Special feed is a mix of corn & soyabean meal)

2. The dietary requirement of the special feed are atleast 0.3 protein.

(protein include corn & soyabean meal)

$$.09x_1 + .6x_2 \geq .3(x_1+x_2)$$

3. The fiber constraint is atmost 0.05

(fiber include corn & soyabean meal)

$$0.02x_1 + 0.06x_2 \leq 0.05(x_1+x_2)$$

LPP Model: $\text{Min. } z = .3x_1 + .9x_2$

Subject to $x_1+x_2 \geq 800$

$$.09x_1 + .6x_2 \geq .3x_1 + .3x_2 \Rightarrow .21x_1 - 0.3x_2 \leq 0$$

$$.02x_1 + 0.06x_2 \leq 0.05x_1 + 0.05x_2 \Rightarrow .03x_1 - 0.01x_2 \geq 0$$

$$x_1, x_2 \geq 0.$$

1. First consider the inequality constraints as equalities.

$$x_1 + x_2 = 800$$
$$\cdot 21x_1 - \cdot 3x_2 = 0$$
$$\cdot 03x_1 - 0\cdot 1x_2 = 0$$
$$x_1 = 0$$
$$x_2 = 0$$

2. when $x_1 = 0$, $x_1 + x_2 = 800$
$x_2 = 800$

(0,800)

$x_2 = 0$, $x_1 + x_2 = 800$
$x_1 = 800$

(800,0)

3. when $x_1 = 0$, $\cdot 21x_1 - \cdot 3x_2 = 0$
$x_2 = 0$

(0,0)

$x_2 = 0$, $\cdot 21x_1 - \cdot 3x_2 = 0$
$x_1 = 0$

(0,0)

As both constraints are origin,
consider $x_1 = 200$ $\quad \cdot 21x_1 - \cdot 3x_2 = 0 \Rightarrow \cdot 21(200) - 3x_2 = 0$
$x_2 = 140$

(200,140)

4. when $x_1 = 0$ $\quad \cdot 03x_1 - 0\cdot 01x_2 = 0$
$x_2 = 0$

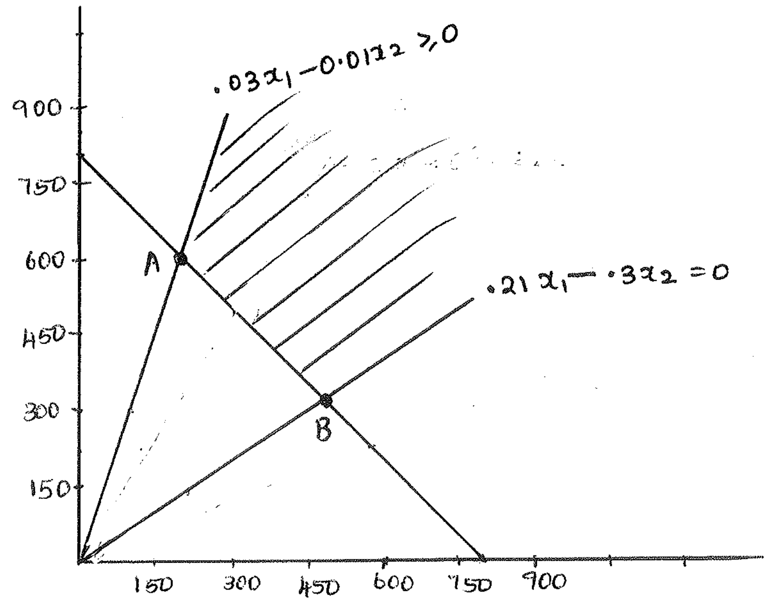
(0,0)

$x_2 = 0$ $\quad \cdot 03x_1 - 0\cdot 01x_2 = 0$
$x_1 = 0$

(0,0)

As both values are (0,0), consider $x_1 = 100$ $\quad \cdot 03(100) - 0\cdot 01x_2 = 0$
$x_2 = 300$

(100,300)



5. The optimum solution is found by the feasible region A, B ...

$Z = \cdot 3x_1 + \cdot 9x_2$

Min. A(600, 200) $\cdot 3(600) + \cdot 9(200) = 600$

B(470·6, 329·4) $\cdot 3(470 \cdot 6) + \cdot 9(329 \cdot 4) = 437 \cdot 64$

6. The optimum solution is $Z = 437 \cdot 64$

$x_1 = 470 \cdot 6 \quad x_2 = 329 \cdot 4$

2. A Solve the following LP problem using graphical method

Min. $Z = 2x_1 + 3x_2$

Subject to $x_1 + x_2 \geq 6$

$7x_1 + x_2 \geq 14$

$x_1, x_2 \geq 0$

1. First consider the inequality constraints as equalities

$x_1 + x_2 = 6$

$7x_1 + x_2 = 14$

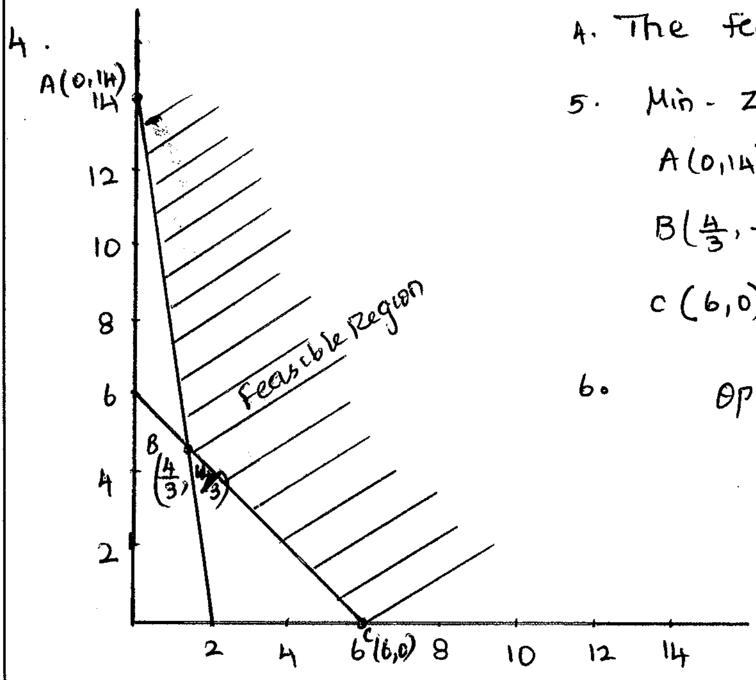
$x_1 = 0$

$x_2 = 0$

2. when $x_1 = 0$, $x_1 + x_2 = 6$, $x_2 = 6$ (0, 6) $x_2 = 0$, $x_1 + x_2 = 6$, $x_1 = 6$ (6, 0)

3. when $x_1 = 0$, $7x_1 + x_2 = 14$, $x_2 = 14$ (0, 14) $x_2 = 0$, $7x_1 + x_2 = 14$, $x_1 = 2$ (2, 0)

4.



4. The feasible region A-B-C

5. Min. $Z = 2x_1 + 3x_2$

A(0, 14) $Z = 2(0) + 3(14) = 42$

B$\left(\frac{4}{3}, \frac{14}{3}\right)$ $Z = 2(4/3) + 3(14/3) = 16 \cdot 67$

C(6, 0) $Z = 2(6) + 3(0) = 12$

6. Optimum solution $Z = 12$

$x_1 = 6$, $x_2 = 0$.

3. A company manufactures two type of cloths using three different colours of wool. One yard length of type A cloth require 4oz of red wool, 5 oz of green wool & 3oz of yellow wool. One yard length of type B cloth requires 5oz of red wool, 2oz of green wool & 8 oz of yellow wool. The wool available for manufactuer is 1000 oz of red wool, 1000 oz of green wool & 1200 oz of yellow wool. The manufactues can make a profit of Rs.5 on one yard of type A cloth & Rs.3 on one yard of type B cloth. Find the best combination of the quantities of type A & type B cloth which gives him maximum profit by solving LPP graphically.

	Type A	Type B	Availability
Red wool	4	5	1000
Green wool	5	2	1000
Yellow wool	3	8	1200
Profit (Rs.)	5	3	

LP Model: Max. $Z = 5x_1 + 3x_2$

Subject to
$$4x_1 + 5x_2 \leq 1000$$
$$5x_1 + 2x_2 \leq 1000$$
$$3x_1 + 8x_2 \leq 1200$$
$$x_1, x_2 \geq 0$$

1. Consider the inequality constraint as equality
$$4x_1 + 5x_2 = 1000$$
$$5x_1 + 2x_2 = 1000$$
$$3x_1 + 8x_2 = 1200$$
$$x_1 = 0$$
$$x_2 = 0$$

2. when $x_1 = 0$, $4x_1 + 5x_2 = 1000$, $x_2 = 200$ $\qquad$ $x_2 = 0$, $4x_1 + 5x_2 = 1000$, $x_1 = 250$

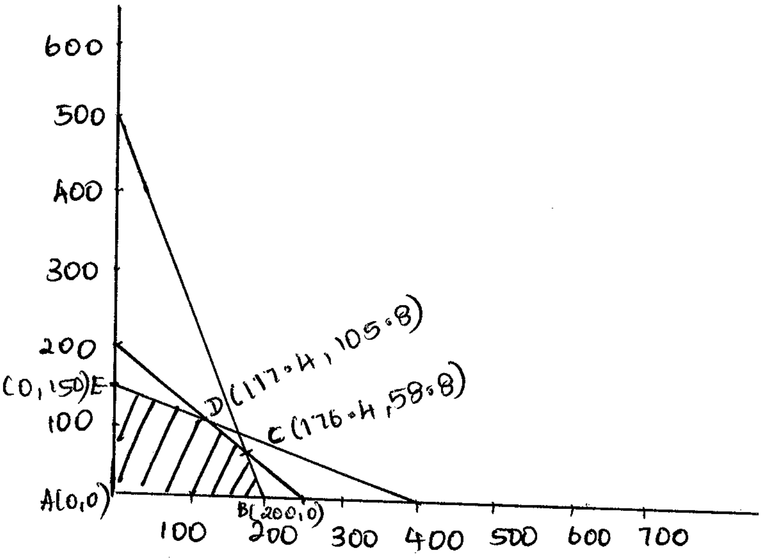
(0, 200) $\qquad$ (250, 0)

3. when $x_1=0$, $5x_1+2x_2=1000$, $x_2=500$ (0,500)

$x_2=0$, $5x_1+2x_2=1000$, $x_1=200$ (200,0)

4. when $x_1=0$, $3x_1+8x_2=1200$, $x_2=150$ (0,150)

$x_2=0$, $3x_1+8x_2=1200$, $x_1=400$ (400,0)



5) The feasible region is A-B-C-D-E

6) A(0,0) $Z=0$

B(200,0) $Z=5x_1+3x_2=1000$

C(176.4,58.8) $Z=5(176.4)+3(58.8)=1058.4$

D(117.4,105.8) $Z=5(117.4)+3(105.8)=904.4$

E(0,150) $Z=5(0)+3(150)=450$

The optimum solution is

$Z=1058.4$

$x_1=176.4$, $x_2=58.8$.

4. A company making cold drinks has two bottling plant located at towns T_1 & T_2. Each plant produces three drinks A, B & C their production capacity per day is given

Cold drinks	Plant	
	T1	T2
A	6000	2000
B	1000	2500
C	3000	3000

The marketing department of the company forecasts a demand of 80000 bottles of A, 22000 bottles of B and 40,000 bottles of C during the month of June. The operating costs per day of plants at T_1 & T_2 are Rs.6000 & Rs.4000 Find graphically the number of days for which each plant must be run in June so as to minimize the operating costs while meeting the market demand.

Cold drinks	T1	T2	Availability
A	6000	2000	80000
B	1000	2500	22000
C	3000	3000	40000
Profit	6000	4000	

LP Model: Min $z = 6000x_1 + 4000x_2$

Subject to $3x_1 + x_2 \geq 40$

$x_1 + 2.5x_2 \geq 22$

$3x_1 + 3x_2 \geq 40$

$x_1, x_2 \geq 0$

Graphical Solution

1. Consider the inequality constraints as equalities

$3x_1 + x_2 = 40$

$x_1 + 2.5x_2 = 22$

$3x_1 + 3x_2 = 40$

$x_1 = 0$

$x_2 = 0$

2. when $x_1 = 0$, $3x_1 + x_2 = 40$, $x_2 = 0$, $3x_1 + x_2 = 40$

$x_2 = 40$ $x_1 = 40/3$

(0, 40) (13.3, 0)

3. when $x_2 = 0$, $x_1 + 2.5x_2 = 22$, $x_2 = 0$, $x_1 + 2.5x_2 = 22$

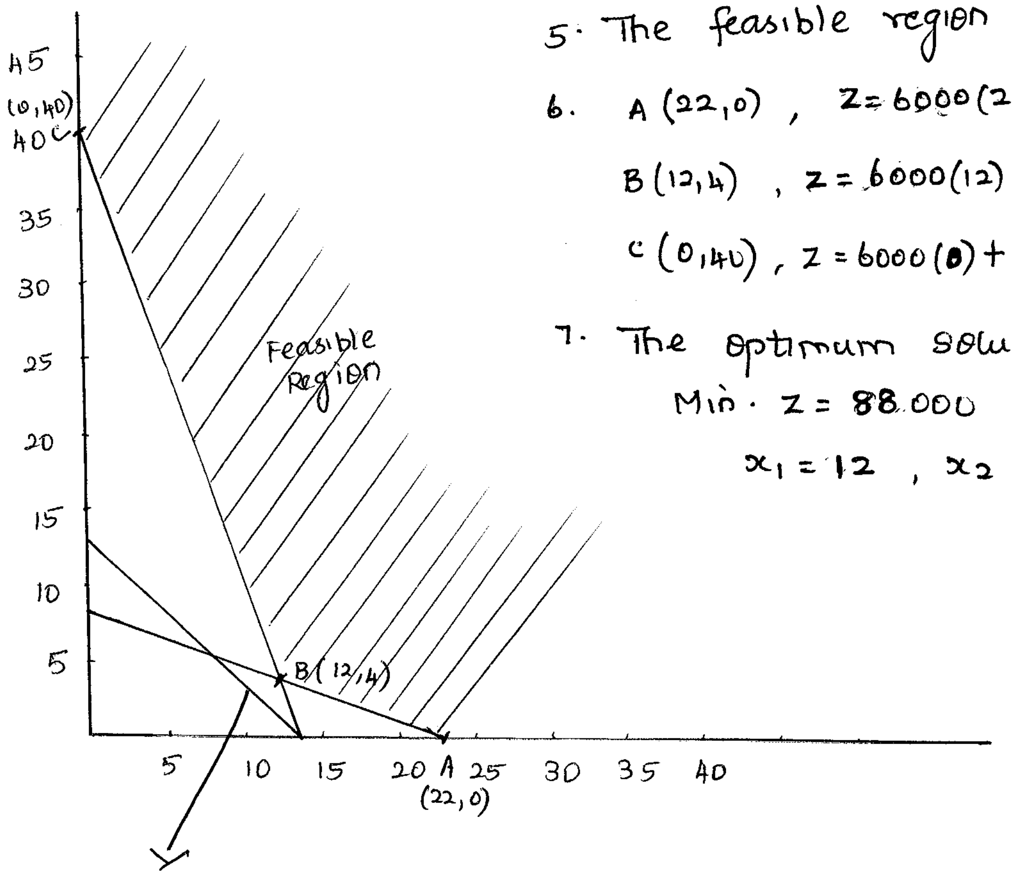
$2.5x_2 = 22$ $x_1 = 22$

(0, 8.8) $x_2 = 8.8$ (22, 0)

4. when $x_1 = 0$, $3x_1 + 3x_2 = 40$ $x_2 = 0$, $3x_1 + 3x_2 = 40$

$3x_2 = 40$ $3x_1 = 40$

(0, 13.3) $x_2 = 13.3$ (13.3, 0) $x_1 = 13.3$



Redundant constraint

5. The feasible region is A-B-C

6. A (22, 0) , Z = 6000(22) + 4000(0) = 132000

B (12, 4) , Z = 6000(12) + 4000(4) = 88000

C (0, 40) , Z = 6000(0) + 40(4000) = 160000

7. The optimum solution is

Min. Z = 88.000

$x_1 = 12$, $x_2 = 4$

Note: Redundant constraint : The constraint if removed does not affect the solution space

5. A firm plans to purchase atleast 200 quintals of scrap containing high quality metal x and low quality metal y. It decides that the scrap to be purchased must contain atleast 100 quintals of x-metal & no more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A & B) in unlimited quantities. The percentage of x & y metals in terms of weight in the scrap supplied by A & B is given below.

Metals	Supplier A	Supplier B
x	25%	75%
y	10%	20%

The price of A's scrap is Rs.200 per quintal & that of B's Rs.400 per quintals. The firm wants to determine the quantities that it should buy from the two suppliers so that the total cost is minimized.

LPP Model: Max. $z = 200x_1 + 400x_2$

Subject to,

1. Atleast 200 quintals of scrap containing high quality x & low quality y.

$$x_1 + x_2 \geq 200$$

2. Scrap contain atleast 100 quintals of x metals & no more than 35 quintals of Y.

$$25\% \text{ of } x_1 + 75\% \text{ of } x_2$$

$$\frac{25}{100}x_1 + \frac{75}{100}x_2 \geq 100$$

$$x_1 + 3x_2 \geq 400$$

$$10\% \text{ of } x_1 + 20\% \text{ of } x_2 = \frac{10}{100}x_1 + \frac{20}{100}x_2 \leq 35$$

$$= x_1 + 2x_2 \leq 350$$

LPP Model: $Z = 200x_1 + 400x_2$

Subject to $x_1 + x_2 \geq 200$

$x_1 + 3x_2 \geq 400$

$x_1 + 2x_2 \leq 350$, $x_1, x_2 \geq 0$

Solution :

1. consider the inequality constraints as equalities.

$x_1 + x_2 = 200$

$x_1 + 3x_2 = 400$

$x_1 + 2x_2 = 350$

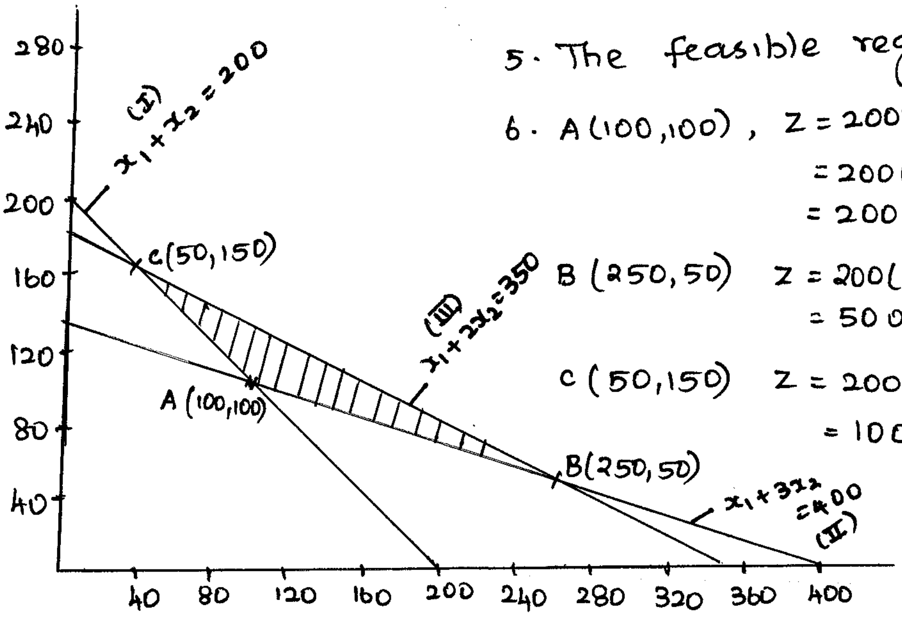
$x_1, x_2 = 0$

2. when $x_1 = 0$, $\overset{x_1+x_2=200}{x_2 = 200}$ (0,200)

$x_2 = 0$, $x_1 = 200$ (200,0)

3. when $x_1 = 0$, $x_1 + 3x_2 = 400$, $x_2 = 133.3$ (0,133.3) , $x_2 = 0$ $x_1 + 3x_2 = 400$ $x_1 = 400$ (400,0)

4. when $x_1 = 0$, $x_1 + 2x_2 = 350$, $x_2 = 175$ (0,175) , $x_2 = 0$ $x_1 + 2x_2 = 350$ $x_1 = 350$ (350,0)



5. The feasible region A, B, C

6. A(100,100), $Z = 200x_1 + 400x_2$

$= 200(100) + 400(100)$

$= 20000 + 40000 = 60000$

B (250,50) $Z = 200(250) + 400(50)$

$= 50000 + 20000 = 70000$

C (50,150) $Z = 200(50) + 400(150)$

$= 10000 + 60000 = 70000$

The optimum solution

Min $Z = 60000$

$x_1 = 100$, $x_2 = 100$

Mixed constraints

Use the graphical method to solve the following LPP

1. A firm makes two products x & y and has a total production capacity of 9 tonnes per day, x, y requires the same production capacity. The firm has a permanent contract to supply atleast 2 tonnes of x and atleast 3 tonnes of y per day to another company. Each tonnes of x requires 20 machine hours production time & each tonnes of y requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All the firm's output can bel sold and the profit made is Rs. 80 per tonnes of x and Rs. 120 per tonnes of y. It is required to determine the production schedule for maximum profit to calculate this profit.

LPP Model :

Max $Z = 80x_1 + 120x_2$

Subject to

(1) Total product capacity of 9 tonnes per day, x & y require the same production capacity. $x_1 + x_2 \le 9$

(2) Supply atleast 2 tonnes of x. $x_1 \geq 2$

(3) Atleast 3 tonnes of y. $x_2 \geq 3$

(4) Each tonne of x requires 20 m/c hours production time & y requires 50 m/c hours production time

But Max. m/c hours possible is 360. $20x_1 + 50x_2 \le 360$

$$\boxed{\begin{array}{ll} \text{Max. } & Z = 80x_1 + 120x_2 \\ \text{Subject to} & x_1 + x_2 \le 9 \\ & x_1 \geq 2 \\ & x_2 \geq 3 \\ & 20x_1 + 50x_2 \le 360 \\ & x_1, x_2 \geq 0 \end{array}}$$

Graphical Solution :

1. Consider the inequalities as equality constraints

$x_1 + x_2 = 9$

$x_1 = 2$

$x_2 = 3$

$20x_1 + 50x_2 = 360$

$x_1 = 0$, $x_2 = 0$

2. when $x_1 = 0$, $x_1 + x_2 = 9$, $x_2 = 0$, $x_1 + x_2 = 9$

$x_2 = 9$ $x_1 = 9$

first constraint is (0,9) , (9,0)

3. when $x_1 = 0$, $20x_1 + 50x_2 = 360$, $x_2 = 0$, $20x_1 + 50x_2 = 360$

$50x_2 = 360$ $20x_1 = 360$

$x_2 = 7.2$ $x_1 = 18$

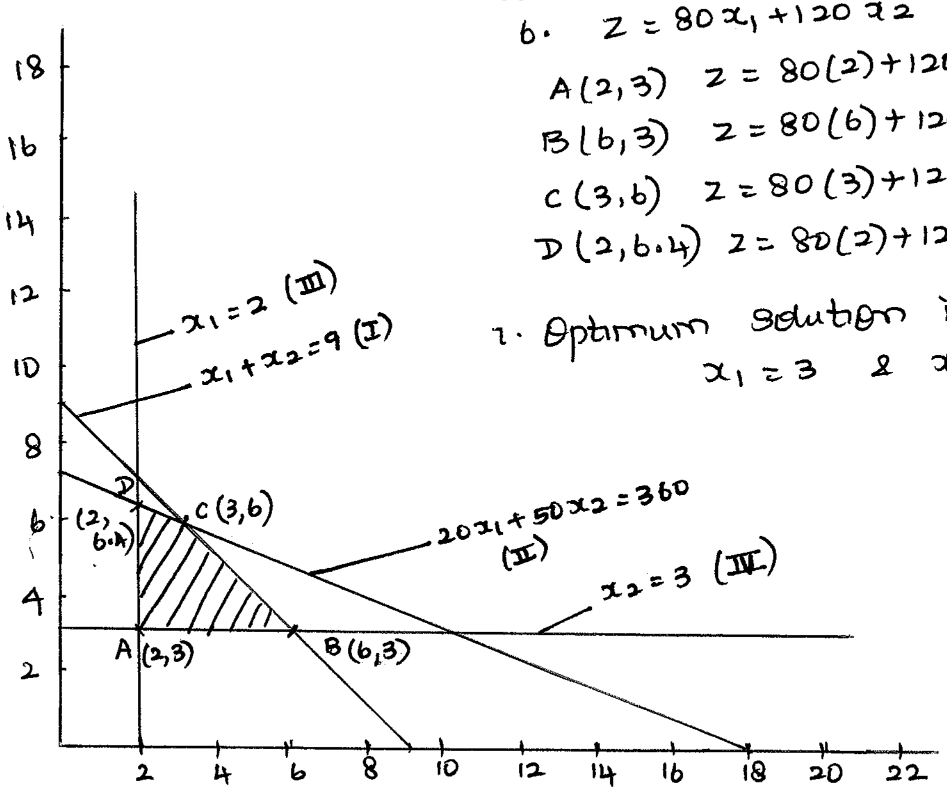
Second constraint, (0, 7.2), (18, 0)

4. Third constraint is $x_1 = 2$ for all values of x_2

$\therefore$ (2,0) (2,1) (2,2)

5. $x_2 = 3$, for values of x_1

fourth constraint (0,3), (1,3), (2,3)



6. $Z = 80x_1 + 120x_2$

A(2,3) $Z = 80(2) + 120(3) = 520$

B(6,3) $Z = 80(6) + 120(3) = 840$

C(3,6) $Z = 80(3) + 120(6) = 960$

D(2,6.4) $Z = 80(2) + 120(6.4) = 928$

7. Optimum solution is $Z = 960$

$x_1 = 3$ & $x_2 = 6$

2) The standard weight of a special purpose brick is 5 kg & it contains two basic ingredients B_1 & B_2. B_1 costs Rs.5 per kg & B_2 costs Rs.8 per kg. Strength considerations dictate the brick contains not more than 4 kg of B_1 & a minimum of 2 kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically the minimum cost of the brick satisfying the above conditions

Let x_1, x_2 be amount of ingredients B_1, B_2.

LP model:

Min. $z = 5x_1 + 8x_2$

Subject to. (1) Strength considerations

brick contains not more than 4 kg of B_1. $x_1 \leq 4$

Minimum of 2 kg of B_2. $x_2 \geq 2$

(2) Standard weight of special purpose brick is 5~~0~~ kg & it contains B_1, B_2. $x_1 + x_2 = 5$

$\therefore$

$$\boxed{\begin{array}{ll} \text{Min.} & z = 5x_1 + 8x_2 \\ \text{Subject to} & x_1 \leq 4 \\ & x_2 \geq 2 \\ & x_1 + x_2 = 5 \\ & x_1, x_2 \geq 0 \end{array}}$$

Graphical solution.

1. Consider the inequality as equality constraints

$x_1 = 4$

$x_2 = 2$

$x_1 + x_2 = 5$

$x_1 = 0$, $x_2 = 0$

2) when $x_1 = 0$, $x_1 + x_2 = 5$, $x_2 = 5$, $x_2 = 0$, $x_1 + x_2 = 0$, $x_1 = 5$

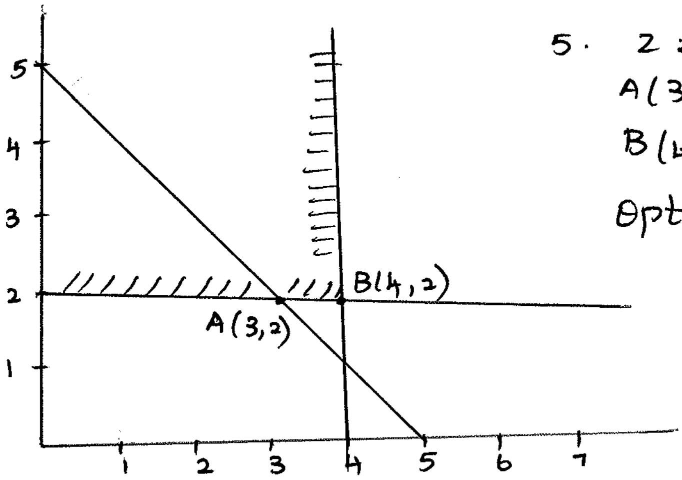
First constraint (0,5), (5,0)

3) ~~when~~ $x_1 = 4$, for all values of x_2

Second constraint (4,0) (4,1) (4,2)

4) $x_2 = 2$ for all values of x_1

Third constraint (0,2) (1,2) (2,2)



5. $z = 5x_1 + 8x_2$

$A(3,2) \quad z = 5(3)+8(2) = 31$

$B(4,2) \quad z = 5(4)+8(2) = 36$

Optimum solution

Min. $z = 31$

$x_1 = 3, x_2 = 2$

3. A manufacturer produces two different models x, y of the same product. The raw materials r_1 & r_2 are required for production. Atleast 18 kg of r_1 & 12 kg of r_2 must be used daily. Also atmost 34 hrs of labour are to be utilized 2kg of r_1 are needed for each model x and 1 kg of r_1 for each model y. for each model of x & y, 1 kg of r_2 is required. It takes 3 hrs to manufacture a model x and 2 hrs to manufacture a model y. The profit is Rs.50 for each model x and Rs.30 for each model y. How many units of each model should be produced to maximize the profit.

	Model X	Model Y	Availability
Material R1	2	1	18
Material R2	1	1	12
Labour hours	3	2	34
profit (Rs)	50	30	

x_1, x_2 - No. of units of model x & y to be produced.

LPP:

Maximize $z = 50x_1 + 30x_2$

Subject to $2x_1 + x_2 \geq 18$

$x_1 + x_2 \geq 12$

$3x_1 + 2x_2 \leq 34$

$x_1, x_2 \geq 0$

1. Consider the inequality constraints as equality constraints

$2x_1 + x_2 = 18$

$x_1 + x_2 = 12$

$3x_1 + 2x_2 = 34$

$x_1 = 0$, $x_2 = 0$

2. when $x_1 = 0$, $2x_1 + x_2 = 18$, $x_2 = 18$ $\quad$ $x_2 = 0$, $2x_1 + x_2 = 18$, $x_1 = 9$

First constraint (0,18), (9,0)

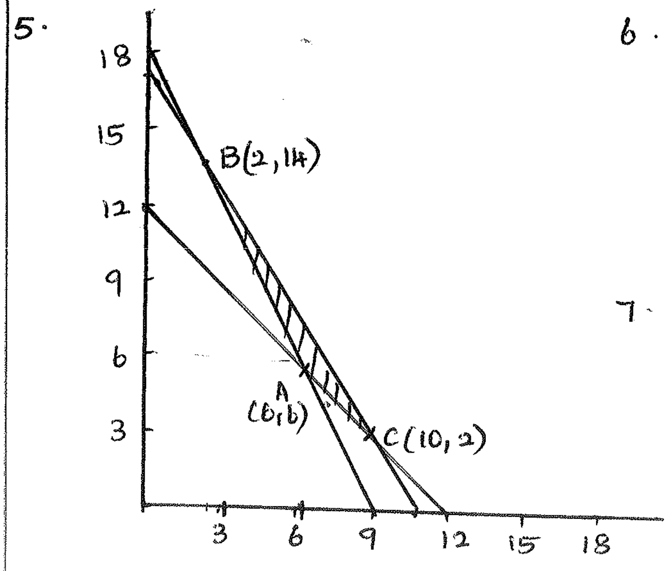
3. when $x_1 = 0$, $x_1 + x_2 = 12$, $x_2 = 12$ $\quad$ $x_2 = 0$, $x_1 + x_2 = 12$, $x_1 = 12$

Second constraint (0,12), (12,0)

4. when $x_1 = 0$, $3x_1 + 2x_2 = 34$, $x_2 = 17$ $\quad$ $x_2 = 0$, $3x_1 + 2x_2 = 34$, $x_1 = 34/3$

Third Constraint (0,17), (11.3,0)

5.



6. Feasible solution, $z = 50x_1 + 30x_2$

A (6,6) = 50(6) + 30(16) = 480

B (2,14) = 50(2) + 30(14) = 520

C (10,2) = 50(10) + 30(2) = 560

7. Optimum solution

$Z = 50x_1 + 30x_2$

is $Z = 560$

$x_1 = 10$

$x_2 = 2$

4. A firm manufactures two products A and B on which the profit earned per unit are Rs.3 & Rs.4 respectively. Each product is processed on two machines M_1 & M_2 product. Product A requires one minute of processing time on M_1 and two minutes on M_2 while B requires one minute on M_1 and one Minute on M_2. Machine M_1 is available for not more than 7 hours 30 minutes while machine M_2 is available for 10 hours during any working day. find the number of units of product A & B to be manufactured to get maximum profit. formulate the above as a LPP & solve by graphical method.

A

7 hours 30 min = 450 min

10 hours = 600 min.

LP Model:

Maximize $z = 3x_1 + 4x_2$

Subject to $x_1 + x_2 \leq 450$

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$

1. Consider the inequality constraints as equality

$x_1 + x_2 = 450$ $\quad x_1 = 0$

$2x_1 + x_2 = 600$ $\quad x_2 = 0$

2. when $x_1 = 0$ $\quad x_1 + x_2 = 450$, $\quad x_2 = 0$, $\quad x_1 + x_2 = 450$

$x_2 = 450$ $\quad x_1 = 450$

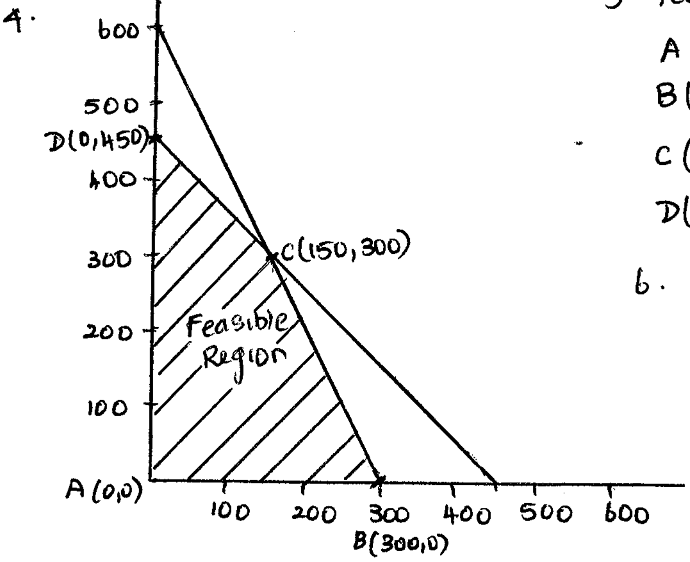
first constraint (0,450), (450,0)

3. when $x_1 = 0$ $\quad 2x_1 + x_2 = 600$, $\quad x_2 = 0$ $\quad 2x_1 + x_2 = 600$

$x_2 = 600$ $\quad x_1 = 300$

Second constraint (0,600) (300,0)

4.



5. Feasible solution, $z = 3x_1 + 4x_2$

A (0,0) $\quad z = 3(0) + 4(0) = 0$

B (300,0) $\quad z = 3(300) + 4(0) = 900$

C (150,300) $\quad z = 3(150) + 4(300) = 1650$

D (0,450) $\quad z = 3(0) + 4(450) = 1800$

6. Optimum Solution,

$z = 1800$

$x_1 = 0, \quad x_2 = 450$.

Multiple solution

5. Solve the following LPP graphically

$$\text{Max}, \; z = 100x_1 + 40x_2$$

Sub. to $5x_1 + 2x_2 \leq 1000$

$3x_1 + 2x_2 \leq 900$

$x_1 + 2x_2 \leq 500$

$x_1, x_2 \geq 0$

1. Consider the inequality constraints as equality.

$5x_1 + 2x_2 = 1000$ $\quad x_1 = 0$

$3x_1 + 2x_2 = 900$ $\quad x_2 = 0$

$x_1 + 2x_2 = 500$

2. when $x_1 = 0$, $5x_1 + 2x_2 = 1000$, $x_2 = 500$; $x_2 = 0$, $5x_1 + 2x_2 = 1000$, $x_1 = 200$

first constraint (0, 500), (200, 0)

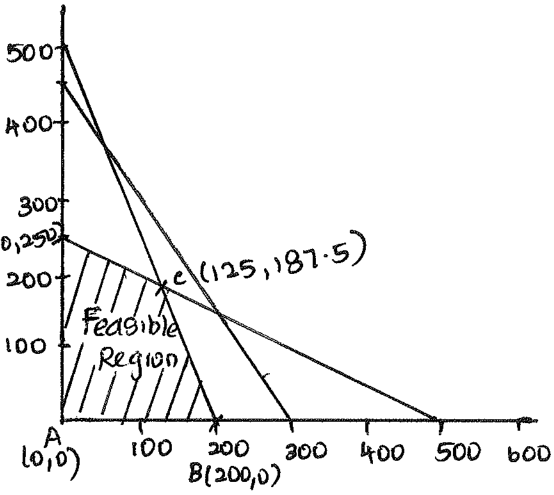
3. when $x_1 = 0$, $3x_1 + 2x_2 = 900$, $x_2 = 450$; $x_2 = 0$ $3x_1 + 2x_2 = 900$, $x_1 = 300$

Second constraint (0, 450), (300, 0)

4. when $x_1 = 0$, $x_1 + 2x_2 \leq 500$, $x_2 = 250$; $x_2 = 0$, $x_1 + 2x_2 = 500$, $x_1 = 500$

Third constraint (0, 250), (500, 0)

5.



solving

$$\begin{array}{rl} 5x_1 + 2x_2 &= 1000 \\ -\,x_1 + -\,2x_2 &= -\,500 \\ \hline 4x_1 &= 500 \\ x_1 &= 125 \end{array}$$

sub $x_1 = 125$,

$x_1 + 2x_2 = 500$

$125 + 2x_2 = 500$

$2x_2 = 375$

$x_2 = 187.5$

$\therefore x_1 = 125, x_2 = 187.5$

6. Feasible solutions, $z = 100x_1 + 40x_2$

A(0,0) $z = 100(0) + 40(0) = 0$

B(200,0) $z = 100(200) + 40(0) = 20000$

C(125, 187.5). $z = 100(125) + 40(187.5) = 20000$

D(0, 250) $z = 100(0) + 40(250) = 10000$

Max value for z occurs at two vertices B & C

7. Optimum solution at two points

$z = 20000$, $x_1 = 200, x_2 = 0$

$x_1 = 125$ $x_2 = 187.5$

LPP having more than 1 optimal solution is said to have multiple optimal soln.

6. UNBOUNDED SOLUTION

Using graphical Method, Solve

Max. $z = 2x_1 + 3x_2$

Sub. to $x_1 - x_2 \le 2$

$x_1 + x_2 \ge 4$

$x_1, x_2 \ge 0$

Step 1 : consider inequality constraints as equalities

$x_1 - x_2 = 2$ $\quad x_1 = 0$

$x_1 + x_2 = 4$ $\quad x_2 = 0$

Step 2, when $x_1 = 0$, $x_1 - x_2 = 2$, $x_2 = -2$ $\qquad x_2 = 0$, $x_1 - x_2 = 2$, $x_1 = 2$

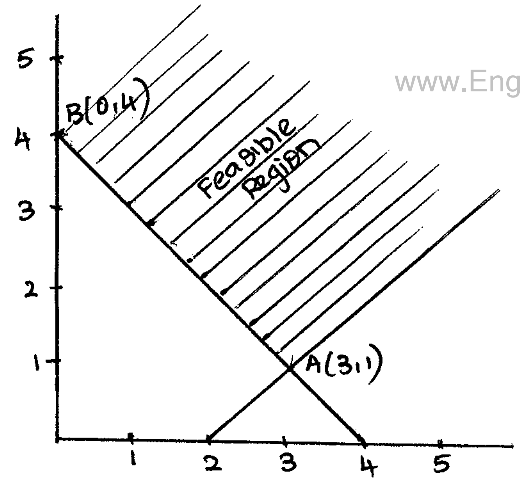
point cannot be negative

∴ Let $x_1 = 3$ $\quad x_2 = 1$ $\quad$ first constraint (3,1), (2,0)

Step 3; when $x_1 = 0$, $x_1 + x_2 = 4$, $x_2 = 4$ $\qquad x_2 = 0$, $x_1 + x_2 = 0$, $x_1 = 4$

Second constraint (0,4), (4,0)

4.



5. Feasible Solution, $z = 2x_1 + 3x_2$

A(3,1), $z = 2(3) + 3(1) = 9$

B(0,4) $z = 2(0) + 3(4) = 12$

The shaded region is unbounded. There are number of points in the shaded region for which the value is more than 12.

The max value of z occurs at infinity.

∴ The problem has an unbounded solution.

7. INFEASIBLE SOLUTION

If it is not possible to find a feasible solution that satisfies all the constraints then the LP problem is said to have an Infeasible Solution.

Solve Graphically the following LPP

Max. $z = 4x_1 + 3x_2$

Sub. to $x_1 - x_2 \leq -1$

$-x_1 + x_2 \leq 0$

$x_1, x_2 \geq 0$

Step 1: consider the inequalities as equality constraints

$x_1 - x_2 = -1$ $\quad x_1 = 0$

$-x_1 + x_2 = 0$ $\quad x_2 = 0$

Step 2: when $x_1 = 0$, $x_1 - x_2 = -1$, $x_2 = 1$ $\qquad x_2 = 0$ $\quad x_1 - x_2 = -1$, $x_1 = -1$

$(0,-1)$ can't be negative pts.

So, let $x_2 = 2$ $\quad x_1 - x_2 = -1$, $x_1 = 1$

First constraint $(0,1)$, $(1,2)$

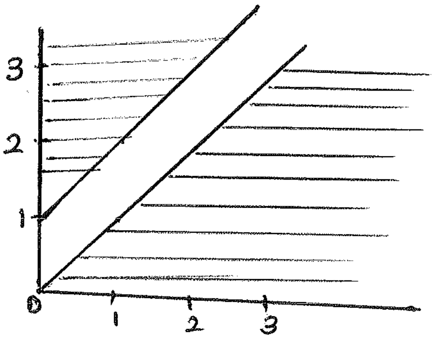
Step 3: when $x_1 = 0$ $\quad -x_1 + x_2 = 0$, $x_2 = 0$ $\qquad x_2 = 0$ $\quad -x_1 + x_2 = 0$, $-x_1 = 0$, $x_1 = 0$

To find another point, we find that $-x_1 + x_2 = 0$

$\therefore x_1 = x_2$

Second constraint $(0,0)$ $(1,1)$ $(2,2)$ ...

4)



5) There is no point common to both the shaded region.

$\therefore$ The LP problem has no feasible solution.

8. Use the graphical method to solve LPP

$$\text{Max } z = x_1 + x_2/2$$

Subject to $3x_1 + 2x_2 \leq 12$

$5x_1 \leq 10$

$x_1 + x_2 \geq 8$

$-x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Step1 : consider the inequalities as equality constraints

$3x_1 + 2x_2 = 12$ $\quad x_1 = 0$

$5x_1 = 10$ $\quad x_2 = 0$

$x_1 + x_2 = 8$

$-x_1 + x_2 = 4$

Step 2 : when $x_1 = 0$, $3x_1 + 2x_2 = 12$, $x_2 = 6$; $x_2 = 0$ $3x_1 + 2x_2 = 12$, $x_1 = 4$

First constraint (0,6), (4,0)

Step 3 : when $5x_1 = 10$, $x_1 = 2$ for all values of x_2

Second constraint (2,0), (2,1) (2,2)

Step 3 : when $x_1 = 0$ $x_1 + x_2 = 8$, $x_2 = 8$; $x_2 = 0$, $x_1 + x_2 = 8$, $x_1 = 8$

Third constraint (0,8), (8,0)

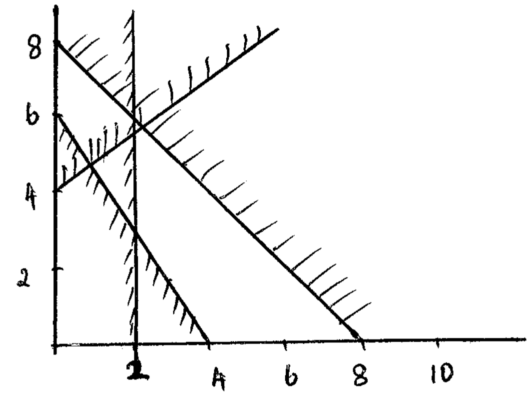
Step 4 : when $x_1 = 0$, $-x_1 + x_2 = 4$, $x_2 = 4$; $x_2 = 0$ $-x_1 + x_2 = 4$, $-x_1 = 4$, $x_1 = -4$

point can't be negative

Assume $x_2 = 5$, $x_1 = 1$

Fourth constraint (0,4), (1,5)

5)



6) Cannot find any point in the shaded regions which can satisfy all the constraints.

$\boxed{\text{Solution Infeasible}}$

9. Solve the LPP by graphical Method.

Max. $z = 100x_1 + 40x_2$

sub. to $5x_1 + 2x_2 \leq 800$

$3x_1 + 2x_2 \leq 600$

$x_1, x_2 \geq 0$.

Step 1 : convert inequalities to equalities

$5x_1 + 2x_2 = 800$

$3x_1 + 2x_2 = 600$

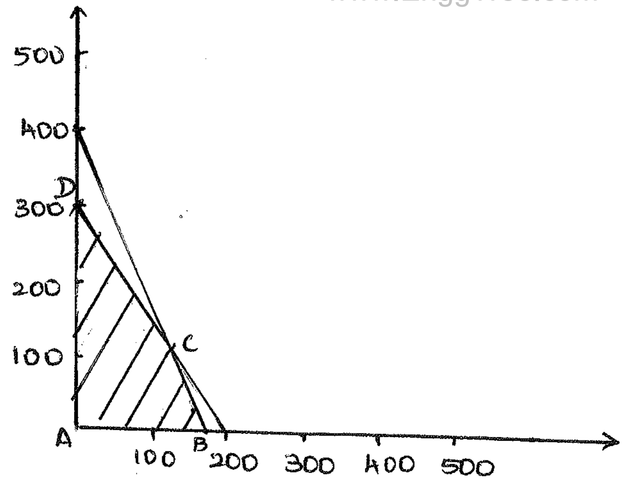
$x_1 = 0 \quad x_2 = 0$

Step 2 : when $x_1 = 0$ $\quad 5x_1 + 2x_2 = 800$ $\quad x_2 = 400$ $\quad (0,400)$

$x_2 = 0$, $\quad 5x_1 + 2x_2 = 800$ $\quad x_1 = 180$ $\quad (180,0)$

Step 3 : when $x_1 = 0$ $\quad 3x_1 + 2x_2 = 600$ $\quad x_2 = 300$ $\quad (0,300)$

$x_2 = 0$, $\quad 3x_1 + 2x_2 = 600$ $\quad x_1 = 200$ $\quad (200,0)$



To find C

$5x_1 + 2x_2 = 800$

$3x_1 + 2x_2 = 600$

$2x_1 = 200$

$x_1 = 100$

$x_2 = 150$

A $(0,0)$ $\quad z = 0$

B $(180,0)$ $\quad z = 18000$

C $(100,150)$ $\quad z = 100(100) + 40(150) = 16000$

D $(0,300)$ $\quad z = 100(0) + 40(300) = 12000$

optimal Solution $\boxed{\text{Max } z = 18000 \quad x_1 = 180 \quad x_2 = 0}$

SIMPLEX METHOD

Simplex method solves the linear programming problem in iterations. It moves the solution to a new corner points that has a potential to improve the value of the objective function. This procedure is repeated & since the number of vertices is finite, this method leads to an optimal solution in a finite number of steps. or indicates the existence of an unbounded solution.

Definitions:

1. Given a system of "m" linear equations with "n" variables (m<n). The solution obtained by setting (n-m) variables equal to zero & solving for the remaining m variables is called a basic solution.
2. The m variables are called basic variables and they form the basic solution. The (n-m) variables which are put to zero are called non-basic variables.
3. A basic solution is said to be a non-degenerate basic solution if none of the basic variables is zero.
4. A basic solution is said to be a generate basic solution if one or more of the basic variables are zero
5. A feasible solution which is also basic is called a basic feasible solution.

Steps for computing optimum solution using simplex algorithm

1. formulation of the mathematical model

$$\text{Min } z = -\text{Max}(-z)$$

2. Check whether all b_i $(i=1,2,..,m)$ are positive. If any b_i's is negative, multiply the inequation of the constraint by -1 so as to get all b_i to be positive.

3. Express the problem in standard form by introducing slack/surplus variables to convert the inequalities into equations.

4. Obtain an initial basic feasible solution in the form $x_B = B^{-1}b$ and put it in the first column of the simplex table.

$c_j =$

		c_j	c_1	c_2	c_3	$\cdots$	0	0	$\cdots$
c_B	S_B	x_B	x_1	x_2	$x_3 \cdots$	x_n	s_1	s_2	$\cdots s_n$
c_{B1}	s_1	b_1	a_{11}	a_{12}	a_{13}	$\cdots a_{1n}$	1	0	$\cdots 0$
c_{B2}	s_2	b_2	a_{21}	a_{22}	a_{23}	$\cdots a_{2n}$	0	1	$\cdots 0$

where C_j - coefficients of the variables in the objective fn.

C_B - coefficients of the basic variables in the objective fn.

S_B - Basic variables

x_B - Values of the basic variables.

5. Compute the net evaluations $(c_j - z_j)$ by using the location $c_j - z_j = c_B(c_j - a_j)$. Examine the sign of $c_j - z_j$

(a) If all $c_j - z_j \geq 0$, then the initial basic feasible solution x_B is an optimum basic solution.

(b) If atleast one $c_j - z_j < 0$, then proceed to next step as the solution is not optimal.

6. Find the entering variable.

The entering variable is the non-basic variable corresponding to the most positive value of $(c_j - z_j)$. Let it be $c_r - z_r$ for some $r=j$. This gives the entering variable x_r and is indicated by an arrow at the bottom of the rth column. If there are more than one variables having the same most positive $c_j - z_j$ then, anyone of them can be selected arbitarily as the entering variable.

7. Find the leaving variable
compute the ratio $\theta = \text{Min}\left\{\frac{x_{Bi}}{a_{ir}}, a_{ir} > 0\right\}$

a) If all $a_{ir} \le 0$, then there is an unbounded solution to the given LPP.
b) If atleast one $a_{ir} > 0$, then the leaving variable is the basis variable corresponding tof the minimum ratio θ.

If $\theta = x_{Bi}/a_{ir}$, $a_{ir} > 0$, then the basic variable (pivot eqn) x_k to leave the basis called key row & the element at the intersection of key row & key column is called the key element (pivot element).

8. Form a newbasis by dropping the leaving variable & introducing the entering variable along with the associated value under C_B column. Convert the pivot element to unity by dividing the pivot equation by pivot element and all other elements in its column to zero by using Gauss elimination method.

$$\text{New element} = \text{old element} - \frac{\text{Product of element in the key row \& column}}{\text{Key element}}$$

9. Repeat the procedure until either an optimum solution is obtained or there is an indication of unbounded solution.

For Maximization Problem
1) If all $(c_j - z_j) \ge 0$, then the current feasible solution is optimal.
2) If atleast are $(c_j - z_j) < 0$, then the current basic feasible solution is not optimal.
3) The entering variable is the non-basic variable corresponding to the most positive value of $(c_j - z_j)$

For Minimization Problem
1) If all $(c_j - z_j) \le 0$, then the current basic feasible solution is optimal.
2) If atleast one $(c_j - z_j) > 0$, then the current basic feasible solution is not optimal

3) The entering variable is the non-basic variable corresponding to the most negative value of $(c_j - z_j)$

SIMPLEX PROBLEMS

1. Use simplex method to solve the LPP.

Max. $Z = 3x_1 + 2x_2$

Subject to $x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

Step 1 : Introduce the slack variables s_1, s_2 & then convert the inequality constraints to equality

LP Model : $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to $x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$x_1 ; x_2, s_1, s_2 \geq 0$

Step 2 : The non-basic variables x_1, x_2 is made zero. Then the initial basic feasible solution is

$s_1 = 4$, $s_2 = 2$

Initial Simplex Table

C_{B_i}	c_j	3	2	0	0	Solution	Min. Ratio
	Basic Variable S_B	x_1	x_2	s_1	s_2	x_B	x_B / x_n
0	s_1	1	1	1	0	4	$4/1 = 4$
0	s_2	①	-1	0	1	2	$2/1 = 2$ ←
	Z_j	0	0	0	0	0	
	$c_j - Z_j$	3 ↑	2	0	0		

$\boxed{Z_j = \sum C_{B_i} a_{ij}}$

Leaving variable s_2 , Entering variable x_1

Since some $c_j - Z_j > 0$, the current feasible solution is not optimum.

The leaving variable row is called the pivot row or key row.

∴ Pivot Element = 1

a) New pivot row element = $\frac{\text{old pivot row}}{\text{pivot element}}$

b) New row element = old element − $\left[\frac{\text{Product of elements in pivot row \& pivot column}}{\text{pivot element}}\right]$

New row element s_1

$x_1 = 1 - \frac{1*1}{1} = 0$ $\qquad$ $x_2 = +1 - \frac{(-1*1)}{1} = 2$

$s_1 = 1 - \frac{0*1}{1} = 1$ $\qquad$ $s_2 = 0 - \frac{1*1}{1} = -1$

$x_B = 4 - \frac{2*1}{1} = 2$

C_{Bi}	C_j	3	2	0	0	Solution	Min. Ratio
	Basic Variable S_B	x_1	x_2	S_1	S_2	x_B	x_B/x_n
0	S_1	0	②	1	−1	2	$2/2 = 1$
3	x_1	1	−1	0	1	2	$2/-1 = -2$
	z_j	3	−3	0	3	6	
	$c_j - z_j$	0	5 ↑	0	−3		

The leaving variable is S_1

Entering variable is $x_2 = 2$

Since $C_j - z_j > 0$, the current feasible solution is not optimum

The leaving variable row is called pivot row or key-row.

Pivot element = 2

$$\text{New pivot row element} = \frac{\text{Old pivot row}}{\text{pivot element}}$$

$$\text{New row element} = \text{Old element} - \left[\frac{\text{Prod. of element in pivot row \& pivot column}}{\text{pivot element}}\right]$$

∴ New row element x_1

$x_1 = 1 - \frac{0 * -1}{2} = 1$ $x_2 = -1 - \frac{2 * -1}{2} = 0$

$S_1 = 0 - \frac{1 * -1}{2} = 1/2$ $S_2 = 1 - \frac{(-1 * -1)}{2} = 1/2$

$X_B = 2 - \frac{2 * -1}{2} = 3$

C_{Bi}	c_j	3	2	0	0	Solution X_B
	Basic Variable S_B	x_1	x_2	S_1	S_2	
2	x_2	0	1	1/2	-1/2	1
3	x_1	1	0	1/2	1/2	3
	z_j	3	2	5/2	1/2	11
	$c_j - z_j$	0	0	-5/2	-1/2	

Since $c_j - z_j$ all are either zero or less than zero, the optimal solution is reached.

Optimal Solution

$z = 11$ $x_1 = 3$, $x_2 = 1$

$\therefore z = 3x_1 + 2x_2$

$= 3(3) + 2(1) = 11$

2. Use the simplex method to solve the following LPP.

Max. $z = 3x_1 + 5x_2 + 4x_3$

Subject to $2x_1 + 3x_2 \leq 8$

$2x_2 + 5x_3 \leq 10$

$3x_1 + 2x_2 + 4x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

Solution:

Step 1: Introduce the slack variables s_1, s_2, s_3 & then convert the inequality constraints as equalities.

Max. $z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$

$2x_1 + 3x_2 + s_1 = 8$

$2x_2 + 5x_3 + s_2 = 10$

$3x_1 + 2x_2 + 4x_3 + s_3 = 15$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Step 2:

The non-basic variables are set $x_1 = 0, x_2 = 0, x_3 = 0$.

The initial basic feasible solution is

$s_1 = 8$

$s_2 = 10$

$s_3 = 15$

C_{Bi}	c_j	3	5	4	0	0	0	Soln.	Min. ratio
	Basic variable s_B	x_1	x_2	x_3	s_1	s_2	s_3	X_B	X_B/X_n
0	s_1	2	③	0	1	0	0	8	$8/3 = 2.6$ ←
0	s_2	0	2	5	0	1	0	10	$10/2 = 5$
0	s_3	3	2	4	0	0	1	15	$15/2 = 7.5$
	z_j	0	0	0	0	0	0	0	
	$c_j - z_j$	3	5 ↑	4	0	0	0		

Leaving variable s_1

Entering variable $x_2 = 5$

$c_j - z_j \geq 0$, Current

Solution not optimum

C_{B_i}	C_j	3	5	4	0	0	0	Solution x_B	Min. ratio x_B/x_n
	Basic Variable S_B	x_1	x_2	x_3	s_1	s_2	s_3		
5	x_2	2/3	1	0	1/3	0	0	8/3	(8/3)/0 = -
0	s_2	-4/3	0	$\textcircled{5}$	-2/3	1	0	14/3	(14/3)/5
0	s_3	5/3	0	4	-2/3	0	1	29/3	(29/3)/4
	Z_j	10/3	5	0	5/3	0	0	40/3	
	$C_j - Z_j$	-1/3	0	+4 ↑	-5/3	0	0		

New row element, s_2

$x_1 = 0 - \frac{2*2}{3} = -4/3$

$x_2 = 2 - \frac{2*3}{3} = 0$

$x_3 = 5 - \frac{0*2}{3} = 5$

$s_1 = 0 - \frac{1*2}{3} = -2/3$

$s_2 = 1 - \frac{0*2}{3} = 1$

$s_3 = 0 - \frac{0*2}{3} = 0$

$x_B = 10 - \frac{8*2}{3} = 14/3$

New row element, s_3

$x_1 = 3 - \frac{2*2}{3} = 5/3$ $x_2 = 2 - \frac{3*2}{3} = 0$

$x_3 = 4 - \frac{0*2}{3} = 4$ $s_1 = 0 - \frac{1*2}{3} = -2/3$

$s_2 = 0 - \frac{0*2}{3} = 0$ $s_3 = 1 - \frac{0*2}{3} = 1$

$x_B = 15 - \frac{8*2}{3} = 29/3$

Entering var = x_3, Leaving Variable = s_2.

New row element; x_2

$x_1 = 2/3 - \frac{-4/3*0}{5} = 2/3$ $x_2 = 1 - \frac{0*0}{5} = 1$

$x_3 = 0 - \frac{5*0}{5} = 0$ $s_1 = +\frac{1}{3} - \frac{-2/3*0}{5} = 1/3$ $s_2 = 0 - \frac{1*0}{5} = 0$ $s_3 = 0 - \frac{0*0}{5} = 0$

$x_B = \frac{8}{3} - \frac{\frac{14}{3}*0}{5} = 8/3$.

C_{B_i}	C_j	3	5	4	0	0	0	Soln. x_B	Min. ratio. x_B/x_n
	Basic var. S_B	x_1	x_2	x_3	s_1	s_2	s_3		
5	x_2	2/3	1	0	1/3	0	0	8/3	(8/3)/(2/3)
4	x_3	-4/15	0	1	-2/15	1/5	0	14/15	(14/15)/(-4/15)
0	s_3	$\textcircled{41/15}$	0	0	-2/15	-4/5	1	89/15	(89/15)/(41/15)
	Z_j	34/15	5	4	17/15	4/5	0	256/15	
	$C_j - Z_j$	11/15 ↑	0	0	-17/15	-4/5	0		

New row element, s_3

$x_1 = \frac{5}{3} - \frac{-4/3*4}{5} = \frac{25+16}{15} = 41/15$

$x_2 = 0 - \frac{0*4}{5} = 0$

$x_3 = 4 - \frac{5*4}{5} = 0$

$s_1 = \frac{-2}{3} - \frac{\frac{-2}{3}*4}{5} = \frac{-10+8}{15} = -2/15$

$s_2 = 0 - \frac{1*4}{5} = -4/5$

$s_3 = 1 - \frac{0*4}{5} = 1$

$x_B = \frac{29}{3} - \frac{14}{3} * \frac{4}{5} = \frac{89}{15}$

Entering variable x_1 leaving variable = s_3

The current solution is not feasible.

New row element, x_2

$x_1 = \frac{2}{3} - \frac{41/15 * 2/3}{41/15} = 0$ $x_2 = 1 - \frac{0*2/3}{41/15} = 1$

$x_3 = 0 - \frac{0*2/3}{41/15} = 0$ $s_1 = \frac{1}{3} - \frac{-2/15 * 2/3}{41/15} = \frac{1}{3} + \frac{4}{3\times41} = \frac{15}{41}$

$s_2 = 0 - \frac{-4/5 * 2/3}{41/15} = 8/41$ $s_3 = 0 - \frac{1*2/3}{41/15} = \frac{-10}{41}$

$x_B = \frac{8}{3} - \frac{89/15 * 2/3}{41/15} = 50/41$

New row element, x_3

$x_1 = \frac{-4}{15} - \frac{\frac{-4}{15} * 41/15}{41/15} = 0$ $x_2 = 0 - \frac{-4/15 * 0}{41/15} = 0$ $x_3 = 1 - \frac{-4/15 * 0}{41/15} = 1$

$s_1 = \frac{-2}{15} - \frac{\frac{-4}{15} * -2/15}{41/15} = \frac{-6}{41}$ $s_2 = \frac{[illegible]}{05} - \frac{\frac{-4}{15} * \frac{-4}{05}}{41/15} = \frac{5}{41}$ $s_3 = 0 - \frac{1 * -4/15}{41/15} = 4/41$

$x_B = \frac{14}{15} - \frac{89/15 * -4/15}{41/15} = 62/41$

	cj	3	5	4	0	0	0		
C_{B_i}	Basic variable S_B	x_1	x_2	x_3	s_1	s_2	s_3	Soln. x_B	Ratio
5	x_2	0	1	0	15/41	8/41	-10/41	50/41	
4	x_3	0	0	1	-6/41	5/41	4/41	62/41	
3	x_1	1	0	0	-2/41	-12/41	15/41	89/41	
	zj	3	5	4	45/41	24/41	11/41	765/41	
	cj - zj	0	0	0	-45/41	-24/41	-11/41		

Since all $c_j - z_j \leq 0$ for non-basic variables, Optimum Solution is reached.

$\therefore x_1 = 89/41$ $x_2 = 50/41$ $x_3 = 62/41$ $Z = 765/41$

ie. $Z = 3x_1 + 5x_2 + 4x_3$

$Z = 3(89/41) + 5(50/41) + 4(62/41)$

$Z = 765/41$

3. An automobile manufactures makes auto-mobiles & trucks in a factory that is divided into two s . . shop A, which performs the basic assembly operation must work 5 man-days on each truck but only 2 man-days on each truck but only 2 man-days on each automobile. Shop B, which performs finishing operation must work 3 man-days for each truck or automobile that it produces. Because of men and machine limitations shop A has 180 man-days per week available while shop B has 135 man-days per week. If the manufacturer makes a profit of Rs. 30 on each truck & Rs. 200 on each automobile, how many of each should he produce to maximize the profit?

	Truck	Automobile	Limitations
Shop A	5	2	180
Shop B	3	3	135
Profit (Rs.)	300	200	

LPP Model: Max $z = 300x_1 + 200x_2$

Subject to $5x_1 + 2x_2 \leq 180$

$3x_1 + 3x_2 \leq 135$

$x_1, x_2 \geq 0$

Step 1 : Introduce a slack variable s_1, s_2 & then convert the inequality constraint as equality

$$\text{Max } z = 300x_1 + 200x_2 + 0s_1 + 0s_2$$

$$5x_1 + 2x_2 + s_1 = 180$$

$$3x_1 + 3x_2 + s_2 = 135$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Step 2 : The non-basic variable $x_1, x_2 = 0$. The initial basic feasible solution $s_1 = 180$, $s_2 = 135$.

Initial Simplex table

C_{Bi}	Basic Var S_B c_j	300 x_1	200 x_2	0 s_1	0 s_2	Soln. x_B	Min. ratio
0	s_1	$\textcircled{5}$	2	1	0	180	180/5=36 ←
0	s_2	3	3	0	1	135	135/3=45
	z_j	0	0	0	0	0	
	$c_j - z_j$	300 ↑	200	0	0		

Leaving variable s_1 Entering variable $x_1 = 300$

As $c_j - z_j > 0$, the current solution is not optimum

New row element, s_2

Old s_2 row	3	3	0	1	135
Value of s_2 in pivot col × New x_1 row (ie) 3 × New x_1 row	3	6/5	3/5	0	108
Subtract New s_2 row	0	9/5	-3/5	1	27

	c_j	300	200	0	0		
C_{Bi}	S_B	x_1	x_2	s_1	s_2	X_B	Min. ratio
300	x_1	1	2/5	1/5	0	36	$\frac{36}{2/5}=90$
0	s_2	0	$\boxed{9/5}$	-3/5	01	27	$\frac{27}{9/5}=15$ ←
	z_j	300	120	60	0	10800	
	c_j-z_j	0	80 ↑	-60	0		

As $c_j - z_j \geq 0$, the current soln. is not optimum.

Entering Variable = x_2 Leaving Variable - s_2

New row x_1

Old x_1 row	1	2/5	1/5	0	36
Value of x_2 in pivot column × New x_2 row	0	2/5	-2/15	2/9	6
New x_1 row	1	0	1/3	-2/9	30

	c_j	300	200	0	0		
C_{Bi}	S_B	x_1	x_2	s_1	S_2	X_B	Min. ratio
300	x_1	1	0	1/3	-2/9	30	
200	x_2	0	1	-1/3	5/9	15	
	z_j	300	200	100/3	400/9	12000	
	c_j-z_j	0	0	-100/3	-400/9		

As $c_j - z_j \leq 0$ for basic variables, the current solution is optimal.

Optimal Solution

$Z = 12000$, $x_1 = 30$ & $x_2 = 15$

(ie) $Z = 300(30) + 200(15) = 9000 + 3000 = 12000$

4. Reddy mikks produces both interior & exterior paints from two raw materials, M_1 & M_2. The following table provides the basic data of the problem.

	Exterior paint	Interior paint	Max. Availability
Raw Material M_1	6	4	24
Raw Material M_2	1	2	6
Profit Rs.	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior by more than 1 ton. Also the maximum daily demand of interior paint is 2 tons. Use the simplex method to maximize his profit?

LP Model:

Max. $Z = 5x_1 + 4x_2$

Subject to
$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$-x_1 + x_2 \leq 1$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Step 1: Convert the inequality constraints to equalities by introducing slack variables s_1, s_2, s_3, s_4

Max. $Z = 5x_1 + 4x_2 + 0s_1 + 0s_2$
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$$

Step 2: The non basic variables x_1, x_2, where $x_1 = 0, x_2 = 0$

Initial basic feasible solution is

$s_1 = 24$, $s_2 = 6$, $s_3 = 1$, $s_4 = 2$

Initial Simplex Table

Cj 5 4 0 0 0 0

C_{B_i}	S_B	x_1	x_2	s_1	s_2	s_3	s_4	X_B	Min. Ratio
0	s_1	6	4	1	0	0	0	24	$\frac{24}{6} = 4$ ←
0	s_2	1	2	0	1	0	0	6	$\frac{6}{1} = 6$
0	s_3	-1	1	0	0	1	0	1	$\frac{1}{-1} = -1$
0	s_4	0	1	0	0	0	1	2	$\frac{2}{0} = \infty$
	Z_j	0	0	0	0	0	0	0	
	$C_j - Z_j$	5 ↑	4	0	0	0	0		

Do not consider -ve min. ratio to find the leaving var.

∴ Entering Element = x

Leaving element = s_1

As $C_j - Z_j > 0$, the current solution is not optimum.

Cj 5 4 0 0 0 0

C_{B_i}	S_B	x_1	x_2	s_1	s_2	s_3	s_4	X_B	Min. ratio
5	x_1	1	2/3	1/6	0	0	0	4	$\frac{4}{2/3} = 6$
0	s_2	0	4/3	-1/6	1	0	0	2	$\frac{2}{4/3} = 3/2$ ←
0	s_3	0	5/3	1/6	0	1	0	5	$\frac{5}{5/3} = 3$
0	s_4	0	1	0	0	0	1	2	$2/1 = 2$
	Z_j	5	10/3	5/6	0	0	0	20	
	$C_j - Z_j$	0	2/3 ↑	-5/6	0	0	0		

Entering Var = x_2, Leaving Variable = s_2

As $C_j - Z_j > 0$, the current solution is not optimum.

Cj 5 4 0 0 0 0

C_{B_i}	S_B	x_1	x_2	s_1	s_2	s_3	s_4	X_B	Min. ratio
5	x_1	1	0	1/4	-1/2	0	0	3	
4	x_2	0	1	-1/8	3/4	0	0	3/2	
0	s_3	0	0	3/8	-5/4	1	0	5/2	
0	s_4	0	0	1/8	-3/4	0	1	1/2	
	Z_j	5	4	3/4	1/2	0	0	21	
	$C_j - Z_j$	0	0	-3/4	-1/2	0	0		

As all $C_j - Z_j \leq 0$, the current solution is optimum.

Optimum solution, $Z = 21$

$x_1 = 3$, $x_2 = 3/2$

To find New row s_2

Old s_2 row	1	2	0	1	0	0	6
1 × New x_1 row (sub)	1	2/3	1/6	0	0	0	4
New s_2 row	0	4/3	-1/6	1	0	0	2

To find New row s_3

Old s_3 row	-1	1	0	0	1	0	1
-1 × New x_1 row (-)	-1	-2/3	-1/6	0	0	0	-4
New s_3 row	0	5/3	1/6	0	1	0	5

To find New row s_4

Old s_4 row	0	1	0	0	0	1	2

As x_1 value of s_4 row is zero, no need any calculation.

To find New row x_1

Old x_1 row	1	2/3	1/6	0	0	0	4
$\frac{2}{3}$ × New x_2 Row	0	2/3	-1/12	1/2	0	0	1
New x_1 row	[illegible]	0	1/4	-1/2	0	0	3

To find New row s_3

Old s_3 row	0	5/3	1/6	0	1	0	5
$\frac{5}{3}$ × New x_2 row	0	5/3	$\frac{-5}{24}$	$\frac{5}{4}$	0	0	5/
New s_3 row	0	0	[illegible]	$\frac{-5}{4}$	1	0	5/

To find New row s_4.

Old s_4 row	0	1	0	0	0	1	2
1 × New x_2 row	0	1	-1/8	3/4	0	0	3/2
New s_4 row	0	0	1/8	-3/4	0	1	1/2

5. A gear manufacturing company received an order for three specific types of gears for regular supply. The management is considering to devote the available excess capacity to one or more of the three types A, B & C. The available capacity on the machines which might limit output & the number of machine hours required for each unit of the respective gear is also given below.

Machine Type	Available machine hrs/week	Productivity in machine hours/unit		
		Gear A	Gear B	Gear C
Hobbing m/c	250	8	2	3
Shapping m/c	150	4	3	0
Grinding	50	2	-	1
Profit (Rs)		20	6	8

Find how much of gear the company should produce in order to maximize profit.

LP model: Max. $Z = 20x_1 + 6x_2 + 8x_3$

subject to $8x_1 + 2x_2 + 3x_3 \leq 250$

$4x_1 + 3x_2 \leq 150$

$2x_1 + x_3 \leq 50$

$x_1, x_2, x_3 \geq 0$

Step 1 : Convert inequality constraints to equality constraints by introducing slack variables.

Max. $Z = 20x_1 + 6x_2 + 8x_3 + 0S_1 + 0S_2 + 0S_3$

Subject to. $8x_1 + 2x_2 + 3x_3 + S_1 = 250$

$4x_1 + 3x_2 + S_2 = 150$

$2x_1 + x_3 + S_3 = 50$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Step 2 : The Non basic variables $x_1 = x_2 = x_3 = 0$.

The initial basic feasible solution is

$S_1 = 250$

$S_2 = 150$

$S_3 = 50$.

Initial Simplex Table

c_j 20 6 8 0 0 0

c_{B_i}	S_B	x_1	x_2	x_3	s_1	s_2	s_3	X_B	Min. ratio
0	s_1	8	2	3	1	0	0	250	$\frac{250}{8}$ = 31.2
0	s_2	4	3	0	0	1	0	150	$\frac{150}{4}$ = 37.5
0	s_3	②	0	1	0	0	1	50	$\frac{50}{2}$ = 25 ←
	z_j	0	0	0	0	0	0	0	
	$c_j - z_j$	20 ↑	6	8	0	0	0		

Entering Variable x_1
Leaving variable s_3
As $c_j - z_j > 0$, the current solution is not optimal.

c_j 20 6 8 0 0 0

c_{B_i}	S_B	x_1	x_2	x_3	s_1	s_2	s_3	X_B	Min. ratio
0	s_1	0	2	-1	1	0	-4	50	50/2 = 25
0	s_2	0	③	-2	0	1	-2	50	50/3 = 16.6 ←
20	x_1	1	0	1/2	0	0	1/2	25	25/0 = 0
	z_j	20	0	10	0	0	10	500	
	$c_j - z_j$	0	6 ↑	-2	0	0	-10		

Entering var = x_2, Leaving var = s_2
As $c_j - z_j > 0$, current soln. not optimal.

c_j 20 6 8 0 0 0

c_{B_i}	S_B	x_1	x_2	x_3	s_1	s_2	s_3	X_B	Min. ratio
0	s_1	0	0	(1/3)	1	-2/3	-8/3	50/3	50 ←
6	x_2	0	1	-2/3	0	1/3	-2/3	50/3	-ve (NA)
20	x_1	1	0	1/2	0	0	1/2	25	50
	z_j	20	6	6	0	2	6	600	
	$c_j - z_j$	0	0	2 ↑	0	-2	-6		

Entering variable = x_3, Leaving var. = s_1
As $c_j - z_j > 0$, Current soln. is not optimal

c_j 20 6 8 0 0 0

c_{B_i}	S_B	x_1	x_2	x_3	s_1	s_2	s_3	X_B	Min. ratio
8	x_3	0	0	1	3	-2	-8	50	
6	x_2	0	[illegible]	0	2	5/3	14/3	50	
20	x_1	1	0	0	-3/2	1	9/2	0	
	z_j	20	6	8	6	14	54	700	
	$c_j - z_j$	0	0	0	-6	-14	-54		

As all $c_j - z_j \le 0$, the current solution is optimal.

Optimal solution is

$Z = 700$, $x_1 = 0$, $x_2 = 50$, $x_3 = 50$

To find New row s_1

Old s_1 = 8 2 3 1 0 0 250
8 × New x_1 (−) = 8 0 4 0 0 4 200
New s_1 = 0 2 -1 1 0 -4 50

To find New row s_2

Old s_2 = 4 3 0 0 1 0 150
4 × New x_1 (−) = 4 0 2 0 0 2 100
New s_1 = 0 3 -2 0 1 -2 50

To find New row s_1

Old s_1 = 0 2 -1 1 0 -4 50
2 × New x_2 (−) = 0 2 -4/3 0 2/3 -4/3 100/
New s_1 = 0 0 1/3 1 -2/3 -8/3 50/

To find New row x_1

Old x_1 row contains x_2 value as zero. No need to do changes.

To find new row x_2

Old x_2 = 0 1 -2/3 0 1/3 -2/3 50/3
$\frac{-2}{3}$ × New pivot Row (x_3) (−) = 0 0 -2/3 -2 -4/3 -16/3 $\frac{-100}{3}$
New x_2 = 0 1 0 2 5/3 14/3 50

To find new row x_1

Old x_1 = 1 0 1/2 0 0 1/2 25
$\frac{1}{2}$ × New pivot row (x_3) (−) = 0 0 1/2 3/2 -1 -4 25
New x_3 row = 1 0 0 -3/2 1 9/2 0

Artificial Starting Solution

In certain cases, it is difficult to obtain an initial basic feasible solution of the given LP problem.

The reason is

(i) when the constraints are of the type ≤,

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \ , \ x_j \geq 0$$

and value of few right hand side constants is negative (ie., $b_i < 0$). After adding the non-negative slack variable s_i the initial solution so obtained will be $s_i = -b_i$ $(i=1,2,\cdots m)$, the initial solution so obtained will be This solution is not feasible for a particular resource, i. This solution is not feasible because it does not satisfy the non-negativity conditions of slack variables (ie., $s_i \geq 0$)

(ii) when the constraints are of ≥ type

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \ , \ x_j \geq 0$$

After adding surplus (negative slack) variable s_i, the initial ~~b~~ solution so obtained will be $-s_i = b_i$ or $s_i = -b_i$

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \ , \ x_j \geq 0 \ , \ s_i \geq 0$$

This solution is not feasible because it does not satisfy non negativity conditions of surplus (ie., $s_i \geq 0$). In such a case, artificial variables, A_i $(i=1,2,\cdots,m)$ are added to get an initial basic feasible solution. The resulting system of equations then becomes

$$\sum_{j=1}^{n} a_{ij} x_j - s_i + A_i = b_i$$

$$x_i, s_i, A_i \geq 0 \ , \ i = 1,2,\cdots,m.$$

The solution $A_i = b_i$ $(i=1,2,\cdots m)$ is not the solution to the original system of equations. Therefore, Artificial variables must be removed from optimal solutions. There are 2 methods for removing artificial variables.

(1) Big-M Method or Penalty Method (2) Two-Phase Method.

BIG-M Method [PENALTY METHOD]

- A method to remove artificial variable.
- A large undesirable coefficients to artificial variables are assigned to the objective function.
- If z is to be minimized, a verge large positive price (penalty) is assigned to each artificial variable.
- if z is to be maximized, a very large negative price (penalty) is assigned to each artificial variable.

Algorithm:

Step 1: Express the LP problem in the standard form by adding slack variables, surplus variables and/or artificial variables. Assign to zero coefficient to slack and surplus variables. Then assign a very large coefficient +M (Minimization case) and -M (Maximization) to artificial variable in the objective function.

Step 2: The initial basic feasible solution is obtained by assigning zero value to decision variables, $x_1, x_2, \cdots$ etc.

Step 3: Calculate $c_j - z_j$ in last row of simplex table & examine

(i) if all $c_j - z_j \leq 0$, then the current basic feasible soln. is optimal

(ii) if for a column k, $c_k - z_k$ is most negative & all entries in this column are negative, then the problem has an unbounded optim solution.

(iii) if one or more $c_j - z_j < 0$ (minimization case), then select the variable to enter into the basis (soln. mix) with the largest negative $c_j - z_j$ value.

Step 4: Determine the key row and key element in the same manner as discussed in simplex algorithm for the maximization case.

Step 5: Continue with the procedure to update solution at each iteration till optimal solution is obtained.

1. Solve the following LPP by Penalty Method or BIGM Method

Max $Z = 3x_1 + 2x_2$

Sub. to $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1, x_2 \geq 0$

Step 1: Convert inequality constraints to equality constraints by adding slack, surplus and artificial variables.

Max. $Z = 3x_1 + 2x_2 + 0S_1 - 0S_2 - MA_1$

sub. to $2x_1 + x_2 + S_1 = 2$

$3x_1 + 4x_2 - S_2 + A_1 = 12$

$x_1, x_2, S_1, S_2, A_1 \geq 0$

Step 2: Assign 0 to decision variables and surplus variables to get basic solution.

$S_1 = 2$, $A_1 = 12$

Non-Basic var = x_1, x_2, S_2

Basic var = S_1, A_1

	c_j	3	2	0	0	$-M$		
C_{B_i}	S_B	x_1	x_2	S_1	S_2	A_1	X_B	Min. Ratio
0	S_1	2	$\textcircled{1}$	[illegible]	0	0	2	2 ←
$-M$	A_1	3	4	0	-1	1	12	3
	Z_j	$-3M$	$-4M$	0	M	$-M$		
	$c_j - z_j$	$3+3M$	$2+4M$ ↑	0	$-M$	0		

As $c_j - z_j \geq 0$, Current soln. is not optimal

Entering variable = x_2. Leaving Variable = S_1

	c_j	3	2	0	0	$-M$		
C_{B_i}	S_B	x_1	x_2	S_1	S_2	A_1	X_B	Min ratio
2	x_2	2	1	[illegible]	0	0	2	
$-M$	A_1	-5	0	-4	$-$[illegible]	$+1$	4	
	Z_j	$4+5M$	2	$2+4M$	$+M$	$-M$	$4-4M$	
	$c_j - z_j$	$-1-5M$	0	$-2-4M$	$-M$	0		

New A_1 row

old A_1 row	3	4	0	-1	1	12
$4 \times$ New x_2 row (-)	8	4	4	0	0	8
New A_1 row	-5	0	-4	-1	1	4

$c_j - z_j \leq 0$, But there exist artificial variable in the table.

The LPP posses no feasible solution.

2. Use penalty (Big m) method to solve the following LP problem

Min $z = 5x_1 + 3x_2$

sub to $2x_1 + 4x_2 \leq 12$

$2x_1 + 2x_2 = 10$

$5x_1 + 2x_2 \geq 10$

$x_1, x_2 \geq 0$.

Step 1 : convert Min to Max

Max $z^* = -5x_1 - 3x_2$

sub to $2x_1 + 4x_2 \leq 12$

$2x_1 + 2x_2 = 10$

$5x_1 + 2x_2 \geq 10$

$x_1, x_2 \geq 0$

Step 2 : Introduce slack, surplus & artificial variables

Max $z^* = -5x_1 - 3x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

$2x_1 + 4x_2 + s_1 = 12$

$2x_1 + 2x_2 + A_1 = 10$

$5x_1 + 2x_2 - s_2 + A_2 = 10$

$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$.

Step 3 : Basic solution is $x_1 = x_2 = s_2 = 0$ &

$s_1 = 12$, $A_1 = 10$, $A_2 = 10$

C_j -5 -3 0 0 -M -M

C_B	S_B	x_1	x_2	s_1	s_2	A_1	A_2	X_B	Min ratio
0	s_1	2	4	1	0	0	0	12	$\frac{12}{2} = 6$
-M	A_1	2	2	0	0	1	0	10	$\frac{10}{2} = 5$
-M	A_2	⑤	2	0	-1	0	1	10	$\frac{10}{5} = 2$ →
z_j		-7M	-4M	0	M	-M	-M	-20M	
$c_j - z_j$		-5 +7M	-3 +4M	0	-M	0	0		
		↑							

$c_j - z_j > 0$. Soln. not optimal.

Entering var = x_1 Leaving var = A_2

c_j -5 -3 0 0 -M

C_B	S_B	x_1	x_2	S_1	S_2	A_1	X_B	Min Ratio
0	S_1	0	$\boxed{16/5}$	1	2/5	0	8	5/2 →
-M	A_1	0	6/5	0	2/5	1	6	5
-5	x_1	1	2/5	0	-1/5	0	2	5
z_j		-5	$-\frac{6M}{5}-2$	0	$\frac{-2M}{5}+1$	-M	-6M -10	
c_j-z_j		0	$-1+\frac{6M}{5}$ ↑	0	$1-\frac{2M}{5}$	0		

old S_1	$\underline{2}$	$\underline{4}$	1	0	0	12
$2*x_1$	2	4/5	0	-2/5	0	4
(-)						
	0	16/5	1	2/5	0	8

old A_1	2	2	0	0	1	10
$2*x_1$	2	4/5	0	-2/5	0	4
(-)						
	0	6/5	0	2/5	1	6

$c_j-z_j>0$, soln- not optimal

Ent. var $=x_2$ Leaving var $=S_1$

c_j -5 -3 0 0 -M

C_B	S_B	x_1	x_2	S_1	S_2	A_1	X_B	Ratio
-3	x_2	0	1	5/16	1/8	0	5/2	20
-M	A_1	0	0	-3/8	$\boxed{1/4}$	1	3	12 →
-5	x_1	1	0	-1/8	-1/4	0	1	-
z_j		-5	-3	3M/8 -5/16	7/8 - M/4	-M	-5 -3M	
c_j-z_j		0	0	-3M/8 +5/16	M/4 -7/8 ↑	0		

old A_1	0	$\underline{6/5}$	0	2/5	1	6
$\frac{6}{5}*x_2$	0	6/5	3/8	3/20	0	3
(-)						
	0	0	-3/8	1/4	1	3

old x_1	1	$\underline{2/5}$	0	-1/5	0	2
$\frac{2}{5}*x_2$	0	2/5	1/8	1/20	0	1
-						
	1	0	-1/8	-1/4	0	1

$c_j-z_j>0$, Soln not optimal

Ent- var $=S_2$ Leaving var $=A_1$

c_j -5 -3 0 0

C_B	S_B	x_1	x_2	S_1	S_2	X_B	
-3	x_2	0	1	1/2	0	1	
0	S_2	0	0	-3/2	1	12	
-5	x_1	1	0	-1/2	0	4	
z_j		-5	-3	+1	0	-23	
c_j-z_j		0	0	-1	0		

old x_2	0	1	5/16	$\underline{1/8}$	5/2
$1/8*S_2$	0	0	-3/16	1/8	3/2
	0	1	1/2	0	1

old x_1	1	0	-1/8	-1/4	1
$\frac{-1}{4}*S_2$,	0	0	3/8	-1/4	-3
	1	0	-1/2	0	4

$c_j-z_j\le 0$. Soln Optimal

Max $z^*=-23$ $x_1=4$ $x_2=1$

Min $z=23$ $x_1=4$ $x_2=1$

3. Use penalty Method (Big.M Method)

Min $z = 4x_1 + x_2$

Sub to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

step 1:
Convert Min to Max: Min (z) = - Max (-z)

Max $z^* = -4x_1 - x_2$

Step 2: Convert ineq. to eq. by introducing slack, surplus, Artificial var

Max $z^* = -4x_1 - x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$

Sub. to. $3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - s_1 + A_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

Step 3: Basic Solution.

Non - Basic variables $x_1 = x_2 = s_1 = 0$

$A_1 = 3$ $A_2 = 6$ $s_2 = 4$ (Basic Variables)

C_j -4 -1 0 0 -M -M

CB	Bas Var SB	x_1	x_2	s_1	s_2	A_1	A_2	X_B	Ratio
-M	A_1	3	1	0	0	1	0	3	1 →
-M	A_2	4	3	-1	0	0	1	6	1½
0	s_2	1	2	0	1	0	0	4	4
z_j		-7M	-4M	M	0	-M	-M	-9M	
$c_j - z_j$		-4 +7M ↑	-1 +4M	-M	0	0	0		

$c_j - z_j > 0$. ∴ Soln not optimal.

Ent. Var = x_1 Leaving Var = A_1

C_j −4 −1 0 0 −M

C_B	S_B	x_1	x_2	S_1	S_2	A_2	X_B	Ratio
−4	x_1	1	1/3	0	0	0	1	3
−M	A_2	0	$\enclose{circle}{5/3}$	−1	0	1	2	6/5 →
0	S_2	0	5/3	0	1	0	3	9/5
Z_j		−4	$\frac{-4}{3}-\frac{5M}{3}$	M	0	−M	−4 −2M	
$C_j - Z_j$		0	$\frac{1}{3}+\frac{5M}{3}$ ↑	−M	0	0		

old A_2 $\underline{4}$ 3 −1 0 [illegible] 6

$4 * x_1$ 4 $\frac{4}{3}$ 0 0 0 4

(−)

0 5/3 −1 0 1 2

old S_2 1 2 0 1 0 4

$1 * x_1$ 1 1/3 0 0 0 1

(−)

0 5/3 0 1 0 3

Some $C_j - Z_j > 0$. Soln not optimal

Entering Var = x_2 Leaving variable = A_2

C_j −4 −1 0 0

C_B	S_B	x_1	x_2	S_1	S_2	X_B	Min Ratio
−4	x_1	1	0	1/5	0	3/5	3
−1	x_2	0	1	−3/5	0	6/5	—
0	S_2	0	0	①	1	1	1 →
Z_j		−4	−1	−1/5	0	−18/5	
$C_j - Z_j$		0	0	1/5 ↑	0		

old x_1 1 $\underline{1/3}$ 0 0 1

$\frac{1}{3} * x_2$ 0 1/3 −1/5 0 2/5

(−)

1 0 1/5 0 3/5

old S_2 0 5/3 0 1 3

$5/3 * x_2$ 0 5/3 −1 0 2

(−)

0 0 1 1 1

Some $C_j - Z_j > 0$. Soln. not optimal.

Entering variable = S_1. Leaving Variable = S_2

C_j −4 −1 0 0

C_B	S_B	x_1	x_2	S_1	S_2	X_B
−4	x_1	1	0	0	−1/5	2/5
−1	x_2	0	1	0	3/5	9/5
0	S_1	0	0	1	1	1
	Z_j	−4	−1	0	1/5	−17/5
$C_j -$	Z_j	0	0	0	−1/5	

old x_1 1 0 $\underline{1/5}$ 0 3/5

$\frac{1}{5} * S_1$ 0 0 1/5 1/5 1/5

(−)

1 0 0 −1/5 2/5

old x_2 0 1 $\underline{-3/5}$ 0 6/5

$3/5 * S_1$ 0 0 3/5 3/5 3/5

(+)

0 1 0 3/5 9/5

$C_j - Z_j \leq 0$ for all. Solution is optimal

Optimal Soln Max $Z^* = -17/5$

$\boxed{\text{Min } Z = 17/5 \qquad x_1 = 2/5 \qquad x_2 = 9/5}$

4. Solve using Big M Method

Min $z = 2x_1 + 3x_2$

Sub. to. $x_1 + x_2 \geq 6$

$7x_1 + x_2 \geq 14$

$x_1, x_2 \geq 0$.

Step 1 : Convert Min to Max & introduce surplus & Artificial Var.

Max $z^* = -2x_1 - 3x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Sub. to $x_1 + x_2 - S_1 + A_1 = 6$

$7x_1 + x_2 - S_2 + A_2 = 14$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$.

Step 2 : Basic Solution.

$x_1 = x_2 = S_1 = S_2 = 0$

$A_1 = 6 \quad A_2 = 14$

	C_j	-2	-3	0	0	$-M$	$-M$		
C_B	S_B	x_1	x_2	S_1	S_2	A_1	A_2	X_B	Min ratio
$-M$	A_1	1	1	-1	0	1	0	6	6
$-M$	A_2	$\boxed{7}$	1	0	-1	0	1	14	2 →
	z_j	$-8M$	$-2M$	M	M	$-M$	$-M$	$-20M$	
	$C_j - z_j$	$-2+8M$	$-3+2M$	$-M$	$-M$	0	0		

↑

Some $C_j - z_j > 0$ soln. not optimal.

Entering Var $= x_1$ Leaving Var $= A_2$

	C_j	-2	-3	0	0	$-M$		
C_B	S_B	x_1	x_2	S_1	S_2	A_1	X_B	Min ratio
$-M$	A_1	0	$\boxed{6/7}$	-1	$1/7$	1	4	$\frac{14}{3} \rightarrow$
-2	x_1	1	$1/7$	0	$-1/7$	0	2	14
Z_j		-2	$-\frac{2}{7}-\frac{6M}{7}$	$-M$	$\frac{2}{7}-\frac{M}{7}$	M	$-4-4M$	
C_j-Z_j		0	$-\frac{19}{7}+\frac{6M}{7}$	M	$-\frac{2}{7}+\frac{M}{7}$	0		
			$\uparrow$					

old A_1 $\quad \underline{1} \quad 1 \quad -1 \quad 0 \quad 1 \quad 6$

$1 * x_1$ $\quad 1 \quad 1/7 \quad 0 \quad -1/7 \quad 0 \quad 2$

(−)

$0 \quad 6/7 \quad -1 \quad 1/7 \quad 1 \quad 4$

Some $C_j - Z_j > 0$. Soln not optimal.

Entering var $= x_2$ Leaving var $= A_1$

	C_j	-2	-3	0	0		
C_B	S_B	x_1	x_2	S_1	S_2	X_B	Min Ratio
-3	x_2	0	1	$-7/6$	$\boxed{1/6}$	$14/3$	$28 \rightarrow$
-2	x_1	1	0	$1/6$	$-1/6$	$4/3$	—
Z_j		-2	-3	$+19/6$	$-1/6$	$-50/3$	
C_j-Z_j		0	0	$-19/6$	$1/6$		
					$\uparrow$		

old x_1 $\quad 1 \quad \underline{1/7} \quad 0 \quad -1/7 \quad 2$

$\frac{1}{7} * x_2$ $\quad 0 \quad 1/7 \quad -1/6 \quad +1/42 \quad 2/3$

(−)

$1 \quad 0 \quad 1/6 \quad -1/6 \quad 4/3$

Some $C_j - Z_j > 0$. Soln - not optimal.

Entering var $= S_2$ Leaving variable $= x_2$

	C_j	-2	-3	0	0	
C_B	S_B	x_1	x_2	S_1	S_2	X_B
0	S_2	0	6	-7	1	28
-2	x_1	1	1	-1	0	6
Z_j		-2	-2	2	0	-12
C_j-Z_j		0	-1	-2	0	

old x_1 $\quad 1 \quad 0 \quad 1/6 \quad \underline{-1/6} \quad 4/3$

$\frac{+1}{6} * S_2$ $\quad 0 \quad 1 \quad -7/6 \quad 1/6 \quad 14/3$

(+)

$1 \quad 1 \quad -1 \quad 0 \quad 6$

$C_j - Z_j \le 0$ Soln is optimal

Max $Z^* = -12$

optimal soln. Min $Z = 12$

$x_1 = 6 \quad x_2 = 0$

5 Solve the following LPP using Big·M Method

Min $Z = 10x_1 + 15x_2 + 20x_3$

$2x_1 + 4x_2 + 6x_3 \geq 24$

$3x_1 + 9x_2 + 6x_3 \geq 30$

$x_1, x_2, x_3 \geq 0$

Step 1 : Convert Min to Max
Introduce surplus & artificial variables.

Max $Z^* = -10x_1 - 15x_2 - 20x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

$2x_1 + 4x_2 + 6x_3 - S_1 + A_1 = 24$

$3x_1 + 9x_2 + 6x_3 - S_2 + A_2 = 30$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

Step 2 : Basic Solution

$x_1 = x_2 = x_3 = S_1 = S_2 = 0$ (Non-basic Var)

$A_1 = 24 \quad A_2 = 30$ (Basic Var.)

	Cj	-10	-15	-20	0	0	-M	-M		
CB	SB	x_1	x_2	x_3	S_1	S_2	A_1	A_2	X_B	Min ratio
-M	A_1	2	4	6	-1	0	1	0	24	6
-M	A_2	3	⑨	6	0	-1	0	1	30	$\frac{30}{9} = \frac{10}{3}$ →
	Z_j	-5M	-13M	-12M	M	M	-M	-M	-54M	
$C_j - Z_j$		-10 +5M	-15 +13M ↑	-20 +12M	-M	-M	0	0		

$C_j - Z_j > 0$ for some variables.
So, soln. not optimal.

Entering variable $= x_2$ Leaving variable $= A_2$

C_j -10 -15 -20 0 0 -M

C_B	S_B	x_1	x_2	x_3	s_1	s_2	A_1	X_B	Min ratio
-M	A_1	$\frac{2}{3}$	0	$\boxed{\frac{10}{3}}$	-1	4/9	1	32/3	16/5 $\rightarrow$
-15	x_2	1/3	1	2/3	0	-1/9	0	10/3	5
z_j		$\frac{-2M}{3} - 5$	-15	$\frac{-10M}{3} - 10$	M	$-4M/9$ $+5/3$	-M		
$c_j - z_j$		$-5 + \frac{2M}{3}$	0	$-10 + \frac{10M}{3}$ $\uparrow$	-M	$-5/3 + \frac{4M}{9}$	0		

$$\begin{array}{lccccccc} \text{old } A_1 & 2 & \underline{4} & 6 & -1 & 0 & 1 & 24 \\ 4 * x_2 & \frac{4}{3} & 4 & \frac{8}{3} & 0 & \frac{-4}{9} & 0 & \frac{40}{3} \\ (-) & & & & & & & \\ \hline & \frac{2}{3} & 0 & \frac{10}{3} & -1 & \frac{4}{9} & 1 & \end{array}$$

$C_j - z_j > 0$ for some var. soln. not optimal

E.V = x_3 L.V = A_1

C_j -10 -15 -20 0 0

C_B	S_B	x_1	x_2	x_3	s_1	s_2	X_B
-20	x_3	1/5	0	1	-3/10	2/15	16/5
-15	x_2	1/5	1	0	1/5	-1/5	6/5
z_j		-7	-15	-20	3	1/3	-82
$C_j - z_j$		-3	0	0	-3	-1/3	

$$\begin{array}{lcccccc} \text{old } x_2 & \frac{1}{3} & 1 & \underline{\frac{2}{3}} & 0 & \frac{-1}{9} & \frac{10}{3} \\ \frac{2}{3} * x_3 & \frac{2}{15} & 0 & \frac{2}{3} & \frac{-1}{5} & \frac{-4}{45} & \frac{32}{15} \\ (-) & & & & & & \\ \hline & 1/5 & [illegible] & 0 & 1/5 & \frac{-1}{5} & \end{array}$$

All $C_j - z_j \leq 0$. So soln. is optimal

Max $Z^* = -82$

Min $Z = 82$

$x_1 = 0$ $x_2 = 6/5$ $x_3 = 16/5$

Two Phase Method :

The two phase method is another method to solve a given problem in which some artificial variables are involved.

The solution is obtained in two phases.

PHASE I :

In this phase, the simplex method is applied to a specially constructed auxiliary LPP leading to a final simplex table containing a basic feasible solution to the orginal problem.

Step 1 : Assign a cost -1 to each artificial variable & a cost 0 to all other variables in the objective function. Thus the new objective function is $Z^* = -A_1 - A_2 - A_3 \cdots - A_n$ where A_i's are Artifical variables.

Step 2 : Construct the auxiliary LPP in which the new objective function Z^* is to be maximized subject to to the given set of constraints.

Step 3 : Solve the auxiliary LPP by simplex method until either of the following 3 possibilities arise.

(i) Max $Z^* < 0$ & atleast one artificial variable appears in the optimum basis at a non-zero level. In this case the given LPP does not possess any feasible solution. stop the procedure

(ii) Max $Z^* = 0$ and atleast one artificial variable appears in the optimum at zero level. In this case proceed to phase II

(iii) Max $Z^* = 0$ and no artificial variable appears in the optimum basis, then proceed to phase-II.

PHASE II

Use the optimum basic feasible solution of Phase-I as a starting solution for the orginal LPP. Assign the actual costs to the variables in the objective function and 0 to every artificial var. that appears in the basis at zero level. Use simplex method to the modified simplex table obtained at the end of phase I, till an optimum solution (if any) is obtained.

1. Solve the following LPP using two phase Method.

$$\text{Min } z = x_1 + x_2$$

$$\text{Sub. to } 2x_1 + x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

$$x_1, x_2 \geq 0$$

Step 1: Convert obj. function from Min to Max and also introduce artifical, ~~Stack~~ & surplus variables.

$$\text{Max } z^* = -x_1 - x_2$$

$$\text{Sub to } 2x_1 + x_2 - s_1 + A_1 = 4 \qquad x_1, x_2, s_1, s_2, A_1, A_2 \geq 0.$$

$$x_1 + 7x_2 - s_2 + A_2 = 7$$

Step 2: PHASE I (New objective function with -1 as co.eff for artificial var & 0 as coeff for other var.)

$$\text{Max } z^* = -A_1 - A_2$$

$$\text{Sub. to } 2x_1 + x_2 - s_1 + A_1 = 4$$

$$x_1 + 7x_2 - s_2 + A_2 = 7$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Intial basic feasible solution is

$x_1 = x_2 = s_1 = s_2 = 0$ (Non- Basic Var.)

$A_1 = 4$ $A_2 = 7$ (Basic Var.)

	c_j	0	0	0	0	-1	-1		
C_B	S_B	x_1	x_2	s_1	s_2	A_1	A_2	X_B	Min ratio
-1	A_1	2	1	-1	0	1	0	4	$\frac{4}{1} = 4$
-1	A_2	1	$\textcircled{7}$	0	-1	0	1	7	$\frac{7}{7} = 1 \rightarrow$
	z_j	-3	-8	1	1	-1	-1	-11	
	$c_j - z_j$	3	8 $\uparrow$	-[illegible]	-1	0	0		

as $c_j - z_j > 0$ for some; soln not optimum.

Entering Var $= x_2$ Leaving var. $= A_2$.

Artifical var $\underline{\underline{A_2}}$ can be permanently removed from the table in the next iteration.

	C_j	0	0	0	0	-1		
C_B	S_B	x_1	x_2	s_1	s_2	A_1	X_B	Min ratio
-1	A_1	$\boxed{13/7}$	0	-1	1/7	1	3	21/13 →
0	x_2	1/7	1	0	-1/7	0	1	7
	z_j	-13/7	0	1	-1/7	-1	-3	
	$c_j - z_j$	13/7 ↑	0	-1	1/7	0		

$$\begin{array}{lccccc} \text{old } A_1 & 2 & ① & -1 & 0 & 4 \\ 1 * x_2 & 1/7 & 1 & 0 & -1/7 & 1 \\ \hline & 13/7 & 0 & -1 & 1/7 & 3 \end{array}$$

Some $c_j - z_j > 0$. soln. not optimal.

Ent- variable = x_1 Leaving variable = A_1

As A_1 is the leaving variable, it can be permanently removed from the table in the next iteration.

	C_j	0	0	0	0	
C_B	S_B	x_1	x_2	S_1	S_2	X_B
0	x_1	1	0	-7/13	1/13	21/13
0	x_2	0	1	1/13	-2/13	10/13
	z_j	0	0	0	0	0
	$c_j - z_j$	0	0	0	0	

$$\begin{array}{lccccc} \text{old } x_2 & \textcircled{1/7} & 1 & 0 & -1/7 & 1 \\ \frac{1}{7} * x_1 & 1/7 & 0 & -1/13 & 1/91 & 3/13 \\ (-) & & & & & \\ \hline & 0 & 1 & 1/13 & -2/13 & 10/13 \end{array}$$

$c_j - z_j \leq 0$ & artificial variables are eliminated & $z^* = 0$.

End of phase I.

Take the original objective fn. $\boxed{\text{Max } z^* = -x_1 - x_2}$

PHASE II : Take the original objective fn, and also change the coefficients c_j to new obj. fn, and also C_B in the final table of phase I. Proceed with Simplex Method

	C_j	-1	-1	0	0	
C_B	S_B	x_1	x_2	S_1	S_2	X_B
-1	x_1	1	0	-7/13	1/13	21/13
-1	x_2	0	1	1/13	-2/13	10/13
	z_j	-1	-1	6/13	1/13	-31/13
	$c_j - z_j$	0	0	-6/13	-1/13	

Max $z^* = -31/13$

Optimal solution is
Min $z = 31/13$
$x_1 = 21/13$
$x_2 = 10/13$

All $c_j - z_j \leq 0$. soln. is optimal.

2. Use dual phase Method

$$\text{Min } z = 4x_1 + x_2$$

Sub to $3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Step 1 : Convert Min to Max

$$\text{Max } z^* = -4x_1 - x_2$$

Introduce surplus, slack & artifical variables.

$3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - S_1 + A_2 = 6$

$x_1 + 2x_2 + S_2 = 4$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Step 2 : $\boxed{\text{PHASE-I}}$ There are 2 Art-var. Introduce cost (-1) to Art-var and cost 0 to other variables.

$$\text{Max } z^* = -A_1 - A_2$$

Sub to $3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - S_1 + A_2 = 6$

$x_1 + 2x_2 + S_2 = 4$ $\quad x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Basic feasible soln :

$x_1 = x_2 = S_1 = 0$ (Non-Basic variable)

$A_1 = 3 \quad A_2 = 6 \quad S_2 = 4$ (Basic variables)

Intial Simplex table

c_j 0 0 0 0 -1 -1

C_B	S_B	x_1	x_2	S_1	S_2	A_1	A_2	X_B	Min ratio
-1	A_1	③	1	0	0	1	0	3	3/3 = 1 →
-1	A_2	4	3	-1	0	0	1	6	6/4 = 1.5
0	S_2	1	2	0	1	0	0	4	4/1 = 4
z_j		-7	-4	1	0	-1	-1	-9	
$c_j - z_j$		7 ↑	4	-1	0	0	0		

Some $c_j - z_j > 0$. Not optimal

Entering variable = x_1 Leaving variable = A_1

	c_j	0	0	0	0	-1		
C_B	S_B	x_1	x_2	s_1	s_2	A_2	x_B	Min ratio
0	x_1	1	1/3	0	0	0	1	3
-1	A_2	0	$\boxed{5/3}$	-1	0	1	2	6/5 →
0	s_2	0	5/3	0	1	0	3	9/5
z_j		0	-5/3	1	0	-1	-2	
$c_j - z_j$		0	5/3 ↑	-1	0	0		

$$\begin{array}{lcccccc} \text{old } A_2 & \underline{4} & 3 & -1 & 0 & 1 & 6 \\ 4 * x_1 & 4 & 4/3 & 0 & 0 & 0 & 4 \\ (-) & & & & & & \\ \hline & 0 & 5/3 & -1 & 0 & 1 & 2 \end{array}$$

$$\begin{array}{lcccccc} \text{old } s_2 & \underline{1} & 2 & 0 & 1 & 0 & 4 \\ 1 * x_1 & 1 & 1/3 & 0 & 0 & 0 & 1 \\ (-) & & & & & & \\ \hline & 0 & 5/3 & 0 & 1 & 0 & 3 \end{array}$$

Some $c_j - z_j > 0$. not optimal

Entering variable $= x_2$ Leaving variable $= A_2$

	c_j	0	0	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	x_B
0	x_1	1	0	1/5	0	3/5
0	x_2	0	1	-3/5	0	6/5
0	s_2	0	0	1	1	1
z_j		0	0	0	0	0
$c_j - z_j$		0	0	0	0	

$$\begin{array}{lccccc} \text{old } x_1 & 1 & \underline{1/3} & 0 & 0 & 1 \\ 1/3 * x_2 & 0 & 1/3 & -1/5 & 0 & 2/5 \\ (-) & & & & & \\ \hline & 1 & 0 & 1/5 & 0 & 3/5 \end{array}$$

$$\begin{array}{lccccc} \text{old } s_2 & 0 & \underline{5/3} & 0 & 1 & 3 \\ 5/3 * x_2 & 0 & 5/3 & -1 & 0 & 2 \\ (-) & & & & & \\ \hline & 0 & 0 & 1 & 1 & 1 \end{array}$$

So $c_j - z_j \le 0$. can proceed to phase II because all Art. var. eleminated.

PHASE II

Original objective function

$$\text{Max } z^* = -4x_1 - x_2$$

Take the final table of phase I change the coeff. of obj. function (c_j) to new values and CB column.

Find z_j & $c_j - z_j$.

c_j -4   -1   0   0

C_B	S_B	x_1	x_2	S_1	S_2	X_B	Min ratio
-4	x_1	1	0	1/5	0	3/5	3
-1	x_2	0	1	-3/5	0	6/5	–
0	S_2	0	0	①	1	1	1 →
z_j		-4	-1	-1/5	0	-18/5	
$c_j - z_j$		0	0	1/5	0		

Some $c_j - z_j > 0$ not optimal.

Entering variable = S_1 leaving variable = S_2

c_j -4   -1   0   0

C_B	S_B	x_1	x_2	S_1	S_2	X_B
-4	x_1	1	0	0	-1/5	2/5
-1	x_2	0	1	0	+3/5	9/5
0	S_1	0	0	1	1	1
z_j		-4	-1	0	1/5	-11/5
$c_j - z_j$		0	0	0	-1/5	

old x_1 1   0 $\underline{1/5}$ 0   3/5

$\frac{1}{5} * S_1$ (-) 0   0   1/5   1/5   1/5

1   0   0   -1/5   2/5

old x_2 0   1 $\underline{-3/5}$ 0   6/5

$3/5 * S_1$ + 0   0   3/5   +3/5   +3/5

0   1   0   +3/5   9/5

All $c_j - z_j \leq 0$ soln. optimal.

Max $z^* = -11/5$

Optimal Solution

Min $z = 11/5$

$x_1 = 2/5$

$x_2 = 9/5$

3. Use two-phase method to

$$\text{Max } z = 2x_1 + x_2 + \frac{1}{4}x_3$$

Sub. to. $4x_1 + 6x_2 + 3x_3 \le 8$

$3x_1 - 6x_2 - 4x_3 \le 1$

$2x_1 + 3x_2 - 5x_3 \ge 4$

$x_1, x_2, x_3 \ge 0$

Convert ineq. to equality. Introduce slack, surplus & artifical va

$$\text{Max } z = 2x_1 + x_2 + \frac{1}{4}x_3$$

$4x_1 + 6x_2 + 3x_3 + S_1 = 8$

$3x_1 - 6x_2 - 4x_3 + S_2 = 1$

$2x_1 + 3x_2 - 5x_3 - S_3 + A_1 = 4$

$x_1, x_2, x_3, S_1, S_2, S_3, A_1 \ge 0$

Intial Basic Solution is

$S_1 = 8 \quad S_2 = 1 \quad A_1 = 4$ (Basic)

$x_1 = x_2 = x_3 = S_3 = 0$. (Non-Basic)

PHASE 1: Auxiliary LPP

$\text{Max } z^* = -A_1$

[Assign −1 as coeff for artificial var, 0 as coeff. for slack, surplus, decisio var]

	Cj	0	0	0	0	0	0	-1		
C_B	S_B	x_1	x_2	x_3	S_1	S_2	S_3	A_1	X_B	Min ratio
0	S_1	4	⑥	3	1	0	0	0	8	$\frac{8}{6} = 1.3 \rightarrow$
0	S_2	3	-6	-4	0	1	0	0	1	—
-1	A_1	2	3	-5	0	0	-1	1	4	$\frac{4}{3} = 1.3$
	zj	-2	-3	+5	0	0	1	-1	-4	
	Cj - zj	2	$3 \uparrow$	-5	0	0	-1	0		

Degeneracy occurs. between S_1 & A_1 Leaving variable car be chosen arbitarily Let us choose S_1 as Leaving

Some $C_j - z_j > 0$ soln not optimal E.V = x_2 L.V = S_1

	Cj	0	0	0	0	0	0	-1		
C_B	S_B	x_1	x_2	x_3	S_1	S_2	S_3	A_1	X_B	Min ratio
0	x_2	2/3	1	1/2	1/6	0	0	0	4/3	
0	S_2	7	0	-1	1	1	0	0	9	
-1	A_1	0	0	-13/2	-1/2	0	-1	1	0	
	zj	0	0	13/2	1/2	0	1	-1	0	
	Cj - zj	0	0	-13/2	-1/2	0	-1	0		

Old S_2	3	<u>-6</u>	-4	0	1	0	0	1
$6 * x_2$	4	6	3	1	0	0	0	8
(+)	7	0	-1	1	1	0	0	9
Old A_1	2	<u>3</u>	-5	0	0	-1	1	4
$3 * x_2$	2	3	3/2	1/2	0	0	0	4
(−)	0	0	-13/2	-1/2	0	-1	1	0

$C_j - z_j \le 0$, also $z^* = 0$, One Art. variable exist at 0 level. Proceed to Phase II.

PHASE II

Here, consider the original Max. objective function and assign a cost 0 to the artificial variables & slack, surplus variables.

$$\text{Max } z = 2x_1 + x_2 + \frac{1}{4}x_3 + 0s_1 + 0s_2 + 0s_3 .$$

The initial table is the table obtained at end of phase I.

c_j 2 1 1/4 0 0 0 0

C_B	S_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	X_B	Min ratio
1	x_2	2/3	1	1/2	1/6	0	0	0	4/3	2
0	s_2	$\textcircled{7}$	0	-1	1	1	0	0	9	9/7 →
0	R_1	0	0	-13/2	-1/2	0	-1	1	0	—
c_j		2/3	1	1/2	1/6	0	0	0	4/3	
$c_j - z_j$		4/3 ↑	0	-1/4	-1/6	0	0	0		

Since $c_j - z_j > 0$ for some variables

EV $= x_1$ L.V $= s_2$

c_j 2 1 1/4 0 0 0 0

C_B	S_B	x_1	x_2	x_3	s_1	s_2	s_3	R_1	X_B
1	x_2	0	1	$\frac{25}{42}$	$\frac{1}{14}$	$\frac{-2}{21}$	0	0	$\frac{10}{21}$
2	x_1	1	0	-1/7	1/7	1/7	0	0	9/7
0	R_1	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	-1	1	0
z_j		2	1	$\frac{13}{42}$	5/14	4/21	0	0	64/21
$c_j - z_j$		0	0	-5/84	-5/14	-4/21	0	0	

old x_2 $\frac{2}{3}$ 1 $\frac{1}{2}$ $\frac{1}{6}$ 0 0 0 $\frac{4}{3}$

$\frac{2}{3} * x_1$ $(-)\ \frac{2}{3}$ 0 $\frac{-2}{21}$ $\frac{2}{21}$ $\frac{2}{21}$ 0 0 6/7

0 1 $\frac{+25}{42}$ $\frac{1}{14}$ $\frac{-2}{21}$ 0 0 $\frac{10}{21}$

old R_1 $\underline{\underline{0}}$ Already zero.

$c_j - z_j \le 0$. Solution is optimal

$$\text{Max } z = \frac{64}{21}$$

$x_1 = 9/7$ $x_2 = 10/21$

$x_3 = 0$

4) Max $z = 5x_1 + 3x_2$

$2x_1 + x_2 \leq 1$

$x_1 + 4x_2 \geq 6$

$x_1, x_2 \geq 0$

Introduce slack, surplus, artifical variables.

Max $z = 5x_1 + 3x_2$

$2x_1 + x_2 + S_1 = 1$

$x_1 + 4x_2 - S_2 + A_1 = 6$

$x_1, x_2, S_1, S_2, A_1 \geq 0$.

PHASE I : Auxiliary LPP.

Max $Z^* = -A_1$

	c_j	0	0	0	0	-1		
C_B	S_B	x_1	x_2	S_1	S_2	A_1	X_B	Min ratio
0	S_1	2	$\textcircled{1}$	1	0	0	1	$1/1 = 1 \rightarrow$
-1	A_1	1	4	0	-1	1	6	$6/4 = 1.5$
z_j		-1	-4	0	1	-1	-6	
$c_j - z_j$		1	4 $\uparrow$	0	-1	0		

$c_j - z_j > 0$ for some variable. Soln - not optimal

E.V = x_2 L.V = S_1

	c_j	0	0	0	0	-1	
C_B	S_B	x_1	x_2	S_1	S_2	A_1	X_B
0	x_2	2	1	1	0	0	1
-1	A_1	-7	0	+4	-1	1	2
z_j		+7	0	4	1	-1	-2
$c_j - z_j$		-7	0	-4	-1	0	

Old A_1	1	$\underline{4}$	0	-1	1	6
$4 * x_2$	8	4	4	0	0	4
	-7	0	-4	-1	1	2

$c_j - z_j \leqq 0$, The current basic feasible soln. is optimal to auxiliary LPP.

But $z < 0$, one artifical variable exist at non-zero level.

So there is no feasible solution to this problem.

Disadvantage of Big-M Method over Two-phase method

Even though Big-M Method can always be used to check the existence of a feasible solution it may be computationally inconvenient especially when a digital computer is used because of the manipulation of the constant M.

On the other hand, Two-phase method eliminates the constant M from calculations.

Variants of Simplex Method

→ Degeneracy & cycling

→ Unbounded Solution

→ Multiple solutions

→ Non-Existing feasible Solution

→ Unrestricted Variables

1. Degeneracy & cycling

It may arise

a) at the intial stage when atleast one basic variable is zero in the intial basic feasible solution.

b) at any subsequent iteration when more than one basic variable is eligible to leave the basis and hence one or more variables becoming zero in the next iteration & the problem is said to be degenerate. There is no assurance that the value of the objective function will improve, since new solutions may remain degenerate. As a result, same sequence is repeated without improving the solutions. This concept called cycling or Circling.

Rule to avoid cycling:

(i) Divide each element in the tied rows by the positive co-efficients of the key column in that row.

(ii) Compare the resulting ratios, column by column first in the unit matrix & then in the body matrix from left to right.

(iii) The row which contains the smallest algebraic ratio contains the leaving variable.

2. Unbounded Solution:

In some LPP, the Solution space becomes unbounded so that the value of the objective function also can be increased without a limit. The solution space may be unbounded but the solution may be finite.

3. Multiple solutions (or) Alternate optimum Solutions.

In some LPP, there may be alternative or infinite number of solutions

In simplex method, if $(c_j - z_j) \neq 0$ for all non-basic variables, then the problem is said to have a unique optimal solution.

On the otherhand, if the net evaluation $(c_j - z_j) = 0$ for atleast one non-basic variable, then the problem is said to have an alternative or infinite soln.

4. Non-Existing feasible Solution:

In an LPP, when there is no point belonging to the solution space satisfying all the constraints, then the problem does not have feasible solution.

5. Unrestricted or Unconstrained Variables:

In an LPP, if any variable is unconstrained it can be expresed between 2 non-negative variables.

ex: x_1 unrestricted can be written as $x_1 = x_1' - x_1''$

finally change $x_1' - x_1'' = x$.

I DEGENERACY : Reason for occurs due to redundant constraint

1. Max $z = x_1 + 2x_2 + x_3$

$2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \geq -6 \Longrightarrow 2x_1 - x_2 + 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

i) Introduce slack variables

Max $z = x_1 + 2x_2 + x_3 + 0S_1 + 0S_2 + 0S_3$

$2x_1 + x_2 - x_3 + S_1 = 2$

$2x_1 - x_2 + 5x_3 + S_2 = 6$

$4x_1 + x_2 + x_3 + S_3 = 6$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Basic Solution:

$x_1 = x_2 = x_3 = 0$ (Non-Basic)

$\therefore S_1 = 2 \quad S_2 = 6 \quad S_3 = 6$ (Basic)

	c_j	1	2	1	0	0	0			
C_B	S_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B	Min ratio	
0	S_1	2	①	-1	1	0	0	2	$2/1 = 2$	→
0	S_2	2	-1	5	0	1	0	6	–	
0	S_3	4	1	1	0	0	1	6	$6/1 = 6$	
	z_j	0	0	0	0	0	0	0		
	$c_j - z_j$	1	2 ↑	1	0	0	0			

As $c_j - z_j \not> 0$, soln. not optimum

Entering Variable = x_2

Leaving Variable = S_1

C_j 1 2 1 0 0 0

C_B	S_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B	Min ratio
2	x_2	2	1	-1	1	0	0	2	–
0	S_2	4	0	4	1	1	0	8	2
0	S_3	2	0	$\textcircled{2}$	-1	0	1	4	$2^* \rightarrow$
Z_j		4	2	-2	2	0	0	4	
$C_j - Z_j$		-3	0	3	-2	0	0		–

↑

old S_2 2 $\underline{1}$ 5 0 1 0 6
$1 \times x_2$ (New) 2 1 -1 1 0 0 2
+ + + + + + +
4 0 4 1 1 0 8

old S_3 4 1 1 0 0 1 6
$1 * x_2$ 2 1 -1 1 0 0 2
- - + - - - -
2 0 2 1 0 1 4

$C_j - Z_j \geq 0$. Not an optimal soln.

Since both S_2, S_3 having the same minimum ratio 2, there is a tie in leaving variable. This is degenerate. Divide each entry corresponding to basic variables S_2 & S_3 & then corresponding to non-basic variables $x_1, x_2 x_3, S_1$.

Pivot col val		S_2	S_3	x_1	x_2	x_3	S_1
4	Row 2 S_2	$\frac{1}{4}$	$\frac{0}{4}$	$\frac{4}{4}$	$\frac{0}{4}$	$\frac{4}{4}$	$\frac{1}{4}$
2	Row 3 S_3	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{0}{2}$	$\frac{2}{2}$	$\frac{-1}{2}$

Columnwise comparison of quotients starting with column S_2 gives basic variables S_2 & S_3. we find that algebraically smaller ratio for Row 3. So Row 3 (ie. S_3) is selected as key row for leaving

Entering Variable = x_2 Leaving Variable = S_3

C_j 1 2 1 0 0 0

C_B	S_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B	Min ratio
2	x_2	3	1	0	1/2	0	1/2	4	
0	S_2	0	0	0	3	1	-2	0	
1	x_3	1	0	1	-1/2	0	1/2	2	
Z_j		7	2	1	1/2	0	2	10	
$C_j - Z_j$		-6	0	0	-1/2	0	-2		

old x_2 Row 2 1 -1 1 0 0 2
$1 * x_3$ row 1 0 1 -1/2 0 1/2 2
(+)
3 1 0 1/2 0 1/2 4

old S_2 4 0 4 1 1 0 8
$4 * x_3$ row 4 0 4 -2 0 2 8
- - -
0 0 0 3 1 -2 0

All $C_j - Z_j \leq 0$. So result is optimal.

Optimal Solution is $x_1 = 0$ $x_2 = 4$ $x_3 = 2$ $S_1 = 0$ $S_2 = 0$ $S_3 = 0$

Max $Z = 10$

2. Solve the following LPP

Max $z = 3x_1 + 9x_2$

s.t

$x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$, $x_1, x_2 \geq 0$

Solution

1. By introducing slack variable s_1, s_2, the given LPP looks like

Max

$z = 3x_1 + 9x_2 + 0s_1 + 0s_2$

s.t

$x_1 + 4x_2 + s_1 + 0s_2 = 8$

$x_1 + 2x_2 + 0s_1 + s_2 = 4$, $x_1, x_2, s_1, s_2 \geq 0$

2. Initial Basic feasible solution : $n = 4$, $m = 2$

No. of non basic variables $= n - m \Rightarrow 4 - 2 = 2$ $[x_1 = 0, x_2 = 0]$

No. of basic variables $= m \Rightarrow 2$ $[s_1 = 8, s_2 = 4]$

Initial Iteration

		c_j	3	9	0	0	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
0	s_1	8	1	4	1	0	$8/4 = 2$
0	s_2	4	1	2	0	1	$4/2 = 2$ ← s_2 leaves
	z_j	0	0	0	0	0	
	$c_j - z_j$		3	9 ↑	0	0	

x_2 enters

Since some $c_j - z_j > 0$, the current solution is not optimal

Note: Since s_1, s_2 have the same minimum ratio, there is a tie in finding leaving variable [called as degeneracy].

To find leaving variable among s_1 & s_2

	s_1	s_2
s_1 (Row 1)	$\frac{1}{4}$	$\frac{0}{4}$
s_2 (Row 2)	$\frac{0}{2}$	$\frac{1}{2}$

compare the ratios from left to right columnwise, the minimum ratio ($0/2 = 0$) occurs in S_2 row. Thus variable S_2 leaves the basis.

I iteration [x_2 enters, s_2 leaves]

		c_j	3	9	0	0
CB	YB	XB	x_1	x_2	s_1	s_2
0	s_1	0	-1	0	1	2
9	x_2	2	$1/2$	1	0	$1/2$
	z_j	18	$9/2$	9	0	$9/2$
	$c_j - z_j$		$-3/2$	0	0	$-9/2$

(i) New pivot row (x_2 row) =

$$\begin{array}{ccccc} x_B & x_1 & x_2 & s_1 & s_2 \\ 4 & 1 & 2 & 0 & 1 \\ \hline & & 2 & & \end{array}$$

$$= 2 \quad 1/2 \quad 1 \quad 0 \quad 1/2$$

(ii) New s_1 row =

$$\begin{array}{cccccl} & 8 & 1 & 4 & 1 & 0 \quad \text{(old } s_1 \text{ row)} - \\ -(4)\times & 2 & 1/2 & 1 & 0 & 1/2 \quad 4 * \text{new } x_2 \text{ row} \\ \hline & 0 & -1 & 0 & 1 & 2 \end{array}$$

Since all $c_j - z_j \le 0$, the current feasible solution is optimal.

optimum solution = $\boxed{\text{Max } z = 18 \ @ \ x_1 = 0,\ x_2 = 2}$

3. Use two-phase simplex method to solve the following Lp problem

Max $Z = 3x_1 + 2x_2 + 2x_3$

s.t

$$5x_1 + 7x_2 + 4x_3 \leq 7$$
$$-4x_1 + 7x_2 + 5x_3 \geq -2$$
$$3x_1 + 4x_2 - 6x_3 \geq 29/7, \; x_1, x_2, x_3 \geq 0$$

Solution

1. Since R.H.S of 2nd constraint is -ve, multiply this constraint by -1 on both sides. Introduce slack variable S1, S2, surplus variable S3 and Artificial variable A1, the Lpp is

Max

$$Z = 3x_1 + 2x_2 + 2x_3 + 0S_1 + 0S_2 + 0S_3$$

s.t

$$5x_1 + 7x_2 + 4x_3 + S_1 + 0S_2 + 0S_3 + 0A_1 = 7$$
$$4x_1 - 7x_2 - 5x_3 + 0S_1 + S_2 + 0S_3 + 0A_1 = 2$$
$$3x_1 + 4x_2 - 6x_3 + 0S_1 + 0S_2 - S_3 + A_1 = 29/7$$

$$x_1, x_2, x_3, S_1, S_2, S_3, A_1 \geq 0.$$

Phase-I : Assign cost -1 to Artificial variable and cost 0 to all other variables, we get auxiliary Lpp

Max

$$Z_1 = -A_1$$

s.t

$$5x_1 + 7x_2 + 4x_3 + S_1 + 0S_2 + 0S_3 + 0A_1 = 7$$
$$4x - 7x_2 - 5x_3 + 0S_1 + 0S_2 + 0S_3 + 0A_1 = 2$$
$$3x_1 + 4x_2 - 6x_3 + 0S_1 + 0S_2 - S_3 + A_1 = 29/7$$
$$x_1, x_2, x_3, S_1, S_2, S_3, A_1 \geq 0$$

Find Initial basic feasible solution

Non Basic variables $[x_1 = 0, x_2 = 0, x_3 = 0, S_3 = 0]$

Basic variables $[S_1 = 7, S_2 = 2, A_1 = 29/7]$

Initial iteration

		C_j	0	0	0	0	0	0	-1	
CB	YB	XB	x_1	x_2	x_3	S_1	S_2	S_3	A_1	θ
0	S_1	7	5	7	4	1	0	0	0	$7/7 = 1$ ← S_1 leaves
0	S_2	2	4	-7	-5	0	1	0	0	–
-1	A_1	$\frac{29}{7}$	3	4	-6	0	0	-1	1	$\frac{29}{7} \times \frac{1}{4} = 1.08$
	Z_j	$\frac{-29}{7}$	-3	-4	+6	0	0	1	-1	
	$C_j - Z_j$		3	4 ↑	-6	0	0	-1	1	

x_2 enters

Since some $C_j - Z_j \geq 0$, the current solution is not optimal

I iteration $[x_2$ enters, S_1 leaves$]$

		C_j	0	0	0	0	0	0	-1	
CB	YB	XB	x_1	x_2	x_3	S_1	S_2	S_3	A_1	θ
0	x_2	1	5/7	1	4/7	1/7	0	0	0	7/5
0	S_2	9	9	0	-1	1	1	0	0	1
-1	A_1	1/7	1/7	0	-58/7	-4/7	0	-1	1	1 ← A_1 leaves
	Z_j	-1/7	-1/7	0	58/7	4/7	0	1	-1	
	$C_j - Z_j$		1/7 ↑	0	-58/7	-4/7	0	-1	0	

x_1 enters

(i) New pivot row (x_2 row) $= \frac{7 \quad 5 \quad 7 \quad 4 \quad 1 \quad 0 \quad 0 \quad 0}{7}$

$= 1 \quad 5/7 \quad 1 \quad 4/7 \quad 1/7 \quad 0 \quad 0 \quad 0$

(ii) New S_2 row $= 2 \quad 5 \quad -7 \quad -5 \quad 0 \quad 1 \quad 0 \quad 0$

$-(-7) \times 1 \quad 5/7 \quad 1 \quad 4/7 \quad 1/7 \quad 0 \quad 0 \quad 0$

$9 \quad 9 \quad 0 \quad -1 \quad 1 \quad 1 \quad 0 \quad 0$

(iii) New A_1 row = $\frac{29}{7}$ 3 4 −6 0 0 −1 1

$-(4)*$ 1 $5/7$ 1 $4/7$ $1/7$ 0 0 0

$1/7$ $1/7$ 0 $-58/7$ $-4/7$ 0 −1 1

Since some $c_j - z_j > 0$, the current solution is not optimal

Note: Since there is a tie in finding leaving variable between slack variable S_2 and Artificial variable A_1, select A_1 as a leaving variable.

II iteration [x_4 enters, A_1 leaves]

	c_j		0	0	0	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	x_2	$2/7$	0	1	42	3	0	5
0	S_2	0	0	0	521	37	1	63
0	x_4	1	1	0	−58	−4	0	−7
	z_j	0	0	0	0	0	0	0
	$c_j - z_j$		0	0	0	0	0	0

(i) New x_4 row = $\dfrac{1/7 \quad 1/7 \quad 0 \quad -58/7 \quad -4/7 \quad 0 \quad -1}{1/7}$

= 1 1 0 −58 −4 0 −7

(ii) New x_2 row = 1 $5/7$ 1 $4/7$ $1/7$ 0 0

$-(5/7)*$ 1 1 0 −58 −4 0 −7

$2/7$ 0 1 42 3 0 5

(iii) New S_2 row = 9 9 0 −1 1 1 0 −

(9)* 1 1 0 −58 −4 0 −7 $\Rightarrow$ 0 0 0 521 37 1 63

Since all $c_j - z_j \leq 0$, $z_1 = 0$ & no Artificial variables appears in basis, the current solution is optimal to the given auxilary LPP. Therefore proceed to phase - II

Phase - II : Assign cost 0 to Artificial variables if exist and original cost to all other variables, and solve using simplex method

Initial iteration

	c_j		3	2	2	0	0	0	
CB	YB	XB	x_1	x_2	x_3	s_1	s_2	s_3	θ
2	x_2	2/7	0	1	42	3	0	5	12
0	s_2	0	0	0	521	37	1	63	0 ← s_2 leaves
3	x_1	1	1	0	-58	-4	0	-7	—
	z_j	25/7	3	2	-90	-6	0	-11	
	$c_j - z_j$		0	0	92	6	0	11	

x_3 enters

Since some $c_j - z_j > 0$

I iteration [x_3 enters, s_2 leaves]

	c_j		3	2	2	0	0	0
CB	YB	XB	x_1	x_2	x_3	s_1	s_2	s_3
2	x_2	2/7	0	1	0	9/521	-42/521	-41/521
2	x_3	0	0	0	1	37/521	1/521	63/521
3	x_1	1	1	0	0	$\frac{62}{521}$	$\frac{58}{521}$	$\frac{7}{521}$
	z_j	25/7	3	2	2	278/521	92/521	65/521
			0	0	0	-278/521	-92/521	-65/521

Since all $c_j - z_j \leq 0$ & no artificial variable appears at basis the current solution is optimal

(i) new x_3 row $= \dfrac{0 \quad [illegible] \quad 0 \quad 521 \quad 37 \quad 1 \quad 63}{521}$

$$= 0 \quad 0 \quad 0 \quad 1 \quad 37/521 \quad 1/521 \quad 63/521$$

(ii) new x_2 row $=$

$$\begin{array}{rccccccc} & 2/7 & 0 & 1 & 42 & 3 & 0 & 5 \\ -(42)* & 0 & 0 & 0 & 1 & 37/521 & 1/521 & 63/521 \\ \hline & 2/7 & 0 & 1 & 0 & 9/521 & \frac{-42}{521} & -\frac{41}{521} \end{array}$$

(iii) new x_1 row $=$

$$\begin{array}{rccccccc} & 1 & 1 & 0 & -58 & -4 & 0 & -7 \\ -(-58)* & 0 & 0 & 0 & 1 & 37/521 & 1/521 & 63/521 \\ \hline & 1 & 1 & 0 & 0 & 62/521 & 58/521 & 7/521 \end{array}$$

Optimum solution

$$\boxed{\text{Max } z = 25/7 \ @ \ x_1 = 1, \ x_2 = 2/7, \ x_3 = 0}$$

II UNBOUNDED SOLUTION

1. Max $z = 3x_1 + 5x_2$

s.t $x_1 - 2x_2 \le 6$, $x_1 \le 10$, $x_2 \ge 1$ & $x_1, x_2 \ge 0$.

Solution [solving by Big-M Method]

1. By introducing slack variables S_1, S_2 and surplus variable S_3 and Artificial variable A_1, the above LPP looks like

Max $z = 3x_1 + 5x_2 - MA_1$

s.t

$x_1 - 2x_2 + S_1 = 6$

$x_1 + S_2 = 10$

$x_2 - S_3 + A_1 = 1$, $x_1, x_2, S_1, S_2, S_3, A_1 \ge 0$

2. Find Initial basic feasible solution

Non Basic variables [$x_1 = 0$, $x_2 = 0$, $S_3 = 0$]

Basic variables [$S_1 = 6$, $S_2 = 10$, $A_1 = 1$]

Initial iteration

	C_j		3	5	0	0	0	-M	
CB	YB	XB	x_1	x_2	S_1	S_2	S_3	A_1	θ
0	S_1	6	1	-2	1	0	0	0	–
0	S_2	10	1	0	0	1	0	0	–
-M	A_1	1	0	1	0	0	-1	1	1 ← A_1 leaves
	Z_j	-M	0	-M	0	0	M	-M	
	$C_j - Z_j$		3	5+M ↑	0	0	-M	0	

x_2 enters

Since some $C_j - Z_j > 0$, the current solution is not optimal

I iteration [x_2 enters, A_1 leaves]

	C_j		3	5	0	0	0	
CB	YB	XB	x_1	x_2	S_1	S_2	S_3	θ
0	S_1	8	1	0	1	0	-2	–
0	S_2	10	1	0	0	1	0	–
5	x_2	1	0	1	0	0	-1	–
	Z_j	5	0	5	0	0	-5	–
	$C_j - Z_j$		3	0	0	0	5 ↑	–

S_3 enters

Since all pivot column element is ≤ 0, unable to find leaving variable

∴ The solution is unbounded

(i) New x_2 row $= \dfrac{1 \quad 0 \quad 1 \quad 0 \quad 0 \quad -1}{1}$

$= 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad -1$

(ii) New S_1 row $=$
$$\begin{array}{rrrrrr} 6 & 1 & -2 & 1 & 0 & 0 \\ -(-2)\times\ 1 & 0 & 1 & 0 & 0 & -1 \\ \hline 8 & 1 & 0 & 1 & 0 & -2 \end{array}$$

(iii) New S_2 row $=$ 10 1 0 0 1 0

$-(0)\times$ 1 0 1 0 0 -1 → 10 1 0 0 1 0

2. Solve the following LPP by simplex method.

$\text{Max } z = 2x_1 + x_2$

$x_1 - x_2 \leq 10$

$2x_1 - x_2 \leq 40$

$x_1, x_2 \geq 0$

Step 1: Introduce slack variables.

$\text{Max } z = 2x_1 + x_2 + 0s_1 + 0s_2$

$x_1 - x_2 + s_1 = 10$

$2x_1 - x_2 + s_2 = 40$

Step 2: Initial basic solution. X_B.

$(x_1 = x_2 = 0)$ Non-basic

$(s_1 = 10, s_2 = 40)$ Basic variables.

Cj 2 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
0	s_1	①	-1	1	0	10	10 →
0	s_2	2	-1	0	1	40	20
Z_j		0	0	0	0	0	
$C_j - Z_j$		2 ↑	1	0	0	.	

$C_j - Z_j > 0$. Soln-not optimal. Ent. var $= x_1$, Leaving var $= s_1$

Cj 2 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
2	x_1	1	-1	1	0	10	-
0	s_2	0	①	-2	1	20	20 →
Z_j		2	-2	2	0	20	
$C_j - Z_j$		0	3 ↑	-2	0		

old S_2 2 -1 0 1 40

$1 * S_1$ (new) 2 -2 2 0 20

0 1 -2 1 20

Soln not optimal because $C_j - Z_j$ for x_2 column does not match the condition $C_j - Z_j \leq 0$.

Ent-var $= x_2$ Leaving var $= s_2$

Cj 2 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B
2	x_1	1	0	-1	1	30
1	x_2	0	1	-2	1	20
Z_j		2	1	-4	3	80
$C_j - Z_j$		0	0	4	-3	

old x_1 1 -1 1 0 10

$1 * x_2$ (+) 0 1 -2 1 20

1 0 -1 1 30

The $C_j - Z_j > 0$. Soln not optimal because Min ratio does not exist. The soln. to this problem is unbounded.

III: MULTIPLE SOLUTION

1. Max

$z = 4x_1 + 10x_2$

s.t

$2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90 \quad \forall x_1, x_2 \geq 0$

Solution

1. Introduce slack variable S1, S2, S3. The lpp looks like

Max

$z = 4x_1 + 10x_2$

s.t

$2x_1 + x_2 + S_1 = 50$

$2x_1 + 5x_2 + S_2 = 100$

$2x_1 + 3x_2 + S_3 = 90, \quad S_1, S_2, S_3, x_1, x_2 \geq 0$

2. Intial basic feasible solution

Non basic variables $[x_1 = 0, x_2 = 0]$

Basic Variables $[S_1 = 50, S_2 = 100, S_3 = 90]$

Initial iteration

	Cj		4	10	0	0	0	
CB	YB	XB	x_1	x_2	S1	S2	S3	θ
0	S1	50	2	1	1	0	0	50
0	S2	100	2	5	0	1	0	20
0	S3	90	2	3	0	0	1	30
	Zj	0	0	0	0	0	0	
		Cj-Zj	4	10↑	0	0	0	

← S2 leaves

x_2 enters

I iteration [x_1 enters , A_1 leaves]

		c_j	6	4	0	0	0	
CB	YB	XB	x_1	x_2	S_1	S_2	S_3	θ
0	S_1	24	0	1	1	0	2	12
0	S_2	15	0	-1	0	1	3	5
6	x_1	3	1	1	0	0	-1	–
	z_j	18	6	6	0	0	-6	
		$c_j - z_j$	0	-2	0	0	6 ↑	

S_3 enters

New S_1 row = 30 2 3 1 0 0
$-(2) *$ 3 1 1 0 0 -1

24 0 1 1 0 2

New S_2 row = 24 3 2 0 1 0
$-(3) *$ 3 1 1 0 0 -1

15 0 -1 0 1 3

II iteration [S_3 enters, S_2 leaves]

		c_j	6	4	0	0	0
CB	YB	XB	x_1	x_2	S_1	S_2	S_3
0	S_1	14	0	5/3	1	-2/3	0
0	S_3	5	0	-1/3	0	1/3	1
6	x_1	8	1	2/3	0	1/3	0
	z_j	48	6	4	0	2	0
		$c_j - z_j$	0	0	0	-2	0

New S_1 row = 24 0 1 1 0 2

$-(2)*5$ 0 $-1/3$ 0 $1/3$ 1

---

14 0 $5/3$ 1 $-2/3$ 0

New x_1 row = 3 1 1 0 0 -1

$-(-1)*5$ 0 $-1/3$ 0 $1/3$ 1

---

8 1 $2/3$ 0 $1/3$ 0

Since all $c_j - z_j \leq 0$, the current solution is optimal & the $c_j - z_j$ value of non-basic variable $x_2 = 0$, thus the LPP has Multiple Solution.

∴ The optimum solution is

$\boxed{\text{Max } z = 48, \; x_1 = 8, \; x_2 = 0}$

4. UNRESTRICTED / UNCONSTRAINED VARIABLES.

1. Max $z = 8x_2$

s.t $x_1 - x_2 \geq 0$

$2x_1 + 3x_2 \leq -6$, x_1, x_2 are unrestricted

Solution [Big-M]

1. Since x_1 and x_2 are unrestricted, put $x_1 = x_1' - x_1''$ & $x_2 = x_2' - x_2''$

∴ The given LPP becomes

Max $z = 8(x_2' - x_2'')$

s.t $(x_1' - x_1'') - (x_2' - x_2'') \geq 0$

$2(x_1' - x_1'') + 3(x_2' - x_2'') \leq -6$, $x_1', x_1'', x_2', x_2'' \geq 0$

I iteration [x_2 enters, s_2 leaves]

		c_j	4	10	0	0	0
CB	YB	XB	x_1	x_2	s_1	s_2	s_3
0	s_1	30	8/5	0	1	-1/5	0
10	x_2	20	2/5	1	0	1/5	0
0	s_3	30	4/5	0	0	-3/5	1
	z_j	200	4	10	0	2	0
		$c_j - z_j$	0	0	0	-2	0

New s_1 row = 50 2 1 1 0 0

-(1) * 20 2/5 1 0 1/5 0

30 8/5 0 1 -1/5 0

New s_3 row = 90 2 3 0 0 1

-(3) * 20 2/5 1 0 1/5 0

30 4/5 0 0 -3/5 1

Since all $c_j - z_j \le 0$, the current solution is optimal. Since $c_j - z_j$ value of non-basic variable x_1 is zero, this problem has Multiple solution.

optimum solution $\boxed{\text{Max } z = 200 \ @ \ x_1 = 0, \ x_2 = 20}$

2. Max

$z = 6x_1 + 4x_2$

s.t

$2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$, & $x_1, x_2 \geq 0$

Solution [using Big-M]

1. Introduce slack variable S_1, S_2 & surplus variable S_2 with Artificial variable A_1, the lpp is

Max

$z = 6x_1 + 4x_2 - MA_1$

s.t

$2x_1 + 3x_2 + S_1 = 30$

$3x_1 + 2x_2 + S_2 = 24$

$x_1 + x_2 - S_3 + A_1 = 3$, $x_1, x_2, S_1, S_2, S_3, A_1 \geq 0$

2. Initial basic feasible solution

Non basic variables $[x_1 = 0, x_2 = 0, S_3 = 0]$

Basic variables $[S_1 = 30, S_2 = 24, A_1 = 3]$

Initial iteration

		c_j	6	4	0	0	0	$-M$		
CB	YB	XB	x_1	x_2	S_1	S_2	S_3	A_1	θ	
0	S_1	30	2	3	1	0	0	0	15	
0	S_2	24	3	2	0	1	0	0	8	
$-M$	A_1	3	1	1	0	0	-1	1	3	← A_1 leaves
	Z_j	$-3M$	$-M$	$-M$	0	0	M	$-M$		
		$c_j - z_j$	$6+M$	$4+M$	0	0	$-M$	0		
			↑							

x_1 enters

2. Since 2^{nd} constraint b_i (R.H.S) is –ve, Multiply both sides by (–1). Now the LPP looks like

Max

$$Z = 8x_2' - 8x_2''$$

S.t

$$x_1' - x_1'' - x_2' + x_2'' \geq 0$$

$$-2x_1' + 2x_1'' - 3x_2' + 3x_2' \geq 6, \quad x_1', x_1'', x_2', x_2'' \geq 0$$

3. Introduce surplus variable S_1, S_2 and Artificial Variable A_1, A_2, the LPP becomes

Max

$$Z = 8x_2' - 8x_2''$$

S.t

$$x_1' - x_1'' - x_2' + x_2'' - S_1 + A_1 = 0$$

$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - S_2 + A_2 = 6, \quad x_1', x_1'', x_2', x$$

$$S_1, S_2, A_1\ A_2 \geq$$

4. Initial basic feasible solution

Non basic variables $[x_1' = 0,\ x_1'' = 0,\ x_2' = 0,\ x_2'' = 0,\ S_1 = 0,\ S_2 = 0]$

Basic variables $[A_1 = 0,\ A_2 = 6]$

Initial iteration

		c_j	0	0	8	–8	0	0	–M	–M	
CB	YB	XB	x_1'	x_1''	x_2'	x_2''	S_1	S_2	A_1	A_2	θ
–M	A1	0	1	–1	–1	1	–1	0	1	0	0
–M	A2	6	–2	2	–3	3	0	–1	0	1	2
	Z_j	–6M	–M	–M	4M	–4M	M	M	–M	–M	
		$c_j - z_j$	–M	M	8–4M	–8+4M	–M	–M	0	0	

x_2'' enters

I iteration [x_2'' enters, A_1 leaves]

CB	YB	XB	x_1'	x_1''	x_2'	x_2''	s_1	s_2	A_2	θ
	C_j		0	0	8	-8	0	0	-M	
-8	x_2''	0	1	-1	-1	1	-1	0	0	—
-M	A_2	6	-5	5	0	0	3	-1	1	6/5
	Z_j	-6M	5M-8	8-5M	-8	-8	8-3M	M	-M	
		C_j-Z_j	+8-5M	-8+5M	0	0	-8+3M	-M	0	

$$\text{New } A_2 \text{ row} = \begin{array}{cccccccc} 6 & -2 & 2 & -3 & 3 & 0 & -1 & 1 \\ -(3)* \; 0 & 1 & -1 & 1 & 1 & -1 & 0 & 0 \\ \hline 6 & -5 & 5 & 0 & 0 & 3 & -1 & 1 \end{array}$$

II iteration [x_1' enters, A_2 leaves]

CB	YB	XB	x_1'	x_1''	x_2'	x_2''	s_1	s_2
	C_j		0	0	8	-8	0	0
-8	x_2''	6/5	0	0	-1	1	-2/5	-1/5
0	x_1''	6/5	-1	1	0	0	3/5	-1/5
	Z_j	-48/5	0	0	8	-8	16/5	8/5
		C_j-Z_j	0	0	0	0	-16/5	-8/5

The 0.S is

$$\boxed{\begin{array}{l} \text{Max } Z = -48/5 \\ @ \; x_1' = 0, \\ \quad x_1'' = 6/5, \\ \quad x_2' = 0 \\ \quad x_2'' = 6/5 \end{array}}$$

$$\text{New } x_2'' \text{ row} = \begin{array}{ccccccc} 0 & 1 & -1 & -1 & 1 & -1 & 0 \\ -(-1)* \; 6/5 & -1 & 1 & 0 & 0 & 3/5 & -1/5 \\ \hline 6/5 & 0 & 0 & -1 & 1 & -2/5 & -1/5 \end{array}$$

Since all $C_j - Z_j \leq 0$ the current solution is optimal. This LPP has multiple solution because, the non-basic variables x_1', x_2' has its $C_j - Z_j$ value zero.

DUALITY & NETWORKS

Duality:

Dual in general implies two or double. In LP, Duality implies that each LPP can be analyzed in two different ways but would have equivalent solutions.

Any LP problem can be stated in another equivalent form based on same data. The original problem called Primal and the another one called Dual. The Dual of Dual is Primal.

Through dual LPP approach, one may develop a production plan that optimizes resource utilization, so that marginal opportunity cost of each unit of a resource is equal to its marginal return (Shadow Price).

Shadow Price indicates an additional price to be paid to obtain one additional unit of the resources in order to maximize the profit under the resource constraint

If resource is not completely used; i.e, there is slack, then its marginal return is zero

$$\text{Shadow Price} = \frac{\text{change in optimal Objective functional value}}{\text{unit change in the availability of resource}}$$

Note:

- No need to solve both LP problems seperately
- Solving one is equivalent to solving the other
- If optimal solution to one is known, the optimal solution of the other can also be read by $C_j - Z_j$ row.

Importance of Duality:

1. If the primal contains a large number of constraints & a small number of variables, the computation can be reduced by converting it into the dual & solving it.
2. The interpretation of dual variables from the cost or economi point of view, proves extremely useful in making future decision

Formulation of Dual Problem:

Primal - Dual relationship between a pair of LP problems can be expressed as

Primal	Dual
Max $z = \sum_{j=1}^{n} c_j x_j$ Sub-to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$; $i = 1, 2, \cdots, m$ and $x_j \geq 0$; $j = 1, 2, \cdots n$	Min $z_y = \sum_{i=1}^{m} b_i y_i$ Sub to $\sum_{i=1}^{m} a_{ij} y_i \geq c_j$ $j = 1, 2, \cdots, n$ and $y_i \geq 0$ $i = 1, 2, \cdots, m$

Summary

if primal	if Dual
i) Objective is to maximize	i) Objective is to minimize
ii) j^{th} primal variable, x_j	(ii) j^{th} dual constraint
iii) i^{th} primal constraint	iii) i^{th} dual variable, y_i
iv) primal variable x_j unrestricted in sign	iv) Dual constraint j is = type
v) Primal constraint i is = type	v) Dual variable y_i is unrestricted in sign.
vi) Primal constraint $\leq$ type	vi) Dual constraint $\geq$ type

Rules for constructing the dual from primal.

1) A dual variable is defined corresponds to each constraint in the primal LPP & vice-versa. Thus primal LPP with m constraints & n variables, there exists a dual LPP with m var. & n constraints

2) The RHS constants $b_1, b_2, \cdots, b_m$ of primal LPP becomes the coefficients of the dual var. $y_1, y_2, \cdots, y_m$ in the dual objective fn z_y. Also, the coefficients $c_1, c_2, \cdots, c_n$ of the primal var. $x_1, x_2, \cdots, x_n$ in the objective fn becomes RHS constants in the dual LPP.

3) for Max. primal LPP, with all $\leq$ type constraints, there exists Min, dual LPP with all $\geq$ type constraints and vice-versa. Thus inequality constraints reversed in all constraints except non-negativity constraints

4) Matrix of coefficients of variables in the constraints of dual is the transpose of matrix of coefficients of variables in constraints of primal & vice-versa

5) If the objective fn. of primal LPP is MAX, then for dual LPP is MIN & vice versa.

6) If the i^{th} primal constraint is = type, then the i^{th} dual var is unrestricted in sign and vice-versa.

1. Write the dual of the following primal LPP

Max. $Z = x_1 + 2x_2 + x_3$

Subject to $2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \geq -6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Step 1 :

Primal LPP : Max $Z = x_1 + 2x_2 + x_3$

Sub. to $2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \geq -6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Since the problem is not in the canonial form we interchange the inequality of the second constraint

Max $Z = x_1 + 2x_2 + x_3$

Sub. to $2x_1 + x_2 - x_3 \leq 2$

$-2x_1 + x_2 - 5x_3 \leq 6$

$4x_1 + x_2 + x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Step 2 :

$$\boxed{\begin{array}{l} \text{Max. } F = CX \\ \text{Subject to } Ax \leq b \\ \qquad\qquad x \geq 0 \end{array}}$$

$C = (1, 2 \ 1)$ $\quad A = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 1 & -5 \\ 4 & 1 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$ $\quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Step 3:

Since the primal problem is maximization type with $(\leq)$ constraints, with 3 constraints & 3 variables; the dual problem is minimization type with $(\geq)$ constraints with 3 constraints & 3 dual variables y_1, y_2, y_3

Dual problem is

$$\text{Min. } w = b^T Y$$

$$\text{sub to } A^T Y \geq C^T$$

$$Y \geq 0$$

$$\text{Min } w = (2\ 6\ 6)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{Subject to } \begin{pmatrix} 2 & 2 & 4 \\ +1 & -1 & 1 \\ -1 & 5 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq 0$$

$$\text{Min } w = 2y_1 + 6y_2 + 6y_3$$

Subject to $2y_1 + 2y_2 + 4y_3 \geq 1$

$y_1 - y_2 + y_3 \geq 2$

$-y_1 + 5y_2 + y_3 \geq 01$

$y_1, y_2, y_3 \geq 0$

Alternate Method:

Dual	Primal			Min. Z_y
	$x_1 \geq 0$	$x_2 \geq 0$	$x_3 \geq 0$	
$y_1 \geq 0$	2	1	−1	≤ 2
$y_2 \geq 0$	2	−1	5	≤ 6
$y_3 \geq 0$	4	1	1	≤ 6
Max Z_x	$\geq$ 1	$\geq$ 2	$\geq$ 1	

Non negativity constraint in dual.

Non negativity cons. in primal

inequality sign in primal

RHS constant in primal

RHS const. in Dual

Inequality constraint in Dual.

Dual problem

$$\text{Min } Z_y = 2y_1 + 6y_2 + 6y_3$$

sub. to

$2y_1 + 2y_2 + 4y_3 \geq 1$

$y_1 - y_2 + y_3 \geq 2$

$-y_1 + 5y_2 + y_3 \geq 1$

$y_1, y_2, y_3 \geq 0$.

2. write the dual of the LP problem

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Sub-to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0 .$$

Primal LPP : convert all ≤ to ≥ type as the obj. fn is Min

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Sub-to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_2 \geq 4$$
$$-7x_1 + 2x_2 + x_3 \geq -10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$

Primal

Dual	$x_1 \geq 0$	$x_2 \geq 0$	$x_3 \geq 0$	Max zy bi	
$y_1 \geq 0$	3	5	4	≥	7
$y_2 \geq 0$	6	1	3	≥	4
$y_3 \geq 0$	-7	2	1	≥	-10
$y_4 \geq 0$	1	-2	5	≥	3
$y_5 \geq 0$	4	7	-2	≥	2
	≤	≤	≤		
Primal Min Zx	3	-2	4		

Primal	Dual
Min	Max
3 var	3 constraints
5 constraints	5 variable
constraints ≥ type	constraints ≤ type
Non negativity constraint ≥	Non negativity constraints ≥

Dual :

$$\text{Max } zy = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$

Sub-to.

$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3$$
$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$
$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$
$$y_1, y_2, y_3, y_4, y_5 \geq 0 .$$

3) Convert primal to dual

$\text{Min } z = x_1 + 2x_2$

$\text{Sub to } 2x_1 + 4x_2 \leq 160$

$x_1 - x_2 = 30$

$x_1 \geq 10$

$x_1, x_2 \geq 0 .$

Min problem. So convert all constraint to $\geq$

Second constraint is = in primal. So second variable y_2 in dual is unrestricted in sign.

$\therefore \text{Min } z = x_1 + 2x_2$

$\text{Sub to } -2x_1 - 4x_2 \geq -160$

$x_1 - x_2 = 30$

$x_1 \geq 10$

$x_1, x_2 \geq 0$

Dual is

$\text{Max } z_y = -160y_1 + 30y_2 + 10y_3$

$-2y_1 + y_2 + y_3 \leq 1$

$-4y_1 - y_2 \leq 2$

$y_1, y_3 \geq 0$ y_2 unrestricted in sign

4) $\text{Min } z = x_1 - 3x_2 - 2x_3$

$\text{Sub to. } 3x_1 - x_2 + 2x_3 \leq 7$

$2x_1 - 4x_2 \geq 12$

$-4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \geq 0$ x_3 unrestricted

Min problem. So all constraints must be $\geq$ type.

Third constraint is = in primal. So third variable y_3 in dual is unrestricted in sign

Third variable x_3 in primal is unrestricted in sign. So third constraint must be = type.

$\therefore \text{Min } z = x_1 - 3x_2 - 2x_3$

$\text{Sub. to } -3x_1 + x_2 - 2x_3 \geq -7$

$2x_1 - 4x_2 \geq 12$

$-4x_1 + 3x_2 + 8x_3 = 10$

$x_1, x_2 \geq 0$ x_3 unrestricted

Dual

$\text{Max } z_y = -7y_1 + 12y_2 + 10y_3$

$-3y_1 + 2y_2 - 4y_3 \leq 1$

$y_1 - 4y_2 + 3y_3 \leq -3$

$-2y_1 + 8y_3 = -2$

$y_1, y_2 \geq 0$ y_3 unrestricted in sign.

Standard Results on Duality

1) The dual of the dual LP problem is again the primal problem

2) If either the primal or the dual problem has an unbounded solution, the other problem has no feasible solution.

3) If either the primal or dual problem has a finite optimal solution, the other one also possesses the same and the optimal objective function value of the two problems are equal. (ie Max Z_x = Min Z_y)

Dual Problem (Max)	Primal Problem (Min)	
	Feasible	Infeasible
Feasible	Max Z_y = Min Z_x	Max $Z_y \to +\infty$
Infeasible	Min $Z_x \to -\infty$	Unbounded or Infeasible

4) **Complementary slackness property** of primal dual relationship states that for a positive basic variable in the primal, the corresponding dual variable will be equal to zero

1. Use duality to solve the following LPP

Min $z = 2x_1+2x_2$

Sub to $2x_1+4x_2 \geq 1$

$-x_1-2x_2 \leq -1$

$2x_1+x_2 \geq 1$

and $x_1, x_2 \geq 0$.

Given primal LPP is Minimize $z = 2x_1+2x_2$

sub to $2x_1+4x_2 \geq 1$

$x_1+2x_2 \geq 1$

$2x_1+x_2 \geq 1$

$x_1, x_2 \geq 0$.

Its dual is Max $w = y_1+y_2+y_3$

sub - to $2y_1+y_2+2y_3 \leq 2$

$4y_1+2y_2+y_3 \leq 2$

and $y_1, y_2, y_3 \geq 0$.

Solve the dual by introducing slack variables.

$\therefore$ Max $w = y_1 + y_2 + y_3 + 0s_1$

Sub-to $2y_1 + y_2 + 2y_3 + s_1 = 2$

$4y_1 + 2y_2 + y_3 + s_2 = 2$

$y_1, y_2, y_3, s_1, s_2 \geq 0$.

Initial basic feasible solution

$y_1 = y_2 = y_3 = 0$ (Non basic variables)

$s_1 = 2 \quad s_2 = 2$ (Basic variables)

c_j 1 1 1 0 0

C_B	S_B	y_1	y_2	y_3	s_1	s_2	Y_B	Min Ratio
0	s_1	2	1	2	1	0	2	$2/2 = 1$
0	s_2	④	2	1	0	1	2	$2/4 = 0.5 \rightarrow$
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	$+1 \uparrow$	1	1	0	0		

$c_j - z_j > 0$. Soln not optimal

Entering Var $= y_1$

Leaving var $= S_2$

c_j 1 1 1 0 0

C_B	S_B	y_1	y_2	y_3	s_1	S_2	Y_B	Min ratio
0	S_1	0	0	$\boxed{3/2}$	1	$-1/2$	1	$2/3 \rightarrow$
1	y_1	1	$1/2$	$1/4$	0	$1/4$	$1/2$	2
	z_j	1	$1/2$	$1/4$	0	$1/4$	$1/2$	
	$c_j - z_j$	0	$+1/2$	$3/4 \uparrow$	0	$-1/4$		

$c_j - z_j > 0$. Soln- not optimal

Ent- Var $= y_3$ Leaving var $= S_1$

c_j 1 1 1 0 0

C_B	S_B	y_1	y_2	y_3	s_1	s_2	Y_B	Min ratio
1	y_3	0	0	1	$2/3$	$-1/3$	$2/3$	-
1	y_1	1	$(1/2)$	0	$-1/6$	$1/3$	$1/3$	$2/3 \rightarrow$
	c_j	1	$1/2$	1	$1/2$	0	1	
	$c_j - z_j$	0	$+1/2 \uparrow$	0	$-1/2$	0		

$c_j - z_j > 0$ Soln- not optimal

Ent- Var $= y_2$ Leaving Var $= y_1$

	c_j	1	1	1	0	0	
C_B	S_B	y_1	y_2	y_3	s_1	s_2	y_B .
1	y_3	0	0	1	2/3	-1/3	2/3
1	y_2	2	1	0	-1/3	2/3	2/3
	z_j	2	[illegible]	[illegible]	1/3	1/3	4/3
	$c_j - z_j$	-1	0	0	-1/3	-1/3.	

All $c_j - z_j \leq 0$.

The current basic feasible solution is optimal.

Dual Optimal Solution

Max $w = \frac{4}{3}$

$y_1 = 0$

$y_2 = 2/3$

$y_3 = 2/3$

Primal optimal solution.

Here it is observed that the primal variables x_1, x_2 respectively correspond to the slack variables s_1, s_2 of the dual problem.

From the table, $s_1 = -(-1/3) = x_1$ $s_2 = -(-1/3) = x_2$

Min z = Max w

∴ Primal solution is

Min $z = 4/3$

$x_1 = 1/3$

$x_2 = 1/3$.

2. Prove using duality theory that the following linear program is feasible but has no optimal solution.

Min $z = x_1 - x_2 + x_3$

Sub.to $x_1 - x_3 \geq 4$

$x_1 - x_2 + 2x_3 \geq 3$

and $x_1, x_2, x_3 \geq 0$

Dual of primal is Max $w = 4y_1 + 3y_2$

$y_1 + y_2 \leq 1$

$y_2 \geq 1$

$-y_1 + 2y_2 \leq 1$ $y_1, y_2 \geq 0$.

By introducing the slack variables s_1, s_3, surplus variables s_2 and artificial variable R_1

Max $w = 4y_1 + 3y_2 + 0s_1 + 0s_2 + 0s_3 - MA_1$

Sub to $y_1 + y_2 + s_1 = 1$

$y_2 - s_2 + A_1 = 1$

$-y_1 + 2y_2 + s_3 = 1$ $y_1, y_2, s_1, s_2, s_3, A_1 \geq 0$.

Intial Basic feasible soln is $y_1 = y_2 = s_2 = 0$ $s_1 = 1$ $A_1 = 1$ $s_3 = 1$

c_j		4	3	0	0	-M	0		
c_B	s_B	y_1	y_2	s_2	s_1	A_1	s_3	Y_B	Min ratio
0	s_1	1	1	0	1	0	0	1	1
-M	A_1	0	1	-1	0	1	0	1	1
0	s_3	-1	$\textcircled{2}$	0	0	0	1	1	1/2 →
	z_j	0	-M	M	0	-M	0	-M	
	$c_j - z_j$	4	3+M ↑	-M	0	M	0		

$c_j - z_j > 0$ soln.not opt. E.V = y_2 L.V = s_3.

		4	3	0	0	-M	0		
c_B	S_B	y_1	y_2	s_2	s_1	A_1	s_3	Y_B	Min Ratio.
0	s_1	$\textcircled{3/2}$	0	0	1	0	-1/2	1/2	1/3 →
-M	A_1	1/2	0	-1	0	1	-1/2	1/2	1
3	y_2	-1/2	1	0	0	0	1/2	1/2	–
	z_j	$-3/2+\frac{M}{2}$	3	M	0	-M	$\frac{3}{2}+\frac{M}{2}$	$\frac{3}{2}-\frac{M}{2}$	
	$c_j - z_j$	$\frac{11}{2}+\frac{M}{2}$ ↑	0	-M	0	0	$-\frac{3}{2}-\frac{M}{2}$		

$c_j - z_j < 0$ soln not optimal E.V = y_1 L.V = s_1

		4	3	0	0	-M	0	
c_B	S_B	y_1	y_2	s_2	s_1	A_1	s_3	Y_B.
4	y_1	1	0	0	2/3	0	-1/3	1/3
-M	A_1	0	0	-1	-1/3	1	-1/3	1/3
3	y_2	0	1	0	1/3	0	1/3	2/3
	z_j	4	3	M	$\frac{11}{3}+\frac{M}{3}$	-M	$\frac{1}{3}+\frac{M}{3}$	$\frac{10}{3}-\frac{M}{3}$
	$c_j - z_j$	0	0	-M	$\frac{-11}{3}-\frac{M}{3}$	0	$\frac{-1}{3}-\frac{M}{3}$	

All $c_j - z_j \leq 0$ soln optimal

Since $c_j - z_j \leq 0$ and an artificial variable A_1 appears in the basis at non-zero level, the dual problem has no optimal solution

∴ There exists no finite optimum solution to the given primal LPP

Dual - Simplex Method [2·1]

I Difference between Regular Simplex Method and dual Simplex Method.

	Regular Simplex Method	dual Simplex Method
1.	Starts with a basic feasible, but non-optimal solution and works towards optimality	Starts with basic infeasibl but optimal solution and wor towards feasibility
2.	Here, first entering variable then the leaving variable is determined.	Here, first leaving variab and entering variable is determined.
3.	uses Artificial variable.	Avoid Artificial variable

II Working procedure for dual-simplex Method

1. convert the problem into maximization form, if it is in minimization

2. convert ($\geq$) type constraints, into ($\leq$) type by multiplying both sides by (-1).

3. convert the inequality constraints into equalities by introducing slack variables. obtain the Initial Basic feasible Solution and express this information in the simplex table.

4. optimal condition

(i) if all $C_j - Z_j \leq 0$ and all $X_{Bi} \geq 0$, then the Solution is optimal.

(ii) if all $C_j - Z_j \leq 0$ and atleast one $X_{Bi} < 0$, then the

Current solution is not optimal basic feasible solution.
therefore go to next step

(iii) if any $c_j - z_j > 0$, then this method fails.

⑤ Feasibility condition

(5.1) Leaving variable ⇒ is the basic variable corresponding to the most negative value of $X_{Br} = \min\{X_{Bi} ; X_{Bi} < 0\}$

(5.2) Entering variable

$$\min\left\{\frac{c_j - z_j}{y_{rj}}, \; y_{rj} < 0\right\} \text{ for all } j$$

y_{rj} → pivot row elements [ie denominators with -ve value]

⑥ Carry out operations as in regular simplex Method and Repeat steps 4 to 5 until optimum solution is obtained or there is no solution

① Solve by dual simplex Method

Max $z = -3x_1 - 2x_2$

s.t
$x_1 + x_2 \geq 1$
$x_1 + x_2 \leq 7$
$x_1 + 2x_2 \geq 10$,
$x_2 \leq 3$, $x_1, x_2 \geq 0$.

Solution

1. convert all constraints into $\leq$ inequality.

Max $z = -3x_1 - 2x_2$

S.t
$-x_1 - x_2 \leq -1$
$x_1 + x_2 \leq 7$
$-x_1 - 2x_2 \leq -10$
$x_2 \leq 3$ &, $x_1, x_2 \geq 0$.

2. add slack variable S_1, S_2, S_3, S_4

Max $z = -3x_1 - 2x_2$

S.t
$-x_1 - x_2 + S_1 = -1$
$x_1 + x_2 + S_2 = 7$
$-x_1 - 2x_2 + S_3 = -10$
$x_2 + S_4 = 3$, & $x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$

3. Find Initial Basic feasible solution

Non-Basic Variables $[x_1 = 0, \ x_2 = 0]$

Basic Variables $[S_1 = -1, \ S_2 = 7, \ S_3 = -10, \ S_4 = 3]$

Initial iteration

			c_j	-3	-2	0	0	0	0
	C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
	0	S_1	-1	-1	-1	1	0	0	0
	0	S_2	7	1	1	0	1	0	0
S_3 leaves →	0	S_3	-10	-1	-2	0	0	1	0
	0	S_4	3	0	1	0	0	0	1
		Z_j	0	0	0	0	0	0	0
			$c_j - z_j$	-3	-2	0	0	0	0
			0	$\frac{-3}{-1} = 3$	$\frac{-2}{-2} = 1 \uparrow$	-	-	-	-

x_2 enters

Since some $X_{Bi} \leq 0$, the current solution is not optimal.

First iteration [x_2 enters S_3 leaves]

			c_j	-3	-2	0	0	0	0
	C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3	S_4
	0	S_1	4	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0
	0	S_2	2	$\frac{1}{2}$	0	0	1	$+\frac{1}{2}$	0
	-2	x_2	5	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0
S_4 leaves →	0	S_4	-2	$-\frac{1}{2}$	0	0	0	$+\frac{1}{2}$	1
		Z_j	-10	-1	-2	0	0	1	0
			$c_j - z_j$	-2	0	0	0	-1	0
			0	$\frac{-2}{-\frac{1}{2}} = 4 \uparrow$	-	-	-	$\cancel{\frac{-1}{-\frac{1}{2}}} = 2$	-

x_1 enters

New S_1 row

= -1 -1 -1 1 0 0

+ 5 $\frac{1}{2}$ 1 0 0 $-\frac{1}{2}$

4 $-\frac{1}{2}$ 0 1 0 $-\frac{1}{2}$

New S_2 row

= 7 1 1 0 1 0

− 5 $\frac{1}{2}$ 1 0 0 $-\frac{1}{2}$

2 $\frac{1}{2}$ 0 0 1 $+\frac{1}{2}$

New S_4 row

= 3 0 1 0 0 0

− 5 $\frac{1}{2}$ 1 0 0 $-\frac{1}{2}$

−2 $-\frac{1}{2}$ 0 0 0 $+\frac{1}{2}$

II iteration { x_4 enters, S_4 leaves }

		c_j	-3	-2	0	0	0	0
C_B	Y_B	X_B	x_4	x_2	S_1	S_2	S_3	S_4
0	S_1	6	0	0	1	0	-1	-1
0	S_2	0	0	0	0	1	1	1
-2	x_2	3	0	1	0	0	0	1
-3	x_4	4	1	0	0	0	-1	-2
	z_j	-18	-3	-2	0	0	3	4
		$c_j - z_j$	0	0	0	0	-3	-4

Since all $c_j - z_j \leq 0$ & all $X_{Bi} \geq 0$, the current solution is optimal

Max
$z = -18$ @
$x_4 = 4$,
$x_2 = 3$

New s_1 row $= 4 \quad -\frac{1}{2} \quad 0 \quad 1 \quad 0 \quad -\frac{1}{2} \quad 0$
$+\frac{1}{2} * (4 \quad 1 \quad 0 \quad 0 \quad 0 \quad -\cancel{1} \quad -2)$

New s_1 row
$$\begin{array}{rrrrrrrr} = & 4 & -\frac{1}{2} & 0 & 1 & 0 & -\frac{1}{2} & 0 \\ + & 2 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -1 \\ \hline & 6 & 0 & 0 & 1 & 0 & -1 & -1 \end{array}$$

New s_2 row $= 2 \quad \frac{1}{2} \quad 0 \quad 0 \quad 1 \quad \frac{1}{2} \quad 0$
$-\frac{1}{2} * (4 \quad 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad -2)$

$$\begin{array}{rrrrrrrr} = & 2 & \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & 0 \\ - & 2 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -1 \\ \hline & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

New x_2 row $= 5 \quad \frac{1}{2} \quad 1 \quad 0 \quad 0 \quad -\frac{1}{2} \quad 0$
$- \frac{1}{2} * (4 \quad 1 \quad 0 \quad 0 \quad 0 \quad -1 \quad -2)$

$$\begin{array}{rrrrrrrr} = & 5 & \frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ - & 2 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & -1 \\ \hline & 3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

2. Using dual simplex method, solve the lpp

Min $z = x_1 + x_2$

s.t $2x_1 + x_2 \geqslant 2$

$-x_1 - x_2 \geqslant 1$, & $x_1, x_2 \geqslant 0$.

Solution

1. Convert Min → Max

Max $z^* = -x_1 - x_2$

s.t $2x_1 + x_2 \geqslant 2$

$-x_1 - x_2 \geqslant 1$ & $x_1, x_2 \geqslant 0$.

2. Convert all constraints into $\leq$

Max $z^* = -x_1 - x_2$

s.t $-2x_1 - x_2 \leq -2$

$x_1 + x_2 \leq -1$, & $x_1, x_2 \geqslant 0$

3. Introduce s_1, s_2 (slack variables)

Max $z^* = -x_1 - x_2$

s.t $-2x - x_2 + s_1 = -2$

$x_1 + x_2 + s_2 = -1$ & $x_1, x_2, s_1, s_2 \geqslant 0$

4. Find Initial basic feasible solution

Non Basic Variable $[x_1 = 0, \; x_2 = 0]$

Basic Variable $[~~x_1~~ \; s_1 = -2, \; s_2 = -1]$

Initial iteration

s1 leaves →

		c_j	-1	-1	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2
0	S_1	-2	-2	-1	1	0
0	S_2	-1	1	1	0	1
	Z_j^*	0	0	0	0	0
		$c_j - z_j^*$	-1	-1	0	0
		θ	$-1/-2 = .5$, $\frac{-1}{-1} = 1$			

x_1 enters

Since some $X_{Bi} < 0$, the solution is not optional.

I iteration [x_1 enters, s_1 leaves]

s2 leaves →

		c_j	-1	-1	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2
-1	x_1	1	1	1/2	-1/2	0
0	S_2	-2	0	1/2	1/2	1
	z_j^*	-1	-1	-1/2	1/2	0
		$c_j - z_j^*$	0	-1/2	-1/2	0
		θ	–	–	–	–

New S_2 row

$= -1 \quad 1 \quad 1 \quad 0 \quad 1$

$- \; 1 \quad 1 \quad 1/2 \quad -1/2 \quad 0$

$-2 \quad 0 \quad 1/2 \quad 1/2 \quad 1$

Since some $X_{Bi} < 0$, the solution is not optional.

Since y_{ik} [pivot row elements] are positive, we are unable to find the entering variable. Therefore the solution is infeasible

3. using dual simplex Method solve the LPP

Max $z = 6x_1 + 4x_2 + 4x_3$

s.t

$3x_1 + x_2 + 2x_3 \geq 2$

$2x_1 + x_2 - x_3 \geq 1$

$-x_1 + x_2 + 2x_3 \geq 1$ & $x_1, x_2, x_3 \geq 0$.

solution

① convert all constraints into $\leq$ inequality + introduc slack variables ~~s1~~ s_1, s_2, s_3

Max $z = 6x_1 + 4x_2 + 4x_3$

s.t

$-3x_1 - x_2 - 2x_3 + s_1 \leq -2$

$-2x_1 - x_2 + x_3 + s_2 \leq -1$

$x_1 - x_2 - 2x_3 + s_3 \leq -1$, $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

② Find Initial basic feasible solution

Non basic variables $[x_1 = 0, x_2 = 0, x_3 = 0]$

Basic variables $[s_1 = -2, s_2 = -1, s_3 = -1]$

Initial iteration

		c_j	6	4	4	0	0	0
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	-2	-3	-1	-2	1	0	0
0	s_2	-1	-2	-1	1	0	1	0
0	s_3	-1	1	-1	-2	0	0	1
	z_j	0	0	0	0	0	0	0
		$c_j - z_j$	6	4	4	0	0	0

Since $c_j - z_j > 0$, this method fails. ie we cannot solve this problem by ~~using~~ using Simplex Method.

4. Using dual simple Method, solve the given LPP.

Min $z = 2x_1 + x_2$

S.t

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3 \;\; \forall x_1, x_2 \geq 0.$

Solution

① Convert Min → Max + all constraints into ≤ inequality

Max

$z^* = -2x_1 - x_2$

S.t

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 - 2x_2 \leq -3, \;\; x_1, x_2 \geq 0$

② Introduce slacks variables S_1, S_2, S_3

Max $z^* = -2x_1 - x_2$

S.t $-3x_1 - x_2 + S_1 = -3$

$-4x_1 - 3x_2 + S_2 = -6$

$-x_1 - 2x_2 + S_3 = -3, \;\; x_1, x_2, x_3, S_1, S_2, S_3 \geq 0.$

③ Find Initial basic feasible solution.

Non-basic variables $[x_1 = 0, x_2 = 0]$

Basic Variable $[S_1 = -3, S_2 = -6, S_3 = -3]$

Initial iteration

		c_j	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	-3	-3	-1	1	0	0
leaves → 0	S_2	-6	-4	-3	0	1	0
0	S_3	-3	-1	-2	0	0	1
	z_j^*	0	0	0	0	0	0
		$c_j - z_j^*$	-2	-1	0	0	0
		θ	.5	.3	–	–	–

First iteration [S_2 leaves, x_2 enters]

S_1 leaves →

		c_j	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
0	S_1	-1	-5/3	0	1	-1/3	0
-1	x_2	2	4/3	1	0	-1/3	0
0	S_3	1	5/3	0	0	-2/3	1
	z_j^*	-2	-4/3	-1	0	1/3	0
		$c_j - z_j^*$	-2/3	0	0	-1/3	0
		0	·4 ↑	–	–	1	

x_1 enters.

= -3 -3 -1 1 0 0
+ 2 4/3 1 0 -1/3 0

-1 -5/3 0 1 -1/3 0

New S_3 row
= -3 -1 -2 0 0
+2*(2 4/3 1 0 -1/3

1 5/3 0 0 -2/3

II iteration [x_1 enters, S_1 leaves]

		c_j	-2	-1	0	0	0
C_B	Y_B	X_B	x_1	x_2	S_1	S_2	S_3
-2	x_1	3/5	1	0	-3/5	1/5	0
-1	x_2	6/5	0	1	4/5	-3/5	0
0	S_3	0	0	0	1	-1	1
	z_j^*	-12/5	-2	-1	2/5	1/5	0
		$c_j - z_j^*$	0	0	-2/5	-1/5	0

New x_2 row
= 2 4/3 1 0 -1/3 0
-(4/3)* 3/5 1 0 -3/5 1/5 0

6/5 0 1 4/5 -3/5

New S_3 row
= 1 5/3 0 0 -2/3
-(5/3)* 3/5 1 0 -3/5 1/5

0 0 0 1 -1

Since all $c_j - z_j \le 0$, & all $X_{Bi} \geq 0$, the current solution is feasible.

Max $z^* = -12/5$ @ $x_1 = 3/5$, $x_2 = 6/5$

or

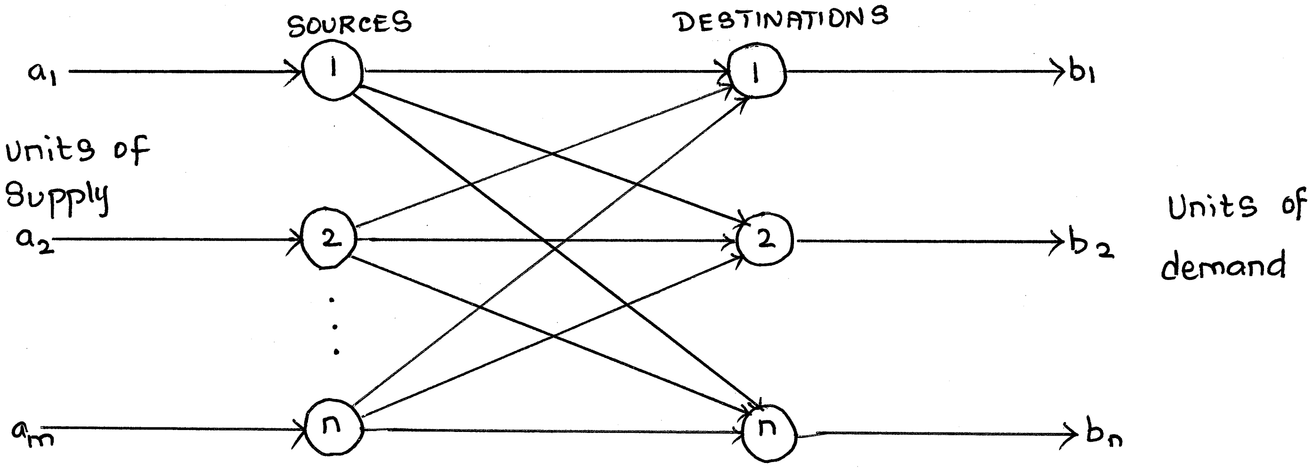
Min $z = 12/5$ @ $x_1 = 3/5$, $x_2 = 6/5$

Transportation & Assignment Model

Transportation Model :

It is a special class of linear programming that deals with shipping a commodity from sources to destinations. The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply & demand limits.

DEFINITION OF THE TRANSPORTATION MODEL:



There are "m" resources and "n" destinations each represented by a node. The arcs represent the route linking the sources & the destinations. Arc(i,j) joining the sources i to destination j carries two pieces of information. The transportation cost per unit c_{ij} and the amount shipped x_{ij}.

The objective of the model is to determine the unknowns x_{ij} that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

Mathematical Model

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Subject to $\sum_{j=1}^{n} x_{ij} = a_i \quad , \quad i = 1, 2, \cdots, m$ (supply)

$\sum_{i=1}^{m} x_{ij} = b_j \quad , \quad j = 1, 2, \cdots, n$ (demand)

$x_{ij} \geq 0$ for all i and j

Generalized format of the transportation problem

Destination (j)

Source (i)

$$\begin{array}{c|ccccccc|c}
 & 1 & 2 & \cdots & j & \cdots & n & & \text{Supply} \\
\hline
1 & c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} & & a_1 \\
2 & c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} & & a_2 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
i & c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} & & a_i \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
m & c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} & & a_m \\
\hline
\text{Demand} & b_1 & b_2 & \cdots & b_j & \cdots & b_n & &
\end{array}$$

Linear Programming formulation of the Transportation Problem

1. Express the following transformation problem as an LP problem

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

Minimize $z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 20x_{34}$

Subject to constraints

a) Supply constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 7$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 9$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 18$$

b) Demand constraints

$$x_{11} + x_{21} + x_{31} = 5$$
$$x_{12} + x_{22} + x_{32} = 8$$
$$x_{13} + x_{23} + x_{33} = 7$$
$$x_{14} + x_{24} + x_{34} = 14$$

and $x_{ij} \geq 0$ for all i and j

Decision Variables : $m \times n = 3 \times 4 = 12$

Constraints : $m + n = 3 + 4 = 7$

Transportation Algorithm

1. Formulate the problem and set up in the matrix form
2. Obtain an initial basic feasible solution

There are three different methods to develop the initial solution.

1. North west corner Method
2. Least cost method
3. Vogel's approximation (or) Penalty method.

The initial solution obtained by any of the three methods must satisfy the following condition

a. The solution must be feasible, it must satisfy all the supply and demand constraints.

b. The number of positive allocations must be equal to $m+n-1$, when m is the number of rows, n is the number of column

Any solution that satisfies the above condition is called non-degenerate basic feasible solution, otherwise degenerate solution

3. Test the initial solution for optimality

Modified Distribution (MODI) method is used to test the optimality of the solution. If the current solution is optimal, then stop. Otherwise, determine the new improved solution.

4. Updating the solution

Repeat step 3 until an optimal solution is reached

Determination of the starting solution

Three Methods

→ North West Corner Method (NWCN)
→ Least Cost Method (LCM)
→ Vogel's Approximation Method (VAM)

NORTH-WEST CORNER METHOD (NWCM)

- This method does not take into account the cost of transportation on any route of transportation.

Algorithm:

Step 1: Start with the cell at the upper left (North-West) corner of the transportation table (or matrix) and allocate commodity equal to the minimum of the rim values for the first row and first column. (ie.) $\min(a_1, b_1)$.

Step 2: (a) If allocation made in Step 1 is equal to the supply available at first source (a_1, in 1st row) then move vertically down to the cell (2,1). ie. 2nd row & 1st col. Apply step 1 again, for next allocation.

(b) If allocation made in Step 1 is equal to the demand of the first destination (b_1 in 1st col.), then move horizontally to the cell (1,2), i.e., 1st row & 2nd col. Apply step 1 again for next allocation.

(c) If $a_1 = b_1$, allocate $x_{11} = a_1$ or b_1 & move diagnolly to the cell (2,2)

Step 3: Continue the procedure step by step till an allocation is made in the south-west corner cell of the matrix.

1. Use North-West corner method (NWCM) to find an initial basic feasible solution to the transportation problem.

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_2	40	8	70	20	18
Demand	5	8	7	14	34 34

Step 1 : check for rim condition, Total supply = Total Demand = 34

It is a balanced transportation problem.

Step 2 : (S_1, D_1) is the north west corner. The rim value for row S_1 and column D_1 are compared. The smaller of two, ie, 5, is assigned as the first allocation. This allocation satisfies demand of D_1, but it leaves the supply of 2 units at S_1.

∴ Move horizontally & allocate as much as possible to cell (S_1, D_2). The rim values of row $S_1 = 2$ and column $D_2 = 8$. Smaller of two is 2. This allocation leaves a demand of 6 in D_2 and allocates the 7 units of supply.

∴ Move vertically & allocate as much as possible to cell (S_2, D_2). The rim values of row $S_2 = 9$ and column $D_2 = 6$. Smaller of two is 6.

Similarly move horizontally & vertically in the same manner to make desired allocation.

These allocations should be equal to $m+n-1 = 3+4-1 = 6$. If yes, the solution is non-degenerate feasible solution.

	D_1	D_2	D_3	D_4	Supply
S_1	19 ⑤	30 ②	50	10	7
S_2	70	30 ⑥	40 ③	60	9
S_3	40	8	70 ④	20 ⑭	18
Demand	5	8	7	14	34 34

$$\text{Total cost} = 5 \times 19 + 2 \times 30 + 6 \times 30 + 3 \times 40 + 4 \times 70 + 14 \times 20$$

$$= \text{Rs} - 1015/-$$

2. Sun Ray transport company ships truckloads of grain from three silos to four mills. The supply in (truckloads) and demand (in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the table. The unit transportation costs, c_{ij} are in hundreds of dollars.

Find the basic feasible solution using North-west corner Method.

Step 1:
Rim condition Satisfied
∴ Total demand = total supply = 50
It is a balanced transportation problem.

	D_1	D_2	D_3	D_4	Supply
S_1	10 ⑤	2 ⑩	20	11	15
S_2	12	7 ⑤	9 ⑮	20 ⑤	25
S_3	4	14	16	18 ⑩	10
Demand	5	15	15	15	50 / 50

Step 2: Allocate units starting from Northwest corner. (S_1, D_1) min (15, 5) = 5. Allocate 5 units to (S_1, D_1). The D_1 demand is fulfilled, but the S_1 contains (15 − 5) = 10 units.
Move horizontally (S_1, D_2).
In the (S_1, D_2) cell, min (10, 15) = 10. Allocate 10 units to (S_1, D_2). D_2 demand not fulfilled (15 − 10) = 5.
The S_1 supply exhausted but
∴ Move vertically down (S_2, D_2).
In the (S_2, D_2) cell [Min (25, 5)] = 5, allocate 5 units to (S_2, D_2). but S_2 supply contains (25 − 5) = 20 units
The D_2 demand is fulfilled
∴ Move horizontally (S_2, D_3)
Proceed till south east corner is reached.

The total cost = 5 × 10 + 10 × 2 + 5 × 7 + 15 × 9 + 5 × 20 + 10 × 18
= $ 520

The allocations should be equal to m + n − 1 = 3 + 4 − 1 = 6.
So the current solution is a non-degenerate basic feasible soln.

3. A dairy firm has 3 plants located in a state. The daily milk production at each plant is as follows.

Plant 1 : 6 million litres, Plant 2 : 1 million litres, Plant 3 : 10 million litres

Each day, the firm must fulfil the needs of its four distribution centres. The minimum requirement of each centre is as follows.

Distribution centre 1 : 7 million litres. Distribution centre 2 : 5 million litres
" " 3 : 3 million litres " " 4 : 2 million litres

cost (in hundreds of rupees) of shipping one million litre for each plant to each distribution centre is given below.

		DISTRIBUTION CENTRES			
		D1	D2	D3	D4
Plant	P1	2	3	11	7
	P2	1	0	6	1
	P3	5	8	15	9

Find the initial basic feasible solution using North-West Corner Method.

Step 1: Check for RIM condition.

Total Supply = Total demand = 17 (satisfied).

So the problem is balanced.

Step 2: Start from North-west (P_1, D_1). Take $\min(6, 7) = 6$. Move vertical.

At (P_2, D_2), $\min(1,1) = 1$

Both supply & demand exhausted.

∴ Move diagonally (P_3, D_2)

Proceed until south-east is reached.

	D_1	D_2	D_3	D_4	Supply
P_1	2 ⑥	3	11	7	6
P_2	1 ①	0	6	1	1
P_3	5	8 ⑤	15 ③	9 ②	10
Demand	7	5	3	2	17 17

The current solution contains only 5 occupied cells but it should be equal to $m+n-1 = 3+4-1 = 6$. ∴ Not equal

The solution is a degenerate solution.

$$\text{Total cost} = (6\times2 + 1\times1 + 5\times8 + 3\times15 + 2\times9)\times100$$
$$= \text{Rs. } 11600.$$

II LEAST COST METHOD (LCM)

- The main objective is to minimize the total transportation cost, transport as much as possible through these routes where the unit transportation cost is lowest.

- This method takes into account the minimum unit cost of transportation for obtaining the initial solution.

Algorithm.

Step 1: Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row & column in which either the supply or demand is fulfilled. If a row and a column are both satisfied simultaneously, then cross off either row or column.

In case, the smallest unit cost is not unique, then select the cell where maximum allocation can be made.

Step 2: After adjusting the supply and demand for all uncrossed row & columns repeat the procedure to select a cell with the next lowest unit cost among the remaining rows & columns of the table and allocate as much as possible to this cell. Then cross off that row and column in which either supply or demand is exhausted.

Step 3: Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied. The solution so obtained need not be non-degenerate.

2-24

1. Solve the problem using Least cost method

Step 1: The transporation problem is balanced as the RIM condition is satisfied. (ie.) Total Supply = Total Demand (34)

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Supply	5	8	7	14	34 34

Step 2: find the least cost

① Least cost is 8 (S3, D2) (Demand ^{D2} = 8, Supply ^{S3} = 18) So Allocate 8 in S3, D2. Omit D2 column as the demand is satisfied.

	D1	D2	D3	D4	
S1	19	30	50	10 ⑦	7 ②
S2	70 ②	30	40 ⑦	60	~~9~~ 2
S3	40 ③	8 ⑧	70	20 ⑦	~~18~~ 10 3
	5	8	7	~~14~~ 7	
		①	④	③	

② Least cost among remaining cells is 10 (S1, D4). (Demand = 14, Supply = So allocate 7 to (S1, D4) Omit S1 row as the supply is allocat

③ Least cost among remaining cells is 20 (S3, D4). (Demand = 7, Supply = 10 Allocate 7. Omit D4 column.

④ Least cost among remaining cells is 40. There is a tie. Choose arbitaril (S2, D3), (S3, D1). Let us choose (S2, D3). Suppy = 9 Demand = 7. So allocate 7. Omit D.

⑤ finally only D1 remaining. Allocate 3 to (S3, D1) and then 2 to (S2 D1)

Total cost = 10 × 7 + 70 × 2 + 40 × 7 + 40 × 3 + 8 × 8 + 20 * (7)
= 70 + 140 + 280 + 120 + ~~5~~64 + 140 = Rs. 814

2) Least cost Method:

	D1	D2	D3	D4	Supply
S1	10	2	20	11	15
S2	12	7	9	20	25
S3	4	14	16	18	10
Demand	5	15	15	15	50 50

Step 1: Rim condition satisfied. Total supply = Total demand.

Step 2: Least cost is 2 (S1, D2), Allocate (15)

Step 3: Least cost is 4 (S3, D1)

Allocate 5

Step 4: Least cost is 9 (S2, D3)

Allocate 15

Step 5: Least cost 18 (S3, D4)

Allocate 5

Step 6: Final Allocation in (S2, D4) ⇒ 10

	D1	D2	D3	D4	Supply
S1	10	2 ⑮	20	11	15 ②
S2	12	7	9 ⑮	20 ⑩	~~25~~ 10
S3	4 ⑤	14	16	18 ⑤	~~10~~ 5 ⑤
	5 ③	15 ②	15 ④	~~15~~ 10	

cost = 2*15 + 9*15 + 20*10 + 4*5 + 18*5

= 30+135+200+20+90 = Rs.475

III VOGEL'S APPROXIMATION METHOD (VAM)

- Vogel's approximation (penalty or regret) is preferred over NWCM and LCM methods.
- In this, an allocation is made on the basis of the opportunity (or penalty or cost) cost that would have been incurred if the allocation in certain cells with minimum transportation cost were missed.
- The allocation is made in such a way that the penalty cost is minimized.
- An initial solution obtained is nearer to an optimal solution or is the optimal solution.

Algorithm:

Step 1: Calculate the penalties for each row (column) by takin the difference between the smallest and next smallest unit transportation cost in the same row (column). This differen indicates the penalty or extra cost that has to be paid if decision maker fails to allocate to the cell with the minimum cost.

Step 2: Select the row/column with the largest penalty and allocate as much as possible in the cell that has the least cost in the selected row/column, and satisfies the rim conditions. If there is a tie in the values of penaltie it can be broken by selecting the cell where maximum allocation can be made.

Step 3: Adjust the supply & demand and cross out the satisfied row/column. If a row and a column are satisfied simultaneous only one of them is crossed out & the remaining row (column is assigned a zero supply (demand). Any row or column with zero supply or demand should not be used in computing future penalties.

Step 4: Repeat steps 1 to 3, until the available supply & demand is satisfied

1. Vogel's Approximation method (VAM)

	D1	D2	D3	D4	Supply	Row Penalty			
S1	19 ⑤	30	50	10 ②	~~7~~ 2	9	9	40	40
S2	70	30	40 ⑦	60 ②	9	10	20	20	20
S3	40	8 ⑧	70	20 ⑩	~~18~~ ~~10~~ ③ 0	12	20	50	-
Demand	5 ②	8 ①	7	14					
Column Penalty	21	$\boxed{22}$	10	10					
	21	-	10	10					
	-	-	10	10					
	-	-	10	50					

Problem is balanced.

1) Compute the penalties for each row & column
22 is the maximum. Choose a minimum cost cell in D_2 col.
8 is the min. cost cell (S_3, D_2). Allocate 8

2) 21 is the maximum. Choose a min-cost cell in D_1 col.
19 is the min cost cell (S_1, D_1). Allocate 5

3) 50 is the max. in 3rd row (S_3). Min cost is 20 in (S_3, D_4)
Allocate 10 in (S_3, D_4).

4) 50 is the max in 4th col. Allocate 2 in $(S_1 D_4)$

5) finally Allocate 7 in (S_2, D_3) and 2 in (S_3, D_4)

cost $= 5 * 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20$

$= \text{Rs.} 779$

2) Find the initial basic feasible solution using VAM

	D_1	D_2	D_3	D_4	
S_1	11	13	17	14	250
S_2	16	18	14	10	300
S_3	21	24	13	10	400
	200	225	275	250	950 950

Soln

The problem is balanced.

	D_1	D_2	D_3	D_4		Row Penalty			
S_1	11 200	13 50	17	14	~~250~~ 50	(2)	(1)	-	-
S_2	16	18 175	14	10 (125)	~~300~~ 125	(4)	(4)	(4)	④
S_3	21	24	13 275	10 125	400	(3)	(3)	(3)	(3)
	200	~~225~~ 175	275	~~250~~ 125					

column penalty	⑤ ↑	(5)	(1)	(0)
	-	⑤	(1)	(0)
	-	⑥	(1)	(0)
	-	-	(1)	(0)

1) Highest penalty col 1, col 2. choose col 1. abitarily. In col 1, 11 is the smallest cost. Allocate (200)

2) Highest Penalty in col 2. choose smallest cost in (13) col 2. Allocate (50)

3) Highest penalty in col 3. choose 18 - smallest cost in col 2. Allocate 175

4) Highest penalty in Row 2. choose 10 the smallest cost. Allocate 125

$cost = 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$

$= 12075/-$

Transportation Algorithm (or) MODI method (MOdified DIstribution method)

Test for optimality.

Step 1: Find the initial basic feasible solution of the given problem by NWCM or LCM or VAM

Step 2: Check the number of occupied cells. If these are less than $(m+n-1)$, there exists degeneracy and we introduce a very small positive assignment of $\epsilon (\simeq 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to $m+n-1$

Step 3: Find the set of values u_i, v_j $(i=1,2,3\cdots m; j=1,2,\cdots n)$ for each occupied cell (i,j) from the relation $c_{ij}=u_i+v_j$ preferably by starting initially with $u_i=0$ or $v_j=0$ for which the corresponding row or column has maximum number of individual allocations.

Step 4: Find u_i+v_j for each unoccupied cell (i,j) and enter at the upper right corner of the corresponding cell (i,j).

Step 5: Find the cell evaluations $d_{ij} = c_{ij} - (u_i+v_j)$ (upper left - upper right) for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding cell (i,j).

Step 6: Examine the cell evaluations d_{ij} for all unoccupied cells (i,j) and conclude that

(i) If all $d_{ij}>0$, then the solution under the test is optimal and unique.

(ii) If all $d_{ij}>0$ with atleast one $d_{ij}=0$, then the solution under the test is optimal & an alternative optimal soln. exists.

(iii) If atleast one $d_{ij}<0$, then the solution is not optimal. Go to next step.

Step 7 : Form a new B.F.S by giving maximum allocation to the cell for which dij is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal & vertical lines beginning and ending at the cell for which dij is most negative and having its other corners at some allocated cells. Along this closed loop indicate +θ and −θ alternatively at the corners. choose minimum of the allocations from the cells having −θ. Add this minimum allocation to the cells with +θ and subtract this minimum allocation from the allocation to the cells −θ.

Step 8 : Repeat steps (2) to (6) to test the optimality of this new basic feasible solution

Step 9 : continue the above procedure till an optimum solution is obtained.

Note : The VAM takes into account not only least cost cij but also the costs that just exceed the least cost cij and therefore yields better initial solution that obtained from other methods in general. So to find the initial solution give preference to VAM unless otherwise specified.

1. Solve the transportation problem

	1	2	3	4	supply
I	21	16	25	13	11
II	17	18	14	23	13
III	32	27	18	41	19
demand	6	10	12	15	43 Σb_j Σa_i

Solution [Phase - I]

① Since $\Sigma a_i = \Sigma b_j$ (43), the given problem is balanced.

∴ This problem has feasible solution

② Find Initial basic feasible solution using VAM'S Model

					supply	penalty					
	21	16	25	13 ⑪	~~11~~ 0	(3)	–	–	–	–	–
	17 ⑥	18 ③	14	23 ④	~~13~~ ~~9~~ ~~3~~ 0	(3)	(3)	(3)	(4)	–	–
	32	27 ⑦	18 ⑫	41	~~19~~ 12	(9)	(9)	(9)	(9)	(9)	(18)
demand	~~6~~ 0	~~10~~ ~~7~~ 0	12	~~15~~ ~~4~~ 0							
	(4)	(2)	(4)	(10)							
	(15)	(9)	(4)	(18)							
Penalty	(15)	(9)	(4)	–							
	–	(9)	(4)	–							
	–	(27)	(18)	–							
	–	–	(18)	–							

Highest penalty row/col choosen

(1) First choose column 4
Allocate 11 in Row1, col4

(2) Second choose column 4,
Allocate 4 in Row 2, col 4

(3) choose column 1,
Allocate 6 in Row 2, col1

(4) choose column 2
Allocate 3 in Row 2, col 2

(5) Allocate 12 in Row 3, col 3
for row penalty high is Row 3

(6) Finally allocate 7 in Row3, col2

③ Check for non-degeneracy

$m+n-1$ = no. of occupied cell

$3+4-1$
(6) = 6 (T)

∴ This problem is non-degenerate.

④ Total Transportation cost

$TC = 13*11 + 17*6 + 18*3 + 23*4 + 27*7 + 18*12$

= Rs 796 /-

Phase-II [Apply Modi Method] ⇒ To find optimal allocation

C_{ij} (u_i+v_j) x_{ij}

d_{ij}

21 7 14	16 8 8	25 −1 26	13 ⑪	$u_1=-10$
17 ⑥	18 ③	14 9 5	23 ④	$u_2=0$
32 26 6	27 ⑦	18 ⑫	41 32 9	$u_3=9$
$v_1=17$	$v_2=18$	$v_3=9$	$v_4=23$	

① Calculate $C_{ij} = u_i + v_j$ for occupied cells

② Calculate u_i+v_j fo unoccupied cell & wri in upper left corner

③ find $d_{ij} = C_{ij} - (u_i+v_j)$ for unoccupied cell & write i lower right corner

Since all $d_{ij} > 0$, the current allocation is optimal.

The optimal allocation is

$x_{14}=11$, $x_{21}=6$, $x_{22}=3$, $x_{24}=4$, $x_{32}=7$, $x_{33}=12$

TC = Rs 796 /-

2) Obtain an optimum basic feasible solution for the following transportation problem

To

from				Available
	7	3	2	2
	2	1	3	3
	3	4	6	5
Demand	4	1	5	10 Σbj
				Σai

Solution

i) Since $\Sigma a_i = \Sigma b_j = 10$, the problem is balanced.
There exist basic feasible solution

ii) By Vogel's Approximation Method, the initial solution is

7	3	2 ②	2	(1)	(5) ←	–
2	1 ①	3 ②	~~3~~ 2	(1)	(1)	(1)
3 ④	4	6 ①	~~5~~ 1	(1)	(3)	(3) ←
4	1	~~5~~ 3				
(1)	(2)↑	(1)				
(1)	–	(1)				
(1)	–	(3)				
–	–	(3)				

① First highest penalty in col 2. Allocate 1 in R2,C2

(2) Second highest penalty in Row 1. Allocate 2 in R1C3

(3) Highest penalty in Row 3. Allocate 4 in R3C1

(4) Finally Allocate 2 in R2C3 and 1 in R3C3

Initial Basic feasible Solution is

Cost = 4+1+6+12+6
= Rs. 29

From the table,

the number of allocations is 5 which is equal to $(m+n-1) = (3+3-1) = 5$

So the solution is non-degenerate basic feasible Solution.

Optimality : Since the solution is non-degenerate, can apply MODI (MOdified DIstribution) Method.

7	3	2 ②	$u_1 = 2$
2	1 ①	3 ②	$u_2 = 3$
3 ④	4	6 ①	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

(i) To start with, the max. no. of allocations is in column 3. So choose $v_3 = 0$.

$c_{ij} = u_i + v_j$

find using allocated cells.

(ii) $C_{13} = U_1 + V_3$

$2 = U_1 + 0$ $\quad U_1 = 2$

$C_{23} = U_2 + V_3$

$3 = U_2 + 0$ $\quad U_2 = 3$

$C_{33} = U_3 + V_3$

$6 = U_3 + 0$ $\quad u_3 = 6$

$C_{22} = u_2 + V_2$

$1 = 3 + V_2$ $\quad V_2 = -2$

$C_{31} = u_3 + V_1$

$3 = 6 + V_1$ $\quad V_1 = -3$

(iv) Find dij for unallocated cells

$d_{ij} = c_{ij} - (u_i + v_j)$

(iii) Find $u_i + v_j$ for all unallocated cells & write it in top right of unallocated. Then find dij by subtracting top left (c_{ij}) and top right $(u_i + v_j)$ values of unallocated cells.

7 -1 8	3 0 3	2 ②	$u_1 = 2$
2 0 2	1 ①	3 ②	$u_2 = 3$
3 ④	4 4 0	6 ①	$u_3 = 6$
$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

(iv) Since all $d_{ij} \geq 0$ with $d_{32} = 0$ the current solution is optimal. & there exists an alternative optimal solution

Optimal Solution is cost = Rs. 29/-

3. Find the optimal transportation cost of the following matrix using <u>Least Cost Method</u> for finding the Critical Solution.

Market

FACTORY	A	B	C	D	E	Available
P	4	1	2	6	9	100
Q	6	4	3	5	7	120
R	5	2	6	4	8	120
Demand	40	50	70	90	90	340 Σb_j Σa_i

Solution

i) Since $\Sigma a_i = \Sigma b_j = 340$, the transportation problem is balanced. There exists initial basic feasible solution.

ii) By using least cost method, the initial solution is as shown below.

4	1 (50)	2 (50)	6	9	~~100~~ ~~50~~
6 (10)	4	3 (20)	5	7 (90)	~~120~~ ~~100~~ 90
5 (30)	2	6	4 (90)	8	~~120~~ ~~30~~ 0
~~40~~ 10	50	~~70~~ 20	90	90	

(1) Least cost is 1 and is in R1 C2. Allocate 50 strike C2, demand over.

(2) Least cost is 2 in R_1C_3 among remaining. Allocate 50. Strike R1, supply over.

(3) Among non striked cells, least cost is 3 in R_2C_3. Allocate 20. Strike C_3.

(4) Among non striked cells, least cost is 4 in R_3C_4. Allocate 90, strike C_4

⑤ Among non-striked cells, least cost in 5 in R_3C_1. Allocate 30. Strike R3

⑥ Allocate 10 in R_2C_1 & 90 in R_2C_5

cost =(1*50) + (2*50) + (6*10) + (3*20) ~~+5*30~~
+ (7*90) + (4*90) +(5×30)

= 50 + 100 + 60 + 60 + 630 + 360 + 150

= Rs. 1410.

No of allocations is 7 which is same as (m+n-1)
((3+5-1) = 7)

So the current solution is Non-degenerate.
Can apply MODI method to find optimal solution.

1) Max. allocations in Row 2.. (ie 3 allocations) so choose $u_2 = 0$

2) Find u_i, v_j values by using allocated cells
$c_{ij} = u_i + v_j$

Since $u_2 = 0$, $v_1 = 6, v_3 = 3, v_5 =$
From v_1, $u_3 = -1$
From u_3, $v_4 = 5$
From v_3, $u_1 = -1$
From u_1, $v_2 = 2$

4 5 +θ -1	1 50	2 -θ 50	6 4 2	9 6 3	$u_1 = -1$
6 10 -θ	4 2 2	3 +θ 20	5 5 0	7 90	$u_2 = 0$
5 30	2 1 1	6 2 4	4 90	8 6 2	$u_3 = -1$
$v_1 = 6$	$v_2 = 2$	$v_3 = 3$	$v_4 = 5$	$v_5 = 7$	

3) Find dij for unallocated cells.

Since d_{11} is negative, the current soln. is not optimal.

4) Choose the most negative dij. This problem contains only one negative value.

Draw a closed loop consisting of horizontal & vertical lines beginning and ending at this cell (1,1) and having its other corners at some occupied cells.

Along this closed loop indicate +θ, & -θ alternatively at the corners.

5) From the cells (1,3) (2,1) having -θ, we find the minimum of the allocations 50, 10 is 10. Add this 10 to the cells with +θ and subtract this 10 to the cells with -θ.

4 10	1 50	2 40	6 3 3	9 6 3	$u_1=0$
6 5 1	4 2 2	3 30	5 4 1	7 90	$u_2=1$
5 30	2 2 0	6 3 3	4 90	8 7 1	$u_3=1$
$v_1=4$	$v_2=1$	$v_3=2$	$v_4=3$	$v_5=6$	

6) No: of allocations is 7 which is same as (m+n-1) = 7. solution is degenerate.

7) Now find u_i, v_j values by starting with $u_1 = 0$. from allocated cells

8) find dij values for unallocated cells.

9) Since all dij ≥ 0. Current soln is optimal

Optimal solution
cost = 1400

4) Apply MODI method ~~the~~ to obtain optimal solution of transportation problem.

19	30	50	10	7
70	30	40	60	9
40	8	70	20	18
5	8	7	14	34 Σb_j Σa_i

Solution:

1. Since $\Sigma a_i = \Sigma b_j = 34$, the problem is balanced. There exist basic feasible solution.

2. Apply Vogel's Approximation method to find intial basic feasible solution.

Row penalty

19 ⑤	30	50	10 ②	~~7~~ 2	9	9	40	40
70	30	40 ⑦	60 ②	~~9~~ 2	10	20	20	20
40	8 ⑧	70	20 ⑩	~~18~~ 10	12	20	← 50	
5	8	~~7~~	~~14~~ ~~4~~ 2					
21	22↑	10	10					
21↑	–	10	10					
–	–	10	10					
–	–	10	50					

(1) High penalty col2 = 22 Allocate 8 in R_3C_2

(2) High penalty in col1 = 21. Allocate 5 in R_1C_1

(3) High penalty in Row3 = 50 Allocate 10 in R_3C_4

(4) High penalty in col4 = 50 Allocate 2 in R_1C_4

(5) Finally allocate 7 in R_2C_3 & 2 in R_2C_4.

Basic feasible solution is

cost $= (19 \times 5) + (10 \times 2) + (40 \times 7) + (60 \times 2) + (8 \times 8) + (20 \times 10)$

$= 779$

Optimality

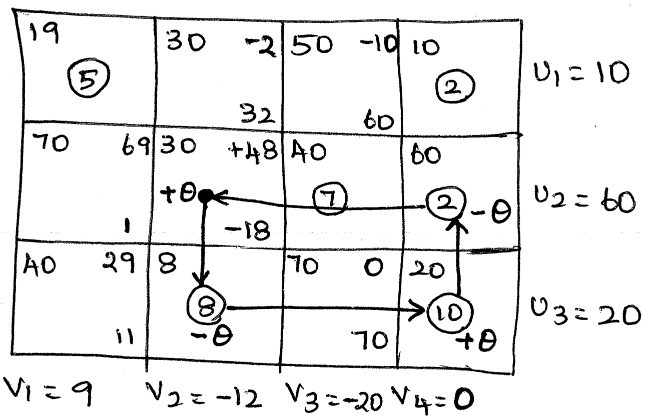
To apply MODI method to test optimality, the solution must be (m+n-1) allocations) must be non-degenerate (ie, there $(m+n-1) = (3+4-1) = 6$.

Here No of allocations = 6

Both are same. So can apply MODI method.

1) calculate U_i & V_j ~~from~~ from allocated cells.

$C_{ij} = U_i + V_j$. Max allocations in C_4 = 3 allocations. So choose V_4=



19 ⑤	30 -2 32	50 -10 60	10 ②	$U_1=10$
70 69 1	30 +48 +θ -18	40 ⑦	60 ② -θ	$U_2=60$
40 29 11	8 ⑧ -θ	70 0 70	20 ⑩ +θ	$U_3=20$
$V_1=9$	$V_2=-12$	$V_3=-20$	$V_4=0$	

2) find d_{ij} for unallocated cells

$$d_{ij} = C_{ij} - (U_i + V_j)$$

3) Since d_{22} is negative the current solution is not optimal.

4) choose the most -ve among -ve values of d_{ij}. $d_{22} = -18$.

5) Form a closed loop starting from d_{22}, consisting of horizontal & vertical lines & ending at (2,2) having its ~~other~~ other corners at some occupied cells.

6) Along this closed loop indicate +θ, -θ alternatively at the corners

7) From the cells (4,2), (3,4) having -θ, we find the minimum of allocations 8, 2 is 2. Add this 2 to the cells with +θ & subtract this 2 to the cells with -θ.

8) Again repeat the optimality test.
- Non-degeneracy check
  6 allocations m+n-1=6.
- Find U_i, V_j from allocated cells
  $C_{ij} = U_i + V_j$.
  Max allocations in R_1, R_2, R_3, C_2, C_4
  choose arbitrarily R_1 ie $U_1=0$
- find d_{ij} for unallocated cells

19 ⑤	30 -2 32	50 8 42	10 ②	$U_1=0$
70 51 19	30 ②	40 ⑦	60 42 18	$U_2=32$
40 29 11	8 ⑥	70 18 52	20 ⑫	$U_3=10$
$V_1=19$	$V_2=-2$	$V_3=8$	$V_4=10$	

Since all d_{ij} are greater than 0, the current solution is optimal.

optimal cost

cost = 19×5 + 10×2 + 30×2 + 40×7 + 8×6 + 20×12

= $\boxed{\text{Rs. 743}}$.

Degeneracy in Transportation problem.

When the number of allocations is less than m+n-1, the transportation problem is said to be a degenerate one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy at Initial stage, we allocate an extremely small amount to one or more empty cells of the transportation table (generally minimum cost cells if possible), so that the total number of occupied cells becomes (m+n-1) at independent positions.

Let us denote this small quantity as ϵ (Epsilon) satisfying the following conditions.

(i) $0 \le \epsilon < x_{ij}$ for all $x_{ij} > 0$

(ii) $x_{ij} + \epsilon = x_{ij}$ for all $x_{ij} > 0$.

The cells containing ϵ are treated as occupied cells. and the problem is solved in usual way. The ϵ's are kept till the optimum solution is attained, Then we let $\epsilon \to 0$

1. Solve the transportation problem to minimize the total cost

14	56	48	27	70
82	35	21	81	47
99	31	71	63	93
70	35	45	60	210 Σbj Σai

Solution

1) Since $\Sigma a_i = \Sigma b_j = 210$, the problem is balanced.

2) Apply Vogel's approximation to find initial basic feasible soln

14 70	56	48	27	~~70~~	13	–	–
82	35 2	21 45	81	~~47~~ 2	14	14	← 46
99	31 33	71	63 60	93	32	32	32
70	35	45	60				
68↑	4	27	36				
–	4	50↑	18				
–	4	–	18				

Basic feasible solution.

cost = (14×70) + (35×2) + (21×45)
(31×33) + (63×60) = 6798

The current solution is degenerate because there are only 5 allocations but there must be $(m+n-1) = (3+4-1) = 6$ allocation.

To resolve degeneracy, allocate a very small quantity ϵ to the cell (1,4). So the number of allocations become 6. Hence the solution is non degenerate.

Optimality

1) The ~~p~~ basic soln is non-degenerate
2) Find u_i, v_j values from allocated cells. Row 1, Row 2, Row 3, col 2, col 4 has max allo. 2. Choose $u_1 = 0$ (Row1)
3) find d_{ij} values for unallocated cells.

14 (70)	56 -5 61	48 -19 67	27 (ϵ)	$u_1 = 0$
82 54 28	35 (2)	21 (45)	81 67 14	$u_2 = 40$
99 50 49	31 (35)	71 17 +54	63 (58)	$u_3 = 36$
$v_1 = 14$	$v_2 = -5$	$v_3 = -19$	$v_4 = 27$	

Since All $d_{ij} > 0$. The current solution is optimal.

$$\text{optimal cost} = (14 \times 70) + (27 \times \epsilon) + (35 \times 2) + (21 \times 45) + (31 \times 35) + (63 \times 58)$$

$$= 6798 + 27\epsilon$$

$$\boxed{= 6789} \quad \text{Let } \epsilon \to 0$$

II Unbalanced Transportation Problem.

If total Supply ≠ total demand, problem is said to be unbalanced. Two cases exists

1) If total supply > total demand, add a column (called dummy demand). The unit cost for the cells in this column is set to 0 & demand for this column is set as the difference between total supply, total dem
2) If total demand > total supply, add a row (called dummy supply). The unit cost is set to zero, supply is set as total demand - total supply.

2) A manufacturer wants to ship 22 loads of his product a shown below. The matrix gives the kilometers from sources of supply to the destinations.

	D1	D2	D3	D4	D5	Supply
S_1	5	8	6	6	3	8
S_2	4	7	7	6	5	5
S_3	8	4	6	6	4	9
	4	4	5	4	8	22 / 25

Σb_j = 22, Σa_i = 25

The shipping cost is Rs.10 per load per km. what shipping schedule should be used in order to minimize the total transportation cost?

Solution:

i) Since $\Sigma a_i \neq \Sigma b_j$ the problem is unbalance.
Total demand is greater than total supply.
So add a new row and assign zero cost to all the cells of new row. Supply for new row = 25-22 = 3.
Now the problem becomes balanced.

5	8	6	6	3	8
4	7	7	6	5	5
8	4	6	6	4	9
0	0	0	0	0	3
4	4	5	4	8	25

2) Now the problem is balanced.

3) Apply vogels Approximation method to find the initial solution.

5	8	6 ⑤	6	3 ③	~~8~~ 3	2	2	2	2	3	3 ←	-
4 ④	7	7	6 ①	5	~~5~~ 1	1	1	1	1	1	1	(4)
8	4 ④	6	6	4 ⑤	~~9~~ 5	0	0	0	0	4 ←	-	-
0	0	0	0 ③	0	~~3~~	0	0	0	-	-	-	-
~~4~~ 1	~~4~~	~~5~~	~~4~~ 1 1	~~8~~ ~~5~~								
4	4	6↑	6	3								
4	4	-	6↑	3								
4↑	3	-	0	1								
1	3↑	-	0	1								
1	-	-	-	1								

5 3 2	8 3 5	6 5 1	6 5 1	3 ⑧	$u_1 = -1$
4 ④	7 4 3	7 6 1	6 ①	5 4 1	$u_2 = 0$
8 4 4	4 ④	6 ②	6 ③	4 (ε)	$u_3 = 0$
0 -2 2	0 -2 2	0 ③	0 0 0	0 -2 2	$u_4 = -6$
$v_1 = 4$	$v_2 = 4$	$v_3 = 6$	$v_4 = 6$	$v_5 = 4$	

Test for optimality

Solution is degenerate

Introduce ε in an unallocated cell. choose Row 3, col 5.

find u_i, v_j values. Max. allocation in Row 3. Choose $u_3 = 0$

find d_{ij} values

As all $d_{ij} \geq 0$, the current solution is optimal

optimal solution

$$\text{cost} = (3*8) + (4*4) + (6*1) + (4*4) + (6*2) + (6*3) + (4*\varepsilon) + (0*3)$$

$$= 24 + 16 + 6 + 16 + 12 + 18 + 4\varepsilon + 0$$

$$= 92 + 4\varepsilon$$

$$\boxed{\text{cost} = 92 \text{ units} \quad \varepsilon \to 0}$$

Alternative optimal solution

The existence of alternative optimal solution can be determined by an inspection of opportunity costs, d_{ij} for the occupied cells. If $d_{ij} = 0$, for an unoccupied cell in an optimal solution, then an alternative optimal solution exists. and can be obtained by bringing such an unoccupied cell in the solution mix without increasing the total transportation cost

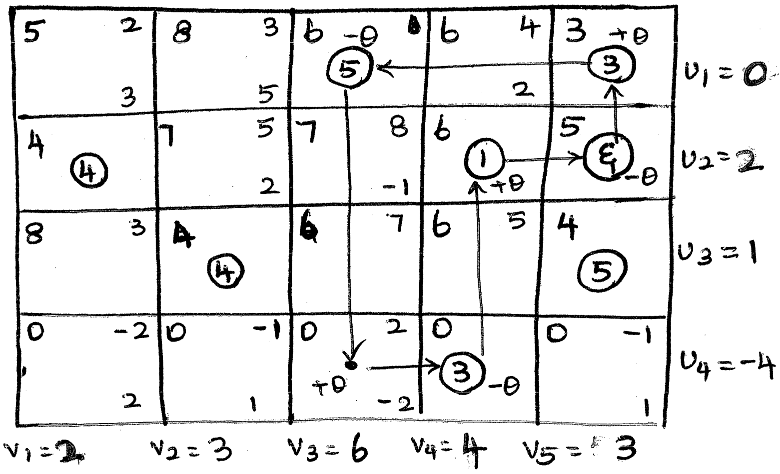
Ex: solved.

	A	B	C	D	Supp	
W	4 0 +4	8 (76)	8 0 +8	0 -16 +16	76	$u_1 = -16$
X	16 16 +θ +0	24 ㉑ -θ	16 ㊶	0 ⑳	82	$u_2 = 0$
Y	8 (72) -θ	16 (05) +θ	24 8 16	0 -8 8	77	$u_3 = -8$
Dem	72	102	41	20		
	$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$		

Here Row2 Col1
$d_{ij} = 0$
Bring it into the solution
min {72, 21}
θ = 21

4 0 4	8 (76)	8 0 8	0 -16 16	$u_1 = -16$
16 ㉑	24 24 0	16 ㊶	0 ⑳	$u_2 = 0$
8 (51)	16 ㉖	24 8 16	0 -8 8	$u_3 = -8$
$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$	

In both cases, case1 cost = Rs 2424
case2 cost = Rs 2424
Alt-opt. solution exists.



5 2 3	8 3 5	6 −θ ⑤	6 4 2	3 +θ ③	$u_1=0$
4 ④	7 5 2	7 8 −1	6 ① +θ	5 (ε) −θ	$u_2=2$
8 3	4 ④	6 7	6 5	4 ⑤	$u_3=1$
0 −2 2	0 −1 1	0 2 +θ −2	0 ③ −θ	0 −1 1	$u_4=-4$
$v_1=2$	$v_2=3$	$v_3=6$	$v_4=4$	$v_5=3$	

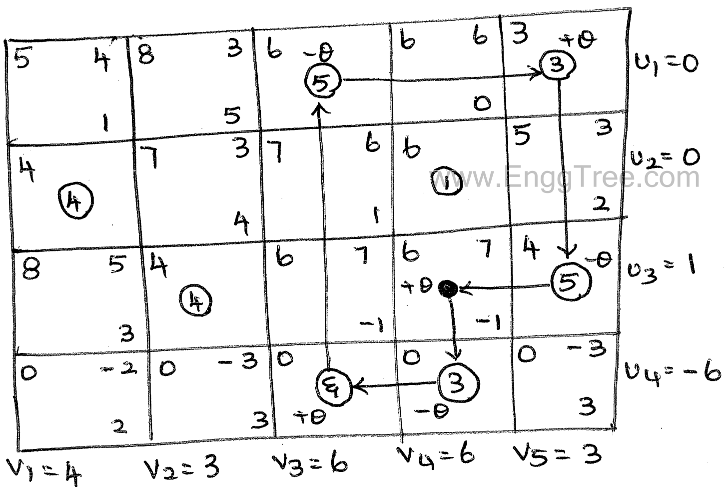
① As the initial soln. is degenerate, we need to remove the degeneracy, by introducing ε to an unoccupied cell with cost. Let us choose (S_2, D_5)

② Then find U_i & V_j values No. of allocations in Row1, Row2,3 is 2 So choose $U_1=0$ arbitarily

3) Now find d_{ij} values for unallocated cells

4) The most negative among d_{ij} is −2 which occurs in R4, C4. Find the closed loop Starting from Row4, col4 ∴ min{3, ε, 5} = ε. Among −θ should choose the minimum. So add ε to +θ cells & subtract ε from −θ cells.

Note
ε+0 = ε
ε−0 = ε



5 4 1	8 3 5	6 −θ ⑤	6 6 0	3 +θ ③	$u_1=0$
4 ④	7 3 4	7 6 1	6 ①	5 3 2	$u_2=0$
8 5 3	4 ④	6 7 −1	6 7 +θ −1	4 −θ ⑤	$u_3=1$
0 −2 2	0 −3 3	0 (ε) +θ	0 ③ −θ	0 −3 3	$u_4=-6$
$v_1=4$	$v_2=3$	$v_3=6$	$v_4=6$	$v_5=3$	

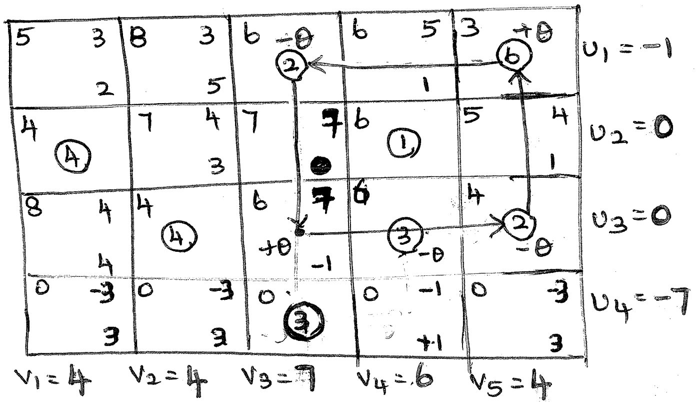
Iteration 2

1) Again Repeat the process. No of allocations = 8 (m+n−1 = 4+5−1=8) ∴ Soln is non-degenerate

2) find U_i & V_j values. Max allocations 2 in R1, R2, R3, R4, C3, C4, C5 choose arbitarily $U_1=0$.

3) Then find d_{ij} values. Again d_{ij} is −ve in R_3C_3 & R_3C_4 choose any one arbitarily & form a closed loop. Let us choose R_3C_4

4) Among −θ cells minimum is min{3, 5, 5} = 3. So add 3 to +θ cells Sub 3 from −θ cells.



5 3 2	8 3 5	6 −θ ②	6 5 1	3 +θ ⑥	$u_1=-1$
4 ④	7 4 3	7 7	6 ①	5 4 1	$u_2=0$
8 4 4	4 ④	6 7 +θ −1	6 ③ −θ	4 ② −θ	$u_3=0$
0 −3 3	0 −3 3	0 ③	0 −1 +1	0 −3 3	$u_4=-7$
$v_1=4$	$v_2=4$	$v_3=7$	$v_4=6$	$v_5=4$	

Iteration 3

1) Soln is non-degenerate

2) find U_i, V_j values

3) find d_{ij} values

4) Again d_{ij} is negative in R_3C_3. Form a closed loop Find min. of −θ {2, 2} = 2. Add 2 to +θ cells Sub 2 from −θ cells.

Maximization Transportation problem :

If there is a transportation problem whose objective is to maximize the total profit, first we have to convert the maximization problem into a minimization problem by subtracting all the entries from the highest entry in the given transportation table.

Ex 1. Solve the following problem to maximize profit.

Profit (Rs) /unit

Destination

		A	B	C	D	Supply
	1	40	25	22	33	100
Source	2	44	35	30	30	30
	3	38	38	28	30	70
Demand		40	20	60	30	

Solution: Since the problem is Maximization type, convert it into a minimization problem by subtracting the cost elements from the highest cost element (here $c_{ij} = 44$)

Minimization problem is

	A	B	C	D	
1	4	19	22	11	100
2	0	9	14	14	30
3	6	6	16	14	70
	40	20	60	30	200 150

The problem is unbalanced
Demand (150) < supply (200)
Add a new col, with zero cost & demand = 50

Solve using VAM
1) R_3C_5 - Allocate 50
2) R_2C_1 - Allocate 30
3) R_3C_2 - Allocate

	A	B	C	D	E					
1	4 ⑩	19	22 (60)	11 ㉚	0	100	4	7	7	7 ←
2	0 ㉚	9	14	14	0	30	0	9 ←	–	–
3	6	6 ⑳	16	14	0 ㊿	~~70~~ 20	6 ←	0	0	–
	~~40~~ 10	20	60	30	50					
	4	3	2	3	0					
	4	3	2	3	–					
	2	13 ↑	6	3	–					
		–			–					

check optimality ·

1) No of allocations (6) less than (m+n−1) = (3+5−1) = 7 (Degenerate)

Assign ε to some unallocated cell · (choose R_3C_3 arbitarily)

2) find u_i, v_j · max allocation $R_1 R_3$

Assume $u_1 = 0$

3) find d_{ij} for unallocated cells ·

4) d_{ij} for some cells are −ve · choose R_3C_5 (most −ve d_{ij}) $= -6$ · form a closed loop

5) Min {60, 50} = 50 ∴ θ = 50 (Take −θ cell values)

4 (10)	19 12 7	22 −θ (60)	11 (30)	0 6 +θ −6	$u_1 = 0$
0 (30)	9 8 1	14 18 −4	14 7 7	0 2 −2	$u_2 = -4$
6 −2 8	6 (20)	16 (ε) +θ	14 5 9	0 (50) −θ	$u_3 = -6$
$v_1 = 4$	$v_2 = 12$	$v_3 = 22$	$v_4 = 11$	$v_5 = 6$	

II iteration

1) Solution is non-degenerate = 7 (m+n−1)

2) find u_i, v_j values · Max allocations = R_1

Assume $u_1 = 0$

3) find d_{ij} for unallocated cells ·

4) Since d_{ij} is −ve for R_2C_3, form a closed loop ·

5) Min {10, 30} = 10 ∴ θ = 10 · (Take −θ cell values)

4 +θ (10)	19 12 7	22 −θ (10)	11 (30)	0 (50)	$u_1 = 0$
0 (30) −θ	9 8 1	14 18 +θ −4	14 7 7	0 −4 4	$u_2 = -4$
6 −2 8	6 (20)	16 (50)	14 5 9	0 −6 6	$u_3 = -6$
$v_1 = 4$	$v_2 = 12$	$v_3 = 22$	$v_4 = 11$	$v_5 = 0$	

III iteration

1) Solution is non-degenerate

2) find u_i, v_j values · max allocation Row1 · Assign $u_1 = 0$

3) find d_{ij}

Since All $d_{ij} > 0$, the current solution is optimal ·

4 (20)	19 8 11	22 18 4	11 (30)	0 (50)	$u_1 = 0$
0 (20)	9 4 5	14 (10)	14 7 7	0 −4 4	$u_2 = -4$
6 2 4	6 (20)	16 (50)	14 9 5	0 −2 2	$u_3 = -2$
$v_1 = 4$	$v_2 = 8$	$v_3 = 18$	$v_4 = 11$	$v_5 = 0$	

To find the ~~cost~~ profit, take the values from original profit matrix ·

40 (20)	25	22	33 (30)	0 (50)
44 (20)	35	30 (10)	30	0
38	38 (20)	28 (50)	30	0

profit = (40×20) + (33×30) + (0×50)
+ (44×20) + (30×10) + (38×20) + (28×50)

$\boxed{\text{profit} = \text{Rs}\cdot 5130}$

ASSIGNMENT PROBLEM

A special case of Transportation Problem with m=n and $s_i=d_j=1$ for all i, and j.
If in a printing press there is one machine and one operator is there to operate. How would you employ the worker?

The immediate answer will be, the available operator will operate the machine Again suppose there are two machines in the press and two operators are engaged at different rates to operate them. Which operator should operate which machine for maximising profit?
Similarly, if there are n machines available and n persons are engaged at different rates to operate them. Which operator should be assigned to which machine to ensure maximum efficiency?

These kinds of problems are referred as Assignment problem. It can be solved using the Hungarian method

$$\begin{array}{cc|ccccc|c} & & & & \text{Destination} & & & \text{Supply} \\ & & 1 & 2 & 3 & \cdots & n & (a_i) \\ \hline & 1 & c_{11} & c_{12} & c_{13} & \cdots & c_{1n} & a_1 \\ & 2 & c_{21} & c_{22} & c_{23} & \cdots & c_{2n} & a_2 \\ & 3 & c_{31} & c_{32} & c_{33} & \cdots & c_{3n} & a_3 \\ \text{Source} & & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ & n & c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} & a_n \\ \hline \text{Demand} & & b_1 & b_2 & b_3 & \cdots & b_n & \end{array}$$

Mathematically, we can express the problem as follows:

$$\text{To minimize } z \text{ (cost)} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}; \ [\ i=1,2,\ldots n;\ j=1,2,\ldots n] \ \ldots(1)$$

$$\text{where } x_{ij} = \begin{cases} 1; \text{ if } i^{th} \text{ person is assigned } j^{th} \text{ work} \\ 0; \text{ if } ith \text{ person is not assigned the } j^{th} \text{ work} \end{cases}$$

with the restrictions

(i) $\sum_{i=1}^{n} x_{ij}=1;\ j=1,2,\ldots n.$, i.e., i^{th} person will do only one work.

(ii) $\sum_{j=1}^{n} x_{ij}=1;\ i=1,2,\ldots n.$,i.e., j^{th} work will be done only by one person.

Difference between Transportation and Assignment Problem

Transportation Problem	**Assignment Problem**
Supply at any source may be any positive quantity a_i	Supply at any source(machine) will be 1 i.e., $a_i=1$
Demand at any destination may be any positive quantity b_j	Supply at any destination (job) will be 1 i.e., $b_j=1$
One or more source to any number of destinations	One source(machine) to only one destination(job)

ASSIGNMENT ALGORITHM

(The Hungarian Method)

Step I

(A) Row reduction:

Subtract the minimum entry of each row from all the entires of the respective row in the cost matrix.

(B) Column reduction:

After completion of row reduction, subtract the minimum entry of each column from all the entires of the respective column.

Step II

Zero assignment:

(A) Starting with first row of the matrix, examine the rows one by one until a row containing exactly one zero is found. Then mental assignment indicated by ( ) is marked to that zero. Now cross all the zeros in the column in which the assignment is made. This procedure should be adopted for each row assignment.

(B) An identical procedure is applied successively to columns. Examine all columns until a column containing exactly one zero is found. Then make an assignment in that position and cross other zeros in the row in which the assignment was made. Continue these successive operations on rows and columns until all zero's have either been assigned or crossed-out.

Now there are two possibilities:

(a) Either all the zeros are assigned or crossed out, i.e., we get the maximal assignment. or

(b) At least two zeros are remained by assignment or by crossing out in each row or column. In this situation we try to exclude some of the zeros by trial and error method. This completes the second step.

After this step we can get two situations.

(i) Total assigned zero's = n

The assignment is optimal.

(ii)Total assigned zero's < n

Use step III and onwards.

Step III:

Draw of minimum lines to cover zero's

Procedure.

(i) Marks (√) to all rows in which the assignment has not been done.

(ii) See the position of zero in marked (√) row and then mark (√) to the corresponding column.

(iii) See the marked (√) column and find the position of assigned zero's and then mark (√) to the corresponding rows which are not marked till now.

(iv) Repeat the procedure (ii) and (iii) till the completion of marking.

Draw the lines through unmarked rows and marked columns.

Step IV: Select the smallest element from the uncovered elements.

(i) Subtract this smallest element from all those elements which are not covered.

(ii) Add this smallest element to all those elements which are at the intersection of two lines.

Step V: Thus we have increased the number of zero's. Now, modify the matrix with the help of step II and find the required assignment.

1. Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows.

$$\begin{array}{cc|ccccc} & & & & \text{JOB} & & \\ & & 1 & 2 & 3 & 4 & 5 \\ \hline & A & 8 & 4 & 2 & 6 & 1 \\ & B & 0 & 9 & 5 & 5 & 4 \\ \text{PERSON} & C & 3 & 8 & 9 & 2 & 6 \\ & D & 4 & 3 & 1 & 0 & 3 \\ & E & 9 & 5 & 8 & 9 & 5 \end{array}$$

Determine the optimum assignment schedule.

Solution:

$$\begin{bmatrix} 8 & 4 & 2 & 6 & 1 \\ 0 & 9 & 5 & 5 & 4 \\ 3 & 8 & 9 & 2 & 6 \\ 4 & 3 & 1 & 0 & 3 \\ 9 & 5 & 8 & 9 & 5 \end{bmatrix}$$

Step 1A : Row subtraction

$$\begin{bmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{bmatrix}$$

Step 1B : column subtraction

Step II 1)
$$\begin{bmatrix} 7 & 3 & 0 & 5 & 0 \\ (0) & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & (0) & 4 \\ 4 & 3 & (0) & \cancel{0} & 3 \\ 4 & 0 & 2 & 4 & 0 \end{bmatrix}$$ Row examination

2)
$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 7 & 3 & \cancel{0} & 5 & (0) \\ B & (0) & 9 & 4 & 5 & 4 \\ C & 1 & 6 & 6 & (0) & 4 \\ D & 4 & 3 & (0) & \cancel{0} & 3 \\ E & 4 & (0) & 2 & 4 & \cancel{0} \end{array}$$ column examination

Step 3 : Since each row & column contains exactly one assignment the current assignment is optimal

A → 5 B → 1 C → 4 D → 3 E → 2

find cost from orginal problem.

$\text{cost} = (1+0+2+1+5) = \boxed{9 \text{ units}}$

Since the number of rows is equal to number of columns the given assignment problem is unbalanced.

Step 1A: Select the smallest cost element from each row & subtra from all the elements of the corresponding row

Step 1B: Select the smallest cost element from each column & subtract from all the elements of the corresponding column

Step 2 : Since each row & column contai atleast one zero, we can make assignments in reduced matrix.

1) Examine rows successively until a row with exactly 1 unmark zero is found. Since the 2nd row contains a single zero, encircle this zero and cross all other zero of its column.

2) The third row contains exactly 1 unmar zero, so encircle & cross all other 0's in its col

3) The fourth row contains exactly 1 unmarked zero, so encircle this ze and cross all other zeros in its colum

4) fifth row contains 2 zeros. Likewise examine columnwise. The 2nd col. contains exactly 1 unmark zero, so encircle it & cross all other zeros in its row.

5) fifth col. contains exactly 1 unmarke zero, encircle it & cross zeros in its ro

2) The processing time in hours for the job when allocated to the different machines are indicated below. Assign the machines for the jobs so that the total processing time is minimum.

Machines

$$\text{Jobs}\quad\begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & 9 & 22 & 58 & 11 & 19 \\ J_2 & 43 & 78 & 72 & 50 & 63 \\ J_3 & 41 & 28 & 91 & 37 & 45 \\ J_4 & 74 & 42 & 27 & 49 & 39 \\ J_5 & 36 & 11 & 57 & 22 & 25 \end{array}$$

Solution

No of rows = No. of columns = 5. Problem is unbalanced.

$$\begin{bmatrix} 9 & 22 & 58 & 11 & 19 \\ 43 & 78 & 72 & 50 & 63 \\ 41 & 28 & 91 & 37 & 45 \\ 74 & 42 & 27 & 49 & 39 \\ 36 & 11 & 57 & 22 & 25 \end{bmatrix}$$

Step 1: Row subbraction
Take smallest element from each row & subbract this from all the elements of the corresponding row.

column subbraction: take smallest element from each column & subbract this from all the elements of the corresponding column.

After Row Subtraction

$$\begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

After column subbraction

$$\begin{bmatrix} \cancel{0} & 13 & 49 & 0 & 0 \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix}$$

Step 2: Assignment
Row Scan
1st row - 2 zeros - skip
2nd row - 1 zero - Assign zero & strike the 0's in that col.
3rd row - 1 zero - Assign zero & strike the 0's in that col.
4th row - 1 zero - Assign it & strice of the 0's in that col

col scan
4th col - 1 zero - Assign it & strike of the 0's in that row

$$\begin{bmatrix} \cancel{0} & 13 & 49 & (0) & \cancel{0} \\ (0) & 35 & 29 & 5 & 10 \\ 13 & (0) & 63 & 7 & 7 \\ 47 & 15 & (0) & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix}$$

All zeros are either assigned or crossed

Step 3: Since row 5 & col 5 has no assigments, the current soln. not optimal

Step 4: Cover zeros by drawing a minimum no. of straight lines
a) Mark (✓) row 5 with no assignments
b) Mark (✓) column 2 which has a zeros in marked row
c) Mark (✓) row 3, that have assignments in marked column.
d) No zeros in row 3. apart from assignment. So stop
e) Draw lines through unmarked rows & marked columns

$$\begin{array}{ccccc|c} \cancel{0} & 13 & 49 & (0) & \cancel{0} & \\ (0) & 35 & 29 & 5 & 10 & \\ 13 & (0) & 63 & 7 & 7 & ✓ \\ 47 & 15 & (0) & 20 & 2 & \\ 25 & \cancel{0} & 46 & 9 & 4 & ✓ \\ & ✓ & & & & \end{array}$$

Step 5: Here 4 is the smallest element not covered by these straight lines. Subtract this 4 from all the uncovered elements and add this 4 to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on these straight lines.

$$\begin{bmatrix} \cancel{0} & \overset{+4}{13} & 49 & (0) & \cancel{0} \\ (0) & \overset{+4}{35} & 29 & 5 & 10 \\ 13-4 & (0) & 63-4 & 7-4 & 7-4 \\ 47 & \underset{+4}{15} & (0) & 20 & 2 \\ 25-4 & \cancel{0} & 46-4 & 9-4 & 4-4 \end{bmatrix} \begin{matrix} \\ \\ \checkmark \\ \\ \checkmark \end{matrix} \Longrightarrow \begin{bmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{bmatrix}$$

$$\begin{matrix} & \checkmark & & & \end{matrix}$$

Repeat step 2 – Assignment.

Row Scanning

1) 1st row – 3 zeros
2nd row – 1 zero – Assign it & cross the 0's in col
3rd row – 1 zero – Assign it & cross 0's in col
4th row – 1 zero – Assign it & cross 0's in column
5th row – 1 zero unassigned Assign it, cross 0's in column

$$\begin{bmatrix} \cancel{0} & 17 & 49 & 0 & \cancel{0} \\ (0) & 39 & 29 & 5 & 10 \\ 9 & (0) & 59 & 3 & 3 \\ 47 & 19 & (0) & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & (0) \end{bmatrix}$$

Column Scanning

Column 4 – 1 zero – Assign it & cross the 0's in row.

$$\begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & \cancel{0} & 17 & 49 & (0) & \cancel{0} \\ J_2 & (0) & 39 & 29 & 5 & 10 \\ J_3 & 9 & (0) & 59 & 3 & 3 \\ J_4 & 47 & 19 & (0) & 20 & 2 \\ J_5 & 21 & \cancel{0} & 42 & 5 & (0) \end{array}$$

In the above matrix, each row & column contains exactly one assignment; therefore the current assignment is optimal.

Optimum Schedule . J_1-M_4 J_2-M_1 J_3-M_2 J_4-M_3 J_5-M_5

cost from original table = 11 + 43 + 28 + 27 + 25 hours
= 134 hours.

3. Four different jobs can be done on four different machines. The set up and down time costs are assumed to be prohibitively high for change overs. The matrix below gives the cost in rupees of processing job i on machine j

Machines

$$\begin{array}{c|cccc} & M_1 & M_2 & M_3 & M_4 \\ \hline J_1 & 5 & 7 & 11 & 6 \\ J_2 & 8 & 5 & 9 & 6 \\ J_3 & 4 & 7 & 10 & 7 \\ J_4 & 10 & 4 & 8 & 3 \end{array}$$

How should the jobs be assigned to various machines so that total cost is minimized?

Solution

No: of rows = No. of columns = 4 ∴ Problem is balanced.

Step 1 : Row Subtraction, column subtraction

$$\begin{bmatrix} 5 & 7 & 11 & 6 \\ 8 & 5 & 9 & 6 \\ 4 & 7 & 10 & 7 \\ 10 & 4 & 8 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix}$$

Step 2 : Assignment

Row Scanning

1) 1st row - only 1 zero Assign & cross 0's in col
2) 2nd row - 2 zeros.
3) 3rd row - No zeros
4) 4th row - 1 zero - Assign it.

$$\begin{bmatrix} (0) & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & (0) \end{bmatrix}$$

Column Scanning

i) 2nd col - 1 zero - Assign it & cross zero in row

$$\begin{bmatrix} (0) & 2 & 2 & 1 \\ 3 & (0) & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & (0) \end{bmatrix}$$

Since there are some rows & columns without assignment the current assignment not optimal.

Step 3: Draw Minimum number of lines.

$$\begin{array}{cccc|c} (0) & 2 & 2 & 1 & \checkmark \\ 3 & (0) & \cancel{0} & 1 & \\ \cancel{0} & 3 & 2 & 3 & \checkmark \\ 7 & 1 & 1 & (0) & \\ \checkmark & & & & \end{array}$$

1) Mark row 3 - unassigned rows
2) Mark col 1, which contains zero in 3rd row.
4) Mark assigned row in col. 1 (ie) Row 1 contains assigned zero. No more zeros in Row 1
5) Now draw lines through all unmarked rows & marked columns

Here smallest element is 1.
Subtract this 1 from all the uncovered elements
Add this 1 to those elements which lie in the intersection
Do not change the remaining elements.

$$\begin{array}{l}\left[\begin{array}{cccc} 0 & 2-1 & 2-1 & 1-1 \\ 3^{+1} & 0 & 0 & 1 \\ 0 & 3-1 & 2-1 & 3-1 \\ 7^{+1} & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \checkmark \\ \\ \checkmark \\ \\ \end{array} \\ \ \checkmark \end{array} \Rightarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 8 & 1 & 1 & 0 \end{bmatrix}$$

Iteration 2 : Now make assignment.

$$\begin{bmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & 0 & 0 & 1 \\ (0) & 2 & 1 & 2 \\ 8 & 1 & 1 & (0) \end{bmatrix}$$

Row scanning

1) 1st row - 2 zeros
2) 2nd row - 2 zeros
3) 3rd row - 1 zero
Assign & cross 0 in col.
4) 4th row. 1 zero - Assign & cross 0 in col

Column scanning

1) 2nd column - 1 zero - assign & cross zero in row

$$\begin{bmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & (0) & \cancel{0} & 1 \\ (0) & 2 & 1 & 2 \\ 8 & 1 & 1 & (0) \end{bmatrix}$$

Since there are rows & columns without assignments, the solution not optimal.

$$\left[\begin{array}{cccc} \cancel{0} & 1-1 & 1 & \cancel{0} \\ 4 & (0) & \cancel{0} & \\ (0) & 2 & 1 & 2 \\ 8 & 1 & 1 & (0) \end{array}\right] \begin{array}{c} \checkmark \\ \\ \checkmark \\ \checkmark \end{array} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{bmatrix}$$
$$\checkmark \qquad\qquad \checkmark$$

Draw Minimum No. of lines
1) Mark 1st row with no assignments
2) Mark col 1, col 4 which has 0's in row 1
3) In col 1, marked zero is in row (3). So mark row 3
4) In col 4, marked zero is in row (4) So mark row 4.
5) Draw lines through unmarked rows & marked columns.
6) Here smallest element is 1 among uncovered element
Subtract 1 from uncovered element
Add 1 to those element in intersection
Do not change remaining element

Iteration 3 - Assignment

$$\begin{array}{c|cccc} & M_1 & M_2 & M_3 & M_4 \\ \hline J_1 & (0) & \cancel{0} & \cancel{0} & \cancel{0} \\ J_2 & 5 & (0) & \cancel{0} & 2 \\ J_3 & 0 & 1 & (0) & 2 \\ J_4 & 8 & \cancel{0} & \cancel{0} & (0) \end{array}$$

As all rows & columns contains more than 1 zero, choose one zero arbitrarily & assign.
1) Take row 2, col 2 - zero & assign
2) choose row 3, col 3 zero & assign
3) In row 4, 1 zero - assign it
4) In row 1 - 1 zero assign it

Now, All rows & columns have one assignments. So solution is optimal.

optimal Assignment J1 - M1 J2 - M2 J3 - M3 J4 - M4
cost = 5 + 5 + 10 + 3 = 23 Rupees.

Unbalanced Assignment Problem

If the number of rows is not equal to the number of columns in the cost matrix of the given assignment problem, then the problem is said to be unbalanced.

Convert the unbalanced into a ~~b~~balanced one by adding dummy rows or dummy columns with zero cost elements in the cost matrix depend~~en~~ing upon whether m<n or m>n.

Example

1. A batch of 4 jobs can be assigned to 5 different machines. The setup time (in hours) for each job on various machines is given below. find an optimal assignment.

$$\begin{array}{cc|ccccc|} & & 1 & 2 & 3 & 4 & 5 \\ & 1 & 10 & 11 & 4 & 2 & 8 \\ \text{Job} & 2 & 7 & 11 & 10 & 14 & 12 \\ & 3 & 5 & 6 & 9 & 12 & 14 \\ & 4 & 13 & 15 & 11 & 10 & 7 \end{array}$$

Solution: No of rows = 4 No of columns = 5 Not equal. Unbalanced

Add a dummy row with zero cost

Step 1:

Row subtraction column subtraction

$$\begin{pmatrix} 10 & 11 & 4 & 2 & 8 \\ 7 & 11 & 10 & 14 & 12 \\ 5 & 6 & 9 & 12 & 14 \\ 13 & 15 & 11 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2

Assignment

$$\begin{pmatrix} 8 & 9 & 2 & (0) & 6 \\ (0) & 4 & 3 & 7 & 5 \\ \cancel{0} & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & (0) \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

Row Scanning

1) 1st row - 1 zero - assign it & cross 0's in col
2) 2nd row = 1 zero - assign it & cross 0's in col
3) 3rd row - No zero
4) 4th row - One zero - assign it & cross 0's in col
5) 5th row - 2 zeros.

column scanning

1) 2nd col - 1 zero - assign it & cross 0's in its row

Since there are some rows & columns with no assignments (ie. 3rd row, 3rd column), the current assignment is not optimal.

Step 3:
Cover all the zero's by drawing minimum number of straight lines

a) unassigned row is 3. Mark row 3
b) Mark col 1 which contains 0 in row 3
c) Mark row 2 which contains assigned 0 in column 1
No more zeros in row 2.
Draw lines through unmarked row and marked column

$$\begin{array}{c}\left(\begin{array}{ccccc} 8 & 9 & 2 & (0) & 6 \\ (0) & 4 & 3 & 7 & 5 \\ \xcancel{0} & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & (0) \\ \xcancel{0} & (0) & \xcancel{0} & \xcancel{0} & \xcancel{0} \end{array}\right)\begin{array}{c} \\ ✓ \\ ✓ \\ \\ \\ \end{array} \\ \begin{array}{ccccc} ✓ & & & & \end{array}\end{array}$$

Step 4: Here 1 is the smallest element not covered by lines
→ Add 1 to the elements in intersection
→ Sub. 1 from all uncovered elements.
→ Do not change remaining elements which lie on the straight li

$$\left(\begin{array}{ccccc} 8^{+1} & 9 & 2 & (0) & 6 \\ 0 & 4-1 & 3-1 & 7-1 & 5-1 \\ 0 & 1-1 & 4-1 & 7-1 & 9-1 \\ 6^{+1} & 8 & 4 & 3 & 0 \\ 0^{+1} & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccccc} 9 & 9 & 2 & 0 & 6 \\ 0 & 3 & 2 & 6 & 4 \\ 0 & 0 & 3 & 6 & 8 \\ 7 & 8 & 4 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now make assignments

$$\begin{array}{cc|ccccc} & & & & \text{M/C} & & \\ & & 1 & 2 & 3 & 4 & 5 \\ \hline & 1 & 9 & 9 & 2 & (0) & 6 \\ & 2 & (0) & 3 & 2 & 6 & 4 \\ \text{Job} & 3 & \xcancel{0} & (0) & 3 & 6 & 8 \\ & 4 & 7 & 8 & 4 & 3 & (0) \\ & 5 & 1 & \xcancel{0} & (0) & \xcancel{0} & \xcancel{0} \end{array}$$

Row Scanning
1) 1st row - 1 zero - assign & cross 0 in col
1) 2nd row - 1 zero - assign & cross 0 in col.
2) 3rd row - 1 zero - assign & cross 0 in col
4) 4th row = 1 zero - assign & cross 0 in co
5) 5th row = 1 zero - assign & cross 0 in a

Since each row & each column contains exactly one assignment the current assignment is optimal.

optimum schedule Job1 - M/C 4, Job2 - M/C 1, Job3 - M/C 2, Job4 - M/C 5
Job5 - M/C 3

Optimum time = 2 + 7 + 6 + 7 + 0
from orginal table = 22 hours.

Maximization Case in Assignment Problem

Maximization problem has to be converted to minimization problem & solved using Hungarian Method.

To convert maximization to Minimization,

Subtract all the cost elements Cij of the cost matrix from the highest cost element in that cost matrix.

Example: A company has a team of four salesman & four districts to start its business. After taking into the capabilities of salesman & nature of districts, the company estimates the profit per day in rupees. Find the assignment of salesman to various districts which will yield maximum profit.

$$\text{Salesman}\quad \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline A & 16 & 10 & 14 & 11 \\ B & 14 & 11 & 15 & 15 \\ C & 15 & 15 & 13 & 12 \\ D & 13 & 12 & 14 & 15 \end{array}$$

Solution

No of rows = No. of columns = 4 Balanced.

Step 1: convert maximization to minimization, Sub. all elements from largest ele. 16

Maxi

$$\begin{pmatrix} 16 & 10 & 14 & 11 \\ 14 & 11 & 15 & 15 \\ 15 & 15 & 13 & 12 \\ 13 & 12 & 14 & 15 \end{pmatrix} \Longrightarrow \overset{\text{Minimization matrix}}{\begin{pmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}}$$

Step 2: Row subtraction

$$\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix} \Longrightarrow \overset{\text{column subtraction}}{\begin{pmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{pmatrix}}$$

Step 3: Assignment ⇒

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline A & (0) & 6 & 2 & 5 \\ B & 1 & 4 & (0) & \cancel{0} \\ C & \cancel{0} & (0) & 2 & 3 \\ D & 2 & 3 & 1 & (0) \end{array}$$

Row scanning

1st row = 1 zero assign & cross 0 in col

2nd row = 2 zeros

3rd row = 1 zero – assign & cross 0 in col

4th row = 1 zero – assign & cross 0 in col

Column scanning

3rd col = 1 zero – assign & cross 0 in row

Since each row & each columns contains exactly 1 zero, soln is optimal

original

$$\begin{pmatrix} \underline{16} & 10 & 14 & 11 \\ 14 & 11 & \underline{15} & 15 \\ 15 & \underline{15} & 13 & 12 \\ 13 & 12 & 14 & \underline{15} \end{pmatrix}$$

optimum schedule

A – 1 B – 3 C – 2 D – 4

Profit = 16 + 15 + 15 + 15 = Rs. 61/-

Restrictions in Assignment

In some cases, it may not be possible to assign a particular task to particular facility due to space, size of the task, process capability of the facility or other restrictions. This can be overcome by assigning a very high processing time or cost element (can be ∞) to the corresponding cell.

Example

A machine shop purchased a drilling machine and two lathes of different capacities. The positioning of the machines among 4 possible location on the shop floor. Determine the optimal location of the machines.

$$\begin{array}{c|cccc} & \multicolumn{4}{c}{\text{Location}} \\ & 1 & 2 & 3 & 4 \\ \hline \text{Lathe 1} & 12 & 9 & 12 & 9 \\ \text{Drilling} & 15 & \text{not suitable} & 13 & 20 \\ \text{Lathe 2} & 4 & 8 & 10 & 6 \end{array}$$

Solution: Since drilling m/c is not suitable for location 2, the corresponding cost element should be taken as ∞

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \end{pmatrix}$$

The No. of rows = 3
No. of col = 4
Unbalanced, so add a new row with zero cost

$$\begin{pmatrix} 12 & 9 & 12 & 9 \\ 15 & \infty & 13 & 20 \\ 4 & 8 & 10 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 1: Row subtraction, column subtraction.

$$\begin{pmatrix} 3 & 0 & 3 & 0 \\ 2 & \infty & 0 & 7 \\ 0 & 4 & 6 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2: Assignment:

$$\begin{pmatrix} 3 & (0) & 3 & \cancel{0} \\ 2 & \infty & (0) & 7 \\ (0) & 4 & 6 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & (0) \end{pmatrix}$$

Row scanning

2nd row – 1 zero – assign it & cross 0 in col
3rd row – 1 zero assig it & cross 0 in col.
In row 1 & row 4 = there are 2 zeros in same position (col 2 & col 4). In this case choose zero's diagonally $\rightarrow \begin{pmatrix} (0) & \cancel{0} \\ \cancel{0} & (0) \end{pmatrix}$

So assign col 2 zero in row 1 / col 4 zero in row 4 or col 4 in row 1 / col 2 in row 4

Since each row & each column contains atleast one zero, solution is optimal.

optimal assignment is

Lathe 1 → Location 2, Drill → Location 3, Lathe 3 → Location
Minimum cost = 9 + 13 + 4 = 26 units

Alternative solution exists.

$$\begin{pmatrix} 3 & \cancel{0} & 3 & (0) \\ 2 & \infty & (0) & 7 \\ (0) & 4 & 6 & 2 \\ \cancel{0} & (0) & \cancel{0} & \cancel{0} \end{pmatrix}$$

Lathe 1 → Location 4
Drill → Location 3
Lathe 2 → Location 1
Min cost = 9 + 13 + 4 = 26 units

Travelling Salesman Problem.

A salesman normally must visit a number of cities starting from the head quarters. The distance between every pair of cities are assumed to be known.

The problem of finding the shortest distance (or minimum cost or minimum time) if the salesman starts from his headquaters and passes through each city exactly once & returns to the headquaters is called Travelling Salesman Problem or A Travelling Salesperson problem.

Some of the additional constraints w.r.t assignment problem are

a) A salesman should go through every city exactly once except the starting city.

b) The salesman starting from one city and comes back to that city

c) Obviously going from any city to the same city directly is not allowed (ie. no assignments should be made along the diagonal line).

Conditions (a) & (b) are called Route conditions

Necessary steps to solve a TSP

i) Assigning an infinitely large element (∞) in each of the squares along the diagonal line in the cost matrix.

ii) Solving the problem as a routine assignment problem.

iii) check if the route conditions are satisfied.

iv) If not, making adjustments in assignments to satisfy the condition with minimum increase in total cost.

1. Solve the following Travelling salesman problem.

$$\text{From}\quad \begin{array}{c|cccc} & \multicolumn{4}{c}{\text{To}} \\ & A & B & C & D \\ \hline A & - & 46 & 16 & 40 \\ B & 41 & - & 50 & 40 \\ C & 82 & 32 & - & 60 \\ D & 40 & 40 & 36 & - \end{array}$$

Solution: Problem balanced. put ∞ to diagonal elements because a person can't go directly from 1 city to same city.

$$\begin{pmatrix} \infty & 46 & 16 & 40 \\ 41 & \infty & 50 & 40 \\ 82 & 32 & \infty & 60 \\ 40 & 40 & 36 & \infty \end{pmatrix}$$

Now solve using assignment problem.

Step 1: Row subtraction

$$\begin{pmatrix} \infty & 30 & 0 & 24 \\ 1 & \infty & 10 & 0 \\ 50 & 0 & \infty & 28 \\ 4 & 4 & 0 & \infty \end{pmatrix} \Longrightarrow$$

column subtraction

$$\begin{pmatrix} \infty & 30 & 0 & 24 \\ 0 & \infty & 10 & 0 \\ 49 & 0 & \infty & 28 \\ 3 & 4 & 0 & \infty \end{pmatrix}$$

Step 2: Make assignments.

$$\begin{pmatrix} \infty & 30 & (0) & 24 \\ (0) & \infty & 10 & \cancel{0} \\ 49 & (0) & \infty & 28 \\ 3 & 4 & \cancel{0} & \infty \end{pmatrix}$$

✓ Since Some rows & columns are without assignments, solution not optimal.

Draw min. no. of lines covering zeros.

Smallest element is 3 among uncovered elements

→ sub. 3 from uncovered elements

→ Add 3 to elements in intersection

→ Do not change remaining elements in line

we have,

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & \infty & 27 & (0) & 21 \\ B & \cancel{0} & \infty & 13 & (0) \\ C & 49 & (0) & \infty & 28 \\ D & (0) & 1 & \cancel{0} & \infty \end{array}$$

Now make assignments

Since each row & each column contains exactly one zero, the current soln is optimal for assignment.

But for Travelling salesman, the route conditions must get satisfied. (ie) If I start from city A, I should end up with city A by visiting all cities only once.

The route condition checking. Starting city is A

From A to C, from C to B, from B to D, from D to A

∴ A → C → B → D → A.

It can be shown that, the route starts from A & ends at A & all cities B, C, D are visited once

Route condition satisfied.

Minimum cost = 16 + 32 + 40 + 40 = 128/- units of cost.

2. Solve the following travelling salesman problem to minimize the cost per cycle.

To

		A	B	C	D	E
	A	–	3	6	2	3
	B	3	–	5	2	3
from	C	6	5	–	6	4
	D	2	2	6	–	6
	E	3	3	4	6	–

cost Matrix is

$$\begin{pmatrix} \infty & 3 & 6 & 2 & 3 \\ 3 & \infty & 5 & 2 & 3 \\ 6 & 5 & \infty & 6 & 4 \\ 2 & 2 & 6 & \infty & 6 \\ 3 & 3 & 4 & 6 & \infty \end{pmatrix}$$

Problem is balanced.

Step 1: Row Subtraction

$$\begin{pmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 3 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{pmatrix} \Longrightarrow$$

Column subtraction

$$\begin{pmatrix} \infty & 1 & 3 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 3 & \infty \end{pmatrix}$$

Step 2: Make assignments.

$$\begin{pmatrix} \infty & 1 & 3 & (0) & 1 \\ 1 & \infty & 2 & \cancel{0} & 1 \\ 2 & 1 & \infty & 2 & (0) \\ (0) & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & (0) & 3 & \infty \end{pmatrix}$$

Row Scanning
1st row – only 1 zero – assign it & cross 0 in col
3rd row – only 1 zero – " "
Column Scanning
3rd col – only 1 zero – Assign it & cross 0 in row.
Row Scanning : 4th row more than 1 zero
Column Scanning : 1st col – 1 zero – Assign & cross 0 in row

Since some rows and columns are without assignment, the current assignment is not optimal.

Cover all zeros with min. no. of lines. Subtract the smallest uncovered cost element 1 from all uncovered, Add 1 to those elements in intersection & do not change the remaining

$$\begin{bmatrix} \infty & 1-1 & 3-1 & (0) & 1-1 \\ 1-1 & \infty & 2-1 & \cancel{0} & 1-1 \\ 2 & 1 & \infty & 2^{+1} & (0) \\ (0) & \cancel{0} & 3 & \infty^{+1} & 4 \\ \cancel{0} & \cancel{0} & (0) & 3_{+1} & \infty \end{bmatrix} \begin{matrix} ✓ \\ ✓ \\ \\ \\ \\ \end{matrix}$$

✓

$$\Longrightarrow \begin{bmatrix} \infty & 0 & 2 & 0 & 0 \\ 0 & \infty & 1 & 0 & 0 \\ 2 & 1 & \infty & 3 & 0 \\ 0 & 0 & 3 & \infty & 4 \\ 0 & 0 & 0 & 4 & \infty \end{bmatrix}$$

Now make assignments

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & \cancel{0} & 2 & (0) & \cancel{0} \\ B & (0) & \infty & 1 & \cancel{0} & \cancel{0} \\ C & 2 & 1 & \infty & 3 & (0) \\ D & \cancel{0} & (0) & 3 & \infty & 4 \\ E & \cancel{0} & \cancel{0} & (0) & 4 & \infty \end{array}$$

Row
1) 3rd row - only 1 zero

col
2) 3rd col - only 1 zero

Now Row 1, Row 2, Row 4
forms a cycle with zeros

Row 1 ⇒ 2nd, 4th col
Row 2 ⇒ 1st, 4th col
Row 4 ⇒ 1st, 2nd col

So arbitarily assign a zero in Row 1 — zero in 4th col - cross in row,
Now Row 2 - contains 1 zero - cross in
Now Row 4 - contains 1 zero

Since each row and column contains exactly one zero, the current assignment is optimal.

A-D, B-A, C-E, D-B, E-C

(ie) A-D-B-A , C-E-C

cost is = 2+3+4+2+4 = 15)- units of cost.

But this solution does not satisfy the <u>ROUTE CONDITION</u>

A-D-B-A , C-E-C.

It starts at A & ends at A, but does not visit all the 4 cities B, C, D, E. It visits only 2 cities D & B.

Now try to find the next best solution which satisfies the route condition also - The min. cost element next to 0 is 1. So try to bring 1 into solution.

But 1 occurs in two places. Consider all cases seperately until the acceptable solution is reached.

Consider the one in (2,3). Start by assigning 1 in (2,3) instead of zero at (2,1); change the assignment at (5,3) to (5,1)

Assignment Becomes

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & \cancel{0} & 2 & (0) & 0 \\ B & \cancel{0} & \infty & (1) & 0 & 0 \\ C & 2 & 1 & \infty & 3 & (0) \\ D & \cancel{0} & (0) & 3 & \infty & 4 \\ E & (0) & \cancel{0} & \cancel{0} & 4 & \infty \end{array}$$

Now
A — D — B — C — E — A
Route condition satisfied.
Route condition satisfied.
cost = 2+4+4+2+4 = 16 units of cost

Alternative solution: choose the other 1 in (3,2) & assign it instead of (3,5). Because of this change the assignment in (4,2) to (4,1).

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & \cancel{0} & 2 & \cancel{(0)} & (0) \\ B & \cancel{0} & \infty & 1 & (0) & \cancel{0} \\ C & 2 & (1) & \infty & 3 & \cancel{0} \\ D & (0) & \cancel{0} & 3 & \infty & 4 \\ E & \cancel{0} & \cancel{0} & (0) & A & \infty \end{array}$$

Alternative path: A - E - C - B - D - A.
cost = 3 + 4 + 5 + 2 + 2 = 16

CS6704 - Resource Management Techniques

UNIT III

Integer Programming

In linear programming, each decision variable, slack and/or surplus variable is allowed to take any discrete or fractional value. However, there are certain real-life problems in which the fractional value of the decision variables has no significance. For example, it does not make sense to say that 1.5 men will be working on a project or 1.6 machines will be used in a workshop.

The integer solution can be obtained by rounding off the optimum value of the variable to the nearest integer value. This approach can be easy in terms of economy, time and the cost that might be required to derive an integer solution. This solution however, may not satisfy all the given constraints. Secondly, the objective function value so obtained may not be optimal. Can overcome all these difficulties by using Integer programming techniques.

IPP has some important applications. They are Capital budgeting, construction scheduling, plant location and size, routing and shipping schedule, capacity expansion etc.

TYPES OF INTEGER PROGRAMMING PROBLEMS.

i) Pure or all Integer Programming problems in which all decision variables are restricted to integer values.

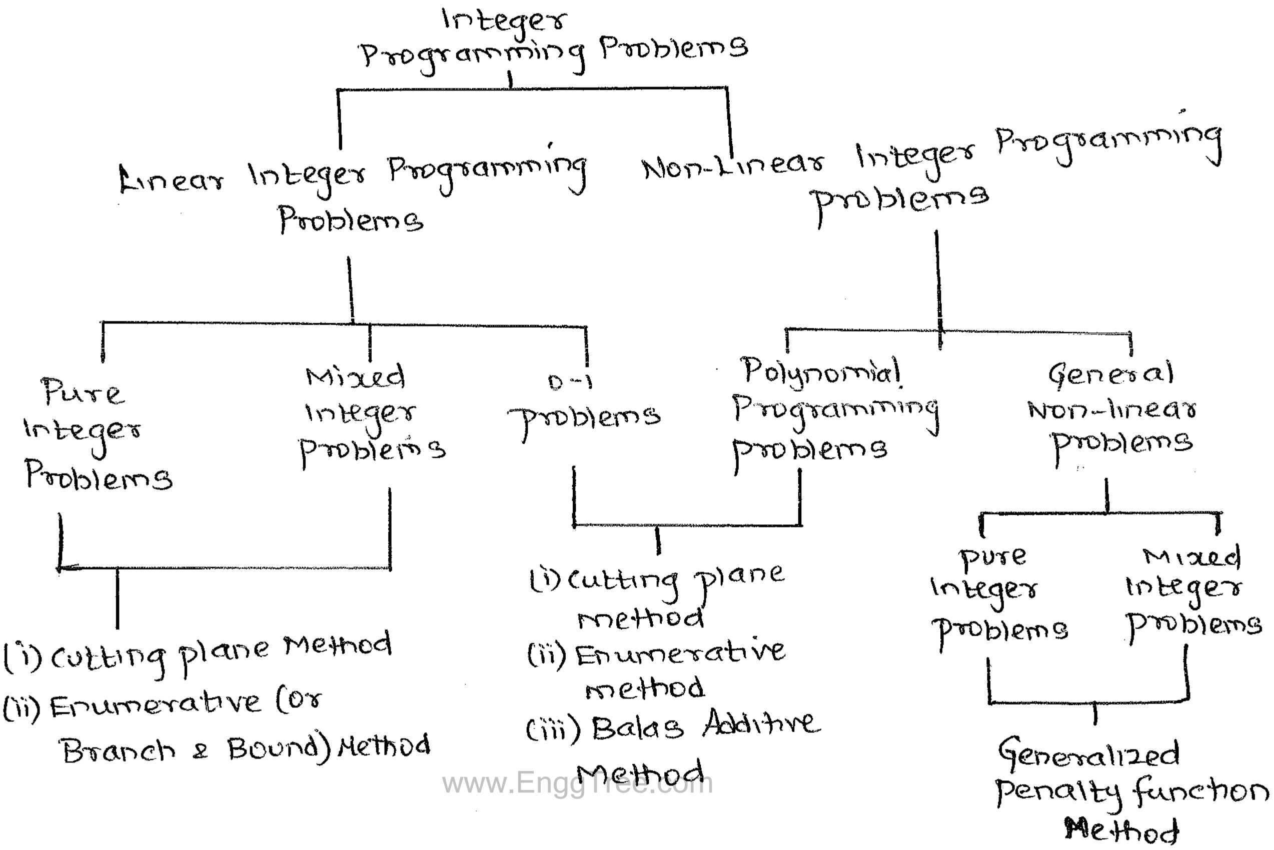
ii) Mixed Integer programming problems in which some but not all of the decision variables are restricted to integer values

iii) Zero-One Integer programming problems in which all decision variables are restricted to integer values of either 0 or 1.

The two methods for solving IPP

(i) Gomory's cutting plane method

(ii) Branch and bound method



Classification of ILP problems & their solution methods

Pure IPP standard form

$$\text{Max } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{Sub. to } a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$$

and $x_1, x_2, \ldots, x_n \geq 0$ and are integers.

Cutting plane solution concept

The cutting plane method to solve integer LP problems was developed by R.E. GOMORY in 1965. This method is based on creating a sequence of linear inequalities called cuts. Such a cut reduces a part of the feasible region of the given LP problem, leaving out a feasible region of the IPP. The hyperplane boundary of a cut is called cutting plane.

Gomory's All Integer Cutting Plane Method

Gomory's Algorithm has the following properties

→ Additional linear constraints never cutoff that portion of the original feasible solution space that contains a feasible integer solution to the original problem.

→ Each new additional constraint (or hyperplane) cuts off the current non-integer optimal solution to the linear programming problem.

Steps for Gomory's All Integer Programming Algorithm

Iterative procedure for the solution of an all integer programming problem is shown below.

Step 1 : Intialization : Formulate the standard Integer LP problem. If there are any non-integer coefficients in the constraint equations, convert them into integer coefficients. Solve the problem by the simplex method, ignoring the integer value requirement of the variables.

Step 2 : Test the optimality :

a) Examine the optimal solution. If all basic variables have integer values, then the integer optimal solution has been obtained and the procedure is terminated.

b) If one or more basic variables with integer value requirement have non-integer solution values, then goto step 3.

Step 3 : Generate Cutting plane : Choose a row r corresponding to a variable x_r that has the largest fractional value f_r and follow the procedure to develop a cut as in $f_r \leq \sum_{j \neq r} f_{rj} x_j$

(or) $\sum_{j \neq r} f_{rj} x_j = f_r + S_g$ (or) $-f_r = S_g - \sum_{j \neq r} f_{rj} x_j$

where $0 \leq f_{rj} \leq 1$ & $0 < f_r < 1$

If there are more than one variable with the largest fraction, then choose the one that has the smallest profit/unit coefficient in the objective fn. of Max. problem

Step 4 : Obtain the new Solution : Add the additional constraint (cut) generated in step 3 to the bottom of the optimal simplex table. Find a new optimal solution by using dual simplex method.

The process is repeated until all basic variables with intege value requirement assume non-negative integer values.

Example 1

1. Solve the following Integer LP problem using Gomory's cutting plane method.

$$\text{Max } z = x_1 + x_2$$

$$\text{Sub. to } 3x_1 + 2x_2 \leq 5$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Initially apply the simplex method.

Convert ineq. to eq. by adding slack variables.

$$\text{Max } z = x_1 + x_2 + 0s_1 + 0s_2$$

$$3x_1 + 2x_2 + s_1 = 5$$

$$x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

Intial Basic feasible solution: $x_1 = x_2 = 0$ (Non basic var)

$s_1 = 5$, $s_2 = 2$ (Basic variables)

c_j 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
0	s_1	③	2	1	0	5	$5/3 \rightarrow$
0	s_2	0	1	0	1	2	-
z_j		0	0	0	0	0	
$c_j - z_j$		1 $\uparrow$	1	0	0		

As $c_j - z_j > 0$ soln not optimal. E.V $= x_1$ L.V $= s_1$

c_j 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
1	x_1	1	2/3	1/3	0	5/3	$\frac{5/3}{2/3} = \frac{5}{2}$
0	s_2	0	1	0	1	2	$2/1 = 2 \rightarrow$
z_j		1	2/3	1/3	0	5/3	
$c_j - z_j$		0	1/3 $\uparrow$	-1/3	0		

As $c_j - z_j > 0$ soln not optimal. E.V $= x_2$ L.V $= s_2$

c_j 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	M.ratio
1	x_1	1	0	1/3	-2/3	1/3	
1	x_2	0	1	0	1	2	
z_j		1	1	1/3	1/3	7/3	
$c_j - z_j$		0	0	-1/3	-1/3		

All $c_j - z_j \leq 0$. Soln is optimal

Max $z = 7/3$ $x_1 = 1/3$ $x_2 = 2$

but x_1 is not an integer.

∴ construct Gomory cut.

Since x_1 is the only basic variable whose value is a non-negative fractional value, therefore consider first row (x_1-row) as source row, to generate Gomory cut

$$\frac{1}{3} = x_1 + 0.x_2 + \frac{1}{3}s_1 - \frac{2}{3}s_2$$

The factoring of numbers shown below

$$\left(0+\frac{1}{3}\right) = (1+0)x_1 + \left(0+\frac{1}{3}\right)s_1 + \left(-1+\frac{1}{3}\right)s_2$$

The non-integer coefficients are factored into integer and fractional parts in such a manner that the fractional part is strictly positive

Rearranging all of the integer coefficients on the left-hand side, we get

$$\frac{1}{3} + (-x_1 + s_2) = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

Since value of variables x_1, s_2 is assumed to be non-negative integer, LHS must satisfy

$$\frac{1}{3} \leq \frac{1}{3}s_1 + \frac{1}{3}s_2$$

Introduce new non-negative integer slack var, sg_1

$$\frac{1}{3} + sg_1 = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

$$\boxed{sg_1 - \frac{1}{3}s_1 - \frac{1}{3}s_2 = \frac{-1}{3}}$$ cut I.

Note :
-ve fractions must be converted to positive.

Example : $-\frac{2}{3}$

$-2/3$ is proper frac

$\therefore -1 + \frac{\text{Den}-\text{num}}{\text{Den}}$

$\Rightarrow -\frac{2}{3} = -1 + \frac{3-2}{3}$

$= \left[-1 + \frac{1}{3}\right]$

Suppose, -ve fraction is improper

Example $\frac{-10}{3}$.

$\frac{-10}{3} = -3 - \frac{1}{3}$

Now take frac part

$-3 - \frac{1}{3} = -3 - 1 + \frac{3-1}{3}$

$\frac{-10}{3} = -4 + 2/3$

Adding this equation (also called Gomory cut) at the bottom of table.

	c_j	1	1	0	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
1	x_1	1	0	1/3	-2/3	0	1/3
1	x_2	0	1	0	10	0	2
0	sg_1	0	0	$\boxed{-1/3}$	-1/3	1	-1/3 $\longrightarrow$
	z_j	1	1	+1/3	+1/3	0	7/2
	$c_j - z_j$	0	0	-1/3	-1/3	0	
	Min ratio	–	–	$\frac{-1/3}{-1/3}=1$	$\frac{-1/3}{-1/3}=1$		

Apply dual simplex method

L.V = sg_1
E.V = s_1

	c_j	1	1	0	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
1	x_1	1	0	0	-1	1	0
1	x_2	0	1	0	1	0	2
0	s_1	0	0	1	1	-3	1
	z_j	1	1	0	0	1	2
	$c_j - z_j$	0	0	0	0	-1	

Here, $c_j - z_j \leq 0$ and all variables are integer optimal soln for ILP is

Max $z = 2$
$x_1 = 0$
$x_2 = 2$

2. Max $z = 3x_1 + 12x_2$

Sub. to $2x_1 + 4x_2 \leq 7$

$5x_1 + 3x_2 \leq 15$

$x_1, x_2 \geq 0$ and integers.

Solve using simplex method by adding slack variables.

$$\text{Max } z = 3x_1 + 12x_2 + 0s_1 + 0s_2$$

Sub to: $2x_1 + 4x_2 + s_1 = 7$

$5x_1 + 3x_2 + s_2 = 15$

$s_1, s_2, x_1, x_2 \geq 0$

Initial basic feasible solution, $x_1 = x_2 = 0$ (Non-basic var)

$s_1 = 7$, $s_2 = 15$ (Basic var).

Initial table.

	Cj	3	12	0	0		
CB	SB	x_1	x_2	S_1	S_2	X_B	Min ratio
0	S_1	2	$\textcircled{4}$	1	0	7	7/4 →
0	S_2	5	3	0	1	15	15/3
	Zj	0	0	0	0	0	
	Cj-Zj	3	12 ↑	0	0		

Cj-Zj > 0, Soln not optimal. E·V = x_2, L·V = S_1

	Cj	3	12	0	0		
CB	SB	x_1	x_2	S_1	S_2	X_B	Min ratio.
12	x_2	1/2	1	1/4	0	7/4	
0	S_2	7/2	0	-3/4	1	37/4	
	Zj	6	12	3	0	21	
	Cj-Zj	-3	0	-3	0		

Cj-Zj ≤ 0. Soln optimal.
But x_2 is non integer

Max $z = 21$ $x_1 = 0$ $x_2 = 7/4$

To obtain the integer valued solution, we proceed to construct Gomory's fractional cut with the help of x_2-row.

$$\frac{7}{4} = \frac{1}{2}x_1 + x_2 + \frac{1}{4}s_1$$

$$\left(1 + \frac{3}{4}\right) = \left(0 + \frac{1}{2}\right)x_1 + (1+0)x_2 + \left(0 + \frac{1}{4}\right)s_1$$

Bring integer coefficients to LHS.

$$\frac{3}{4} + (1 - x_2) = \frac{1}{2}x_1 + \frac{1}{4}s_1$$

$$\frac{3}{4} \leq \frac{1}{2}x_1 + \frac{1}{4}s_1$$ Introduce Gomory slack variable, Sg_1

$$\frac{3}{4} + Sg_1 = \frac{1}{2}x_1 + \frac{1}{4}s_1$$

$$\boxed{Sg_1 - \frac{1}{2}x_1 - \frac{1}{4}s_1 = -\frac{3}{4}}$$ Gomory cut I

		3	12	0	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
12	x_2	1/2	1	1/4	0	0	7/4
0	s_2	7/2	0	-3/4	1	0	39/4
0	sg_1	$\boxed{-1/2}$	0	-1/4	0	1	-3/4 →
Z_j		6	12	3	0	0	21
$C_j - Z_j$		-3	0	-3	0	0	
		$\frac{-3}{-1/2} = 6$ ↑	-	$\frac{-3}{-1/4} = 12$	-	-	

Apply dual simplex method.

L.V = sg_1

E.V = x_1

	C_j	3	12	0	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
12	x_2	0	1	0	0	1	1
0	s_2	0	0	-5/2	1	7	9/2
3	x_1	1	0	1/2	0	-2	3/2
Z_j		3	12	3/2	0	6	33/2
$C_j - Z_j$		0	0	-3/2	0	-6	

$C_j - Z_j \le 0$, but x_1, s_2 both are not integers.

To obtain integer valued solution, we proceed to Gomory's cut

Max$\{9/2, 3/2\}$ = max$\{1/2, 1/2\} = 1/2$

choose arbitarily x_1 row.

<u>x_1 - row.</u> $\frac{3}{2} = x_1 + \frac{1}{2}s_1 - 2sg_1$

$$\left(1+\frac{1}{2}\right) = (1+0)x_1 + \left(0+\frac{1}{2}\right)s_1 + (-2+0)sg_1$$

$$\frac{1}{2} + (1 - x_1 + 2sg_1) = \frac{1}{2}s_1$$

$$\frac{1}{2} \le \frac{1}{2}s_1$$

Introduce sg_2 $\quad \frac{1}{2} + sg_2 = \frac{1}{2}s_1 \Rightarrow \boxed{sg_2 - \frac{1}{2}s_1 = \frac{-1}{2}}$

	C_j	3	12	0	0	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	sg_1	sg_2	X_B
12	x_2	0	1	0	0	1	0	1
0	s_2	0	0	-5/2	1	7	0	9/2
3	x_1	1	0	1/2	0	-2	0	3/2
0	sg_2	0	0	$\boxed{-1/2}$	0	0	1	-1/2
Z_j		3	12	3/2	0	6	0	33/2
$C_j - Z_j$		0	0	-3/2	0	-6	0	
		-	-	$\frac{-3/2}{-1/2} = 3$ ↑		$\frac{-6}{-2} = 3$		

$C_j - Z_j \le 0$ but X_B is -ve

Apply dual simplex method

L.V = sg_2

E.V = s_1

	C_j	3	12	0	0	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	sg_1	sg_2	X_B
12	x_2	0	1	0	0	0	0	1
0	s_2	0	0	0	1	7	-5	7
3	x_1	1	0	0	0	2	1	1
0	s_1	0	0	1	0	0	-2	1
Z_j		3	12	0	0	6	3	15
$C_j - Z_j$		0	0	0	0	-6	-3	

Since all $C_j - Z_j \le 0$, & all variables are integer optimal solution reached

$$\boxed{\begin{array}{c} \text{Max } z = 15 \\ x_1 = 1 \\ x_2 = 1 \end{array}}$$

3. Max $z = x_1 + x_2$

Sub. to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 \geq 0$ and integers.

Intially solve using simplex method by introducing two slack var.

Max $z = x_1 + x_2 + 0s_1 + 0s_2$

Sub. to $2x_1 + 5x_2 + s_1 = 16$

$6x_1 + 5x_2 + s_2 = 30$

$x_1, x_2, s_1, s_2 \geq 0$.

Intial basic solution is $x_1 = x_2 = 0$ (Non-Basic), $s_1 = 16$, $s_2 = 30$ (Basic var)

	c_j	1	1	0	0		
C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
0	s_1	2	5	1	0	16	$16/2 = 8$
0	s_2	6	5	0	1	30	$30/6 = 5 \rightarrow$
	z_j	0	0	0	0	0	
	$c_j - z_j$	1	1	0	0		

$c_j - z_j > 0$, soln. not optimal

E.V $= x_1$

L.V $= s_2$

	c_j	1	1	0	0		
C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
0	s_1	0	$10/3$	1	$-1/3$	6	$9/5 \rightarrow$
1	x_1	1	$5/6$	0	$1/6$	5	6
	z_j	1	$5/6$	0	$1/6$	5	
	$c_j - z_j$	0	$+1/6$	0	$-1/6$		

$c_j - z_j > 0$ soln. not optimal

E.V $= x_2$

L.V $= s_1$

	c_j	1	1	0	0	
C_B	S_B	x_1	x_2	s_1	s_2	X_B
1	x_2	0	1	$3/10$	$-1/10$	$9/5$
1	x_1	1	0	$-1/4$	$1/4$	$7/2$
	z_j	1	1	$1/20$	$3/20$	$53/10$
	$c_j - z_j$	0	0	$-1/20$	$-3/20$	

$c_j - z_j \leq 0$. soln - optimal

Max $z = 53/10$

$x_1 = 7/2$, $x_2 = 9/5$

but x_1, x_2 are not integers.

In order to obtain integer solution, construct a Gomory cut.

Max $\{9/5, 7/2\}$ = fraction part Max $\{4/5, 1/2\} = 4/5$ corresponds to 1st row

$$\frac{9}{5} = x_2 + \frac{3}{10}s_1 - \frac{1}{10}s_2$$

$$1 + \frac{4}{5} = (1+0)x_2 + \left(0 + \frac{3}{10}\right)s_1 + \left(-1 + \frac{9}{10}\right)s_2$$ (Take integers to LHS)

$$\frac{4}{5} + (1 - x_2 + s_2) = \frac{3}{10}s_1 + \frac{9}{10}s_2$$ (Remove integer part by putting inequality sign)

$$\frac{4}{5} \leq \frac{3}{10}s_1 + \frac{9}{10}s_2$$

Introduce sg_1, $\frac{4}{5} + sg_1 = \frac{3}{10}s_1 + \frac{9}{10}s_2$

After rearranging, $\boxed{sg_1 - \frac{3}{10}s_1 - \frac{9}{10}s_2 = \frac{-4}{5}}$

Add this fractional cut constraint at the bottom of optimum simplex table.

	Cj	1	1	0	0	0	
CB	SB	x_1	x_2	s_1	s_2	sg_1	XB
1	x_2	0	1	3/10	-1/10	0	9/5
1	x_1	1	0	-1/4	1/4	0	7/2
0	sg_1	0	0	$\boxed{-3/10}$	-9/10	1	-4/5 →
	zj	1	1	1/20	3/20	0	53/10
	cj-zj	0	0	-1/20	-3/20	0	

$\frac{-1/20}{-3/10} = 1/6 \uparrow$ $\frac{-3/20}{-9/10} = \frac{+1}{6}$

Apply dual simplex method, as XB is negative.

L.V = sg_1

E.V = s_1

	Cj	1	1	0	0	0	
CB	SB	x_1	x_2	s_1	s_2	sg_1	XB
1	x_2	0	1	0	-1	1	1
1	x_1	1	0	0	1	-5/6	25/6
0	s_1	0	0	1	3	-10/3	8/3
	zj	1	1	0	0	1/6	31/6
	Cj-zj	0	0	0	0	-1/6	

Since Cj - zj ≤ 0 & all XB > 0 the current solution is optimal but not integer.

fraction Max $\left\{\frac{25}{6}, \frac{8}{3}\right\}$

Therefore construct Gomory fractional cut. which corresponds to s_1 row.

$\Rightarrow$ fraction max $\left\{\frac{1}{6}, \frac{2}{3}\right\} = \frac{2}{3}$

$$\frac{8}{3} = 0\cdot x_1 + 0\cdot x_2 + s_1 + 3s_2 - \frac{10}{3} sg_1$$

$$2 + \frac{2}{3} = ([illegible] + 0) s_1 + (3+0) s_2 + \left(-4 + \frac{2}{3}\right) sg_1$$

Bring Integer towards LHS.

(Remove integer part)

$$\frac{2}{3} + (2 - s_1 - 3s_2 + 4sg_1) = \frac{2}{3} sg_1$$

$$\frac{2}{3} \leq \frac{2}{3} sg_1 \quad .$$

Introduce new sg_2

$$\frac{2}{3} + sg_2 = \frac{2}{3} sg_1$$

$$\therefore \boxed{sg_2 - \frac{2}{3} sg_1 = \frac{-2}{3}}$$ cut II

	Cj	1	1	0	0	0	0	
CB	SB	x_1	x_2	s_1	s_2	sg_1	sg_2	XB
1	x_2	0	1	0	-1	1	0	1
1	x_1	1	0	0	1	-5/6	0	25/6
0	s_1	0	0	1	3	-10/3	0	8/3
0	sg_2	0	0	0	0	$\boxed{-2/3}$	1	-2/3 →
	zj	1	1	0	0	1/6	0	31/6
	Cj-zj	0	0	0	0	-1/6	0	

$\frac{-1/6}{-2/3} = 1/4 \uparrow$

Cj - zj ≤ 0 but XB is negative. Apply dual simplex method.

L.V = sg_2

E.V = sg_1

	Cj	1	1	0	0	0	0	
CB	SB	x_1	x_2	s_1	s_2	sg_1	sg_2	XB
1	x_2	0	1	0	-1	0	3/2	0
1	x_1	1	0	0	1	0	-5/4	5
0	s_1	0	0	1	3	0	-5	6
0	sg_1	0	0	0	0	1	-3/2	1
	zj	[illegible]	1	0	0	0	1/4	5
	cj-zj	0	0	0	0	0	-1/4	

Since all cj - zj ≤ 0 & all XB ≥ 0 and integers, the current solution is feasible & integer optimal.

Max z = 5

$x_1 = 5 \quad x_2 = 0$

4) Solve the following IPP

$\text{Max } z = x_1 + 2x_2$

Sub. to $2x_2 \le 7$

$x_1 + x_2 \le 7$

$2x_1 \le 11$

$x_1, x_2 \geq 0$ and are integers

Introduce slack to convert inequality to equality.

$\text{Max } z = x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$

$2x_2 + s_1 = 7$

$x_1 + x_2 + s_2 = 7$

$2x_1 + s_3 = 11$

$x_1, x_2, s_1, s_2, s_3 \geq 0$.

Basic feasible solution is $x_1 = x_2 = 0$ (Non-Basic) $s_1 = 7, s_2 = 7, s_3 = 11$ (Basic)

Initial table

C_j 1 2 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	s_3	X_B	Min ratio
0	s_1	0	②	1	0	0	7	7/2 →
0	s_2	1	1	0	1	0	7	7/1
0	s_3	2	0	0	0	1	11	—
	z_j	0	0	0	0	0	0	
	$c_j - z_j$	1	2 ↑	0	0	0		

$c_j - z_j > 0$ Not optimal E.V $= x_2$ L.V $= s_1$

First Iteration

C_j 1 2 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	s_3	X_B	Min ratio
2	x_2	0	1	1/2	0	0	7/2	—
0	s_2	①	0	−1/2	1	0	7/2	7/2 →
0	s_3	2	0	0	0	1	11	11/2
	z_j	0	2	1	0	0	7	
	$c_j - z_j$	1 ↑	0	−1	0	0		

$c_j - z_j > 0$ not optimal. E.V $= x_1$ L.V $= s_2$

Second Iteration

C_j 1 2 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	s_3	X_B
2	x_2	0	1	1/2	0	0	7/2
1	x_1	1	0	−1/2	1	0	7/2
0	s_3	0	0	1	−2	1	4
	z_j	1	2	1/2	1	0	21/2
	$c_j - z_j$	0	0	−1/2	−1	0	

$c_j - z_j \le 0$. ∴ current solution optimal but not integer.

$\text{Max } z = 21/2 \quad x_1 = 7/2 \quad x_2 = 7/2$

To obtain integer soln. construct the gomory cut. Both x_1, x_2 are not int

frac. Max $\{7/2, 7/2\} = \max\{\frac{1}{2}, \frac{1}{2}\}$ choose arbitarily x_1 or x_2

choosing x_2. $\frac{7}{2} = x_2 + \frac{1}{2}s_1 \Rightarrow 3 + \frac{1}{2} = (1+0)x_2 + (0+\frac{1}{2})s_1$

$\frac{1}{2} + (3 - x_2) = 1/2\, s_1$

$\frac{1}{2} \le \frac{1}{2}s_1$ Introduce sg_1. $\frac{1}{2} + sg_1 \le \frac{1}{2}s_1$

Add a new row $\boxed{sg_1 - \frac{1}{2}s_1 = -1/2}$ cut I

C_j 1 2 0 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	s_3	sg_1	X_B
2	x_2	0	1	1/2	0	0	0	7/2
1	x_1	1	0	−1/2	1	0	0	7/2
0	s_3	0	0	1	−2	1	0	4
0	sg_1	0	0	(−1/2)	0	0	1	−1/2 →
	z_j	1	2	1/2	1	0	0	21/2
	$c_j - z_j$	0	0	−1/2	−1	0	0	
		—	—	$\frac{-1/2}{-1/2}$ ↑	—	—	—	

As X_B is −ve, apply dual simplex method

E.V $= s_1$ L.V $= sg_1$

C_j 1 2 0 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	s_3	sg_1	X_B
2	x_2	0	1	0	0	0	1	3
1	x_1	1	0	0	1	0	−1	4
0	s_3	0	0	0	−2	1	2	3
0	s_1	0	0	1	0	0	−2	1
	z_j	1	2	0	1	0	1	10
	$c_j - z_j$	0	0	0	−1	0	−1	

$c_j - z_j \le 0$, X_B is positive & all are integers.

∴ optimal solution is

$$\boxed{\begin{matrix}\text{Max } z = 10 \\ x_1 = 4 \\ x_2 = 3\end{matrix}}$$

Mixed Integer Problem - Gomory's Mixed Integer Method.

Gomory's mixed Integer method is very similar to pure integer method Except the step 3 - Generating Cutting plane.

The special cut called Gomory's Mixed Integer cut or simply mixed-cut where only a subset of variables may assume integer values and the remaining variables (including slack & surplus) remain continuous.

Steps for Gomory's Mixed-Integer programming Algorithm

Step 1 : Initialization: Formulate the standard LPP. Solve it by using simplex method, ignoring integer requirement.

Step 2 : Test of optimality:

a) Examine the optimal solution. If all integer resticted basic variables have integer values, then terminate the procedure. The current optimal solution is the optimal basic feasible solution.

b) If all integer resticted basic variables are not integers, then goto step 3.

Step 3 : Generate Cutting Plane : Choose a row r corresponding to a basic variable x, then has the highest fractional value fr and generate a cutting plane ~~and~~ as explained earlier.

$$sg = -fr + \sum_{j\in R_+} a_{rj}x_j + \left(\frac{fr}{fr-1}\right)\sum_{j\in R_-} a_{rj}x_j \quad \text{where } 0<fr<1$$

Step 4 : Obtain the new solution: Add the cutting plane generated in step 3 to the bottom of the optimal simplex table.

Find a new optimal solution by using the dual simplex method and return to step 2. The process is repeated until all resticted basic variables are integers.

1. Solve the following mixed integer problem:

$$\text{Max } z = x_1 + x_2$$

Sub-to $2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

~~x_1~~, $x_2 \geq 0$, x_1 is non-negative integer.

Introducing slack variables.

$$\text{Max } z = x_1 + x_2 + 0s_1 + 0s_2$$

Sub-to $2x_1 + 5x_2 + s_1 = 16$

$6x_1 + 5x_2 + s_2 = 30$

$x_1, x_2, s_1, s_2 \geq 0$

Basic feasible solution: $x_1 = x_2 = 0$ (Non-basic)

$s_1 = 16$, $s_2 = 30$, (Basic variables).

C_j 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
0	s_1	2	5	1	0	16	16/2
0	s_2	$\circled{6}$	5	0	1	30	30/6 $\rightarrow$
z_j		0	0	0	0	0	
$c_j - z_j$		01 $\uparrow$	1	0	0		

$c_j - z_j > 0$ not optimal E.V = x_1, L.V = s_2

C_j 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
0	s_1	0	$\circled{10/3}$	1	$-1/3$	6	$\frac{6}{\frac{10}{3}} = \frac{9}{5}$ $\rightarrow$
1	x_1	1	5/6	0	1/6	5	$\frac{5}{5/6} = 6$
z_j		1	5/6	0	1/6	5	
$c_j - z_j$		0	1/6 $\uparrow$	0	$-1/6$		

$c_j - z_j > 0$, not optimal E.V = x_2, L.V = s_1

C_j 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B
1	x_2	0	1	3/10	$-1/10$	18/10 = 9/5
1	x_1	1	0	$-1/4$	1/4	7/2
z_j		1	1	1/20	3/20	53/10
$c_j - z_j$		0	0	$-1/20$	$-3/20$	

$c_j - z_j \leq 0$, but x_1 is not an integer.

Max $z = 53/10$, $x_1 = 7/2$ $x_2 = 9/5$

Since x_1 is not an integer, find the Gomory cut.

consider x_1 row.

$$\frac{7}{2} = x_1 \cancel{+} -\frac{1}{4}s_1 + \frac{1}{4}s_2$$

$$\left(3 + \frac{1}{2}\right) = (1+0)x_1 + \left(-\frac{1}{4}\right)s_1 + \left(\frac{1}{4}s_2\right).$$

for negative fractions, multiply by $\left(\frac{f_r}{f_r - 1}\right)$ where f_r is the fraction part of corresponding var.

Here $f_r = 1/2$.

$$\left(3 + \frac{1}{2}\right) = (1+0)x_1 + \left(\frac{1/2}{1/2 - 1}\right)\left(\frac{-1}{4}\right)s_1 + \frac{1}{4}s_2.$$

$$\frac{1}{2} + (3 - x_1) = \frac{1}{4}s_1 + \frac{1}{4}s_2$$

$$\frac{1}{2} \leq \frac{1}{4}s_1 + \frac{1}{4}s_2.$$

Introduce sg_1 (Gomorian slack)

$$\frac{1}{2} + sg_1 = \frac{1}{4}s_1 + \frac{1}{4}s_2 \Longrightarrow \boxed{sg_1 - \frac{1}{4}s_1 - \frac{1}{4}s_2 = \frac{-1}{2}} \text{ cut I}$$

Add this gomonan slack at the bottom of the opthmum table.

C_j 1 1 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
1	x_2	0	1	3/10	-1/10	0	9/5
1	x_1	1	0	-1/4	1/4	0	7/2
0	sg_1	0	0	-1/4	-1/4	1	-1/2 →
z_j		1	1	1/20	3/20	0	53/10
$C_j - z_j$		0	0	-1/20	-3/20	0	

Though $C_j - z_j \le 0$ Since, X_B is negative, Soln. not optimal.

L.V = sg_1 E.V = s_1

— — $\frac{-1/20}{-1/4}$ $\frac{-3/20}{-1/4}$ —

1/5 ↑ 3/5

C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
1	x_2	0	1	0	-2/5	6/5	6/5
1	x_1	1	0	0	1/2	-1	4
0	s_1	0	0	1	1	-4	2
z_j		1	1	0	1/10	1/5	26/5
$C_j - z_j$		0	0	0	-1/10	-1/5	

$C_j - z_j \le 0$, X_B are all +ve & x_1 is an integer.

Current Optimal Soln.

$$\boxed{\begin{array}{l} \text{Max. } z = 26/5 \\ x_1 = 4 \quad x_2 = 6/5 \end{array}}$$

2. Solve the following Mixed Integer Programming problem.

Max $z = x_1 + x_2$

Sub-to $3x_1 + 2x_2 \le 5$

$x_2 \le 2$

and $x_1, x_2 \ge 0$ and x_1 an integer.

Step 1: Solve using Simplex.

Convert to equality. Max $z = x_1 + x_2 + 0s_1 + 0s_2$

$3x_1 + 2x_2 + s_1 = 5$

$x_2 + s_2 = 2$ $x_1, x_2, s_1, s_2 \ge 0$.

Initial basic feasible soln. $x_1 = x_2 = 0$, $s_1 = 5$ $s_2 = 2$

1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
0	s_1	3	2	1	0	5	5/3 →
0	s_2	0	1	0	1	2	—
z_j		0	0	0	0	0	
$C_j - z_j$		1 ↑	1	0	0		

$C_j - z_j > 0$ soln not optimal E.V = x_1 L.V = s_1

C_j 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B	Min ratio
1	x_1	1	2/3	1/3	0	5/3	5/2
0	s_2	0	1	0	1	2	2 →
z_j		1	2/3	1/3	0	5/3	
$C_j - z_j$		0	1/3 ↑	-1/3	0		

$C_j - z_j > 0$, soln not optimal

E.V = x_2 L.V = s_2

cj 1 1 0 0

C_B	S_B	x_1	x_2	s_1	s_2	X_B
1	x_1	1	0	1/3	-2/3	1/3
1	x_2	0	1	0	1	2
	z_j	1	1	1/3	1/3	7/3
	$c_j - z_j$	0	0	-1/3	-1/3	

$c_j - z_j \leq 0$. Soln is optimal

$x_1 = 1/3$ $x_2 = 2$ Max $z = 7/3$

But x_1 is not an integer

From the first row, we have

$$\frac{1}{3} = x_1 + 0x_2 + \frac{1}{3}s_1 - \frac{2}{3}s_2$$

$$\left(0 + \frac{1}{3}\right) = (1+0)x_1 + \left(0 + \frac{1}{3}\right)s_1 + \left(\frac{1/3}{1/3 - 1}\right)\left(\frac{-2}{3}\right)s_1$$

$$\frac{1}{3} + (-x_1) = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

Introduce sg_1 - Gomorian slack.

$$\frac{1}{3} \leq \frac{1}{3}s_1 + \frac{1}{3}s_2$$

$$\frac{1}{3} + sg_1 = \frac{1}{3}s_1 + \frac{1}{3}s_2$$

$$sg_1 - \frac{1}{3}s_1 - \frac{1}{3}s_2 = -\frac{1}{3}$$

Add this Gomorian slack at the bottom of the above optimum table

cj 1 1 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
01	x_1	1	0	1/3	-2/3	0	1/3
1	x_2	0	1	0	1	0	2
0	sg_1	0	0	-1/3	-1/3	1	-1/3 →
	z_j	1	1	1/3	1/3	0	7/3
	$c_j - z_j$	0	0	-1/3	-1/3	0	
		-	-	$\frac{-1/3}{-1/3} = 1$ ↑	$\frac{-1/3}{-1/3} = 1$		

$c_j - z_j \leq 0$ but X_B is negative Apply dual simplex method.

L.V = sg_1

E.V = s_1

cj 1 1 0 0 0

C_B	S_B	x_1	x_2	s_1	s_2	sg_1	X_B
1	x_1	1	0	0	-1	1	0
1	x_2	0	1	0	1	0	2
0	s_1	0	0	1	1	-3	1
	z_j	1	1	0	0	1	2
	$c_j - z_j$	0	0	0	0	-1	

$c_j - z_j \leq 0$ and x_1 is integer

Soln is optimal

Max $z = 2$
$x_1 = 0$
$x_2 = 2$

Branch and Bound

This method is applicable to both pure (all) as well as mixed integer programming problem and involves a continuous version of the problem.

Let the given LPP be

Max $z = CX$

Sub. to $AX \leq b$

$X \geq 0$ and integers.

The given problem is first solved as a continuous LPP by ignoring the integrality condition. If in the optimal solution some of the variables say x_r is not an integer, then x_r^* and x_r^*+1 are consecutive non-negative integer, where $x_r^* < x_r < x_r^*+1$. Hence any feasible integer value of x_r must satisfy one of the two conditions.

$x_r \leq x_r^*$ or $x_r \geq x_r^*+1$

These two conditions are mutually exclusive and both cannot be included in the LPP simultaneously. By adding these two conditions seperately, 2 sub-prob. formed

Sub-Problem 1

Max $z = CX$

Sub. to $AX \leq b$

$x_r \leq x_r^*$

and $X \geq 0$

Sub-Problem 2

Max $z = CX$

Sub. to $AX \leq b$

$x_r \geq x_r^*+1$

and $X \geq 0$

Thus the problem has been branched or partitioned the original problem into two sub-problems. Each of these subproblems is then solved seperately as a LPP.

If any sub-problem yields an optimum integer solution, it is not further branched. But, if any sub-problem yields non-integer solution it is further branched into two sub-problems. This branching process is continued until each problem terminates with either integer optimal solution or there is no evidence of better soln. The integer valued soln among all subproblems which gives the most optimum value of the obj. fn is selected as optimum

Disadvantage

1) This method requires the optimum solution of each subproblem. In large sub problems this could be very tedious job.

Solution:

This can be improved by using the concept of BOUNDING. Whenever the continuous optimum solution of a subproblem yields a value of the objective function lower than that of the best available integer solution (for MAX problem), it is useless to explore the problem any further. This sub-problem is said to be fathomed and is dropped from further consideration.

Thus once a feasible solution (integer soln) is obtained, its associative objective function can be taken as lower bound (for MAX problem) to delete inferior sub-problems.

Note: For minimization problems, the procedure is the same except the upper bounds are used.

1. Solve the following all-integer programming problem using branch and bound method.

Max $z = 2x_1 + 3x_2$

Sub. to $6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_1, x_2 \geq 0$ and integers

Solve using Graphical method.

1) Inequality to equality

$6x_1 + 5x_2 = 25$

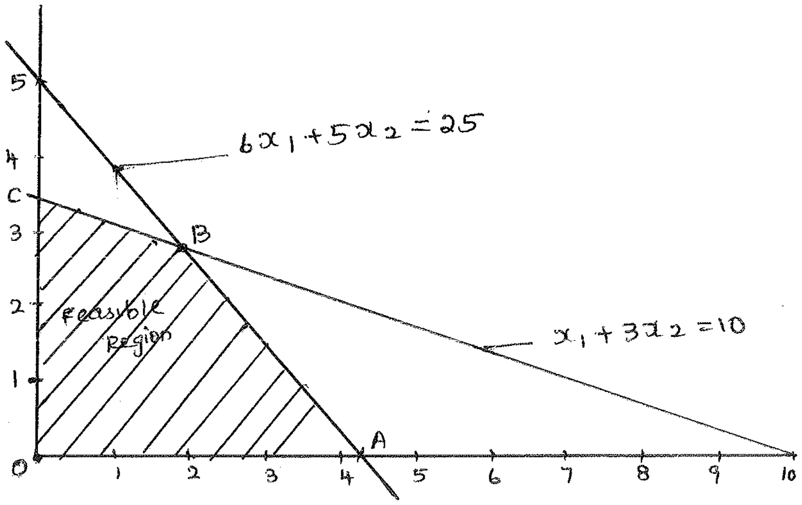
$x_1 = 0$ $x_2 = 25/5 = 5$

$x_2 = 0$ $x_1 = 25/6$

$x_1 + 3x_2 = 10$

$x_1 = 0$ $x_2 = 10/3$

$x_2 = 0$ $x_1 = 10$.



A $x_1 = 25/6$ $x_2 = 0$ $z = 8.33$

B $x_1 = 1.93$ $x_2 = 2.69$ $z = 11.93$

C $x_1 = 0$ $x_2 = 3.33$ $z = 9.99$

Optimal soln. $\boxed{\begin{matrix}\text{Max } z = 11.93 \\ x_1 = 1.93 \quad x_2 = 2.69\end{matrix}}$

pt B
$$6x_1 + 5x_2 = 25$$
$$\underline{-6x_1 \underset{-}{+} 18x_2 = 60}$$
$$13x_2 = 35$$
$$x_2 = 35/13 = 2.69$$

$x_1 + 3(2.69) = 10$

$x_1 = 1.93$

Solution is optimal but x_1 and x_2 are not integers. The value of z represents the initial upper bound $z = 11.93$

Consider the non-integer value of $x_2 = 2.69$ and decompose (branch) into two subproblems by adding the two constraints

$x_2 \leq 2$ & $x_2 \geq 3$. Add this two subproblems.

Subproblem B

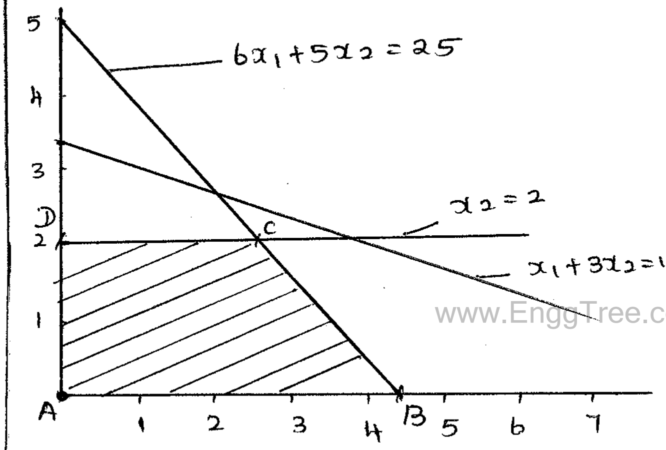
Max $z = 2x_1 + 3x_2$

$6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \leq 2$

$x_1, x_2 \geq 0$



A -(0,0)	z = 0
B -(25/6,0)	z = 8.33
C =(15/6,2)	z = 11
D (0,2)	z = 6

$$\boxed{\begin{array}{l}\text{Max } z = 11 \\ x_1 = 2.5 \quad x_2 = 2\end{array}}$$

Subproblem B contains a non-integer value for x_1. So need to branch further into subproblem D and E.

Another point is that, $z = 11$ of subproblem B is equal to ~~sub~~ lower bound 11.

If z value of subproblem is greater or equal to lower bound it should be branched orelse stopped.

$x_1 \leq 2$ & $x_1 \geq 3$

Subproblem C

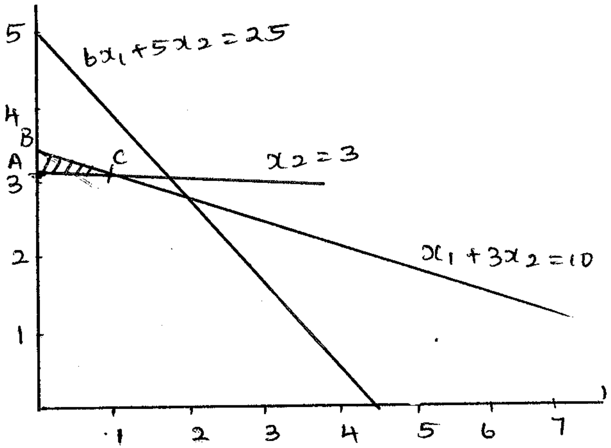
Max $z = 2x_1 + 3x_2$

$6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \geq 3$

$x_1, x_2 \geq 0$.



A (0,3)	z = 9
B (0,3.33)	z = 9.99
C (1 ,3)	z = 11

$$\boxed{\begin{array}{l}\text{Max } z = 11 \\ x_1 = 1 \quad x_2 = 3\end{array}}$$

Subproblem C gives integer solution. No need to branch futher.

Now Max z = 11 becomes new lower bound on the max objective function.

Subproblem D

Max $z = 2x_1 + 3x_2$

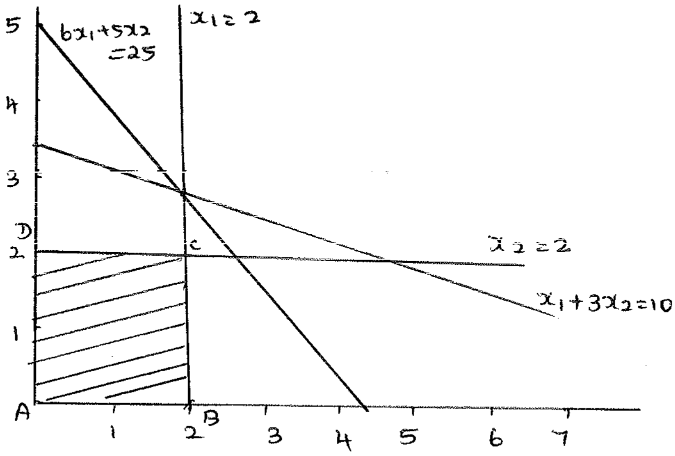
$6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \leq 2$

$x_1 \leq 2$

$x_1, x_2 \geq 0$



A(0,0)	$z = 0$
B(2,0)	$z = 4$
C(2,2)	$z = 10$
D(0,2)	$z = 6$

$\boxed{\begin{array}{l} \text{Max } z = 10 \\ x_1 = 2 \quad x_2 = 2 \end{array}}$

Here the values of x_1, x_2 are integers. No need of further branching. But the $z = 10$ is less than the lower bound $z = 11$

Subproblem E

Max $z = 2x_1 + 3x_2$

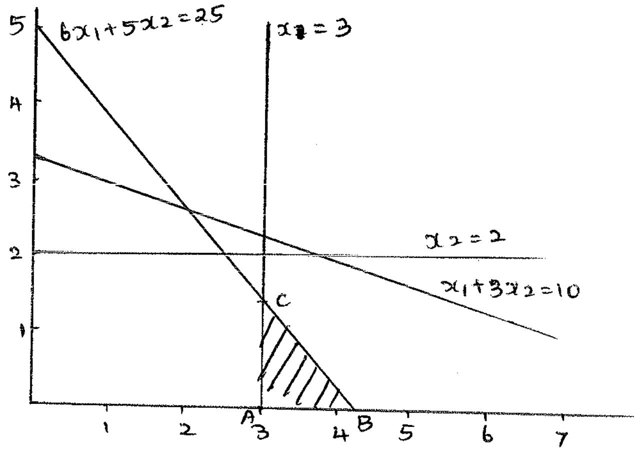
$6x_1 + 5x_2 \leq 25$

$x_1 + 3x_2 \leq 10$

$x_2 \leq 2$

$x_1 \geq 3$

$x_1, x_2 \geq 0$.



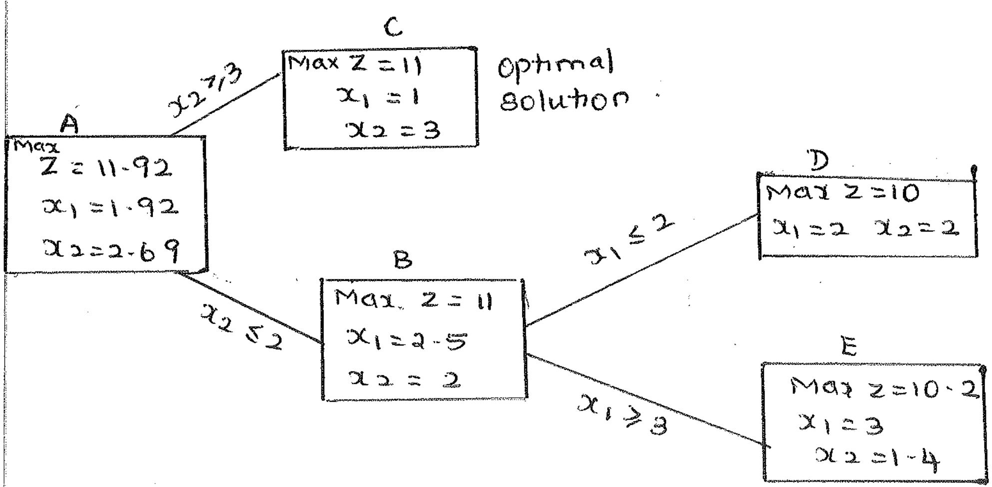
A(3,0)	$z = 6$
B(25/6,0)	$z = 8.33$
C(3,7/5)	$z = 10.2$

$\boxed{\begin{array}{l} \text{Max } z = 10.2 \\ x_1 = 3 \quad x_2 = 1.4 \end{array}}$

x_2 is not an integer. But $z = 10.2$ is less than the lowerbound $z = 11$ we got in subproblem C. No need to branch further.

So the best solution of (integer soln) is from subproblem C

$\boxed{\text{Max } z = 11 \quad x_1 = 1 \quad x_2 = 3}$ optimal Integer solution.



2. Solve using Branch and bound.

Max $z = 2x_1 + 2x_2$

$5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$ and integers.

$5x_1 + 3x_2 = 8$

$5x_1 + 10x_2 = 20$

$-7x_2 = -12$

$x_2 = 12/7$

$x_1 + 2(12/7) = 4$

$x_1 = 4/7$

Solve using Graphical method.

Convert ineq. to equality

$5x_1 + 3x_2 = 8$

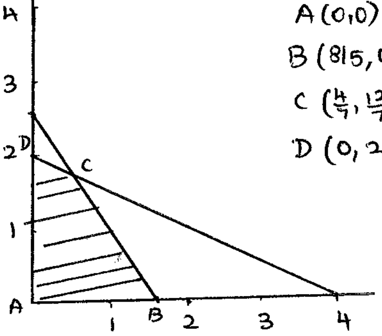
$x_1 = 0 \quad x_2 = 8/3$

$x_2 = 0 \quad x_1 = 8/5$

$x_1 + 2x_2 = 4$

$x_1 = 0 \quad x_2 = 2$

$x_2 = 0 \quad x_1 = 4$



A(0,0) $z = 0$

B(8/5,0) $z = 3.2$

C($\frac{4}{7}, \frac{12}{7}$) $z = 32/7 = 4.57$

D(0,2) $z = 4$

optimal soln is

$$\boxed{\begin{array}{l} \text{Max } z = 32/7 \\ x_1 = 4/7 \quad x_2 = 12/7 \end{array}}$$

In the optimal solution, x_1 and x_2 are not integers. In order to obtain integer solution, we have to branch this problem into two subproblem B and C. with constraints $x_2 \leq 1$, $x_2 \geq 2$

Sub problem B

Max $z = 2x_1 + 2x_2$

$5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_2 \leq 1$

$x_1, x_2 \geq 0$

Sub problem C

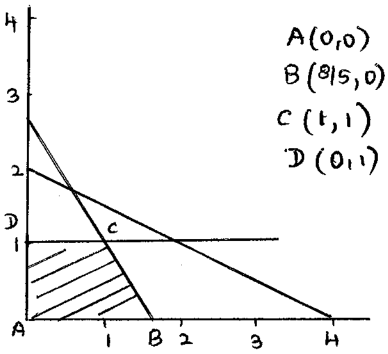
Max $z = 2x_1 + 2x_2$

$5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_2 \geq 2$

$x_1, x_2 \geq 0$.



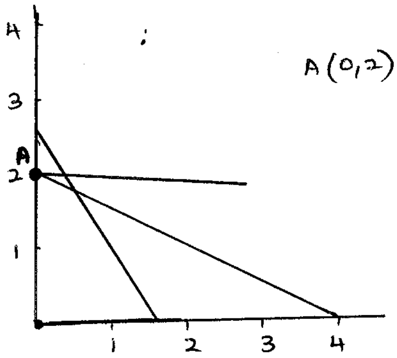
A(0,0) $z = 0$

B(8/5,0) $z = 3.2$

C(1,1) $z = 4$

D(0,1) $z = 3$

$$\boxed{\begin{array}{l} \text{Max } z = 4 \\ x_1 = 1 \\ x_2 = 1 \end{array}}$$



A(0,2) $z = 4$

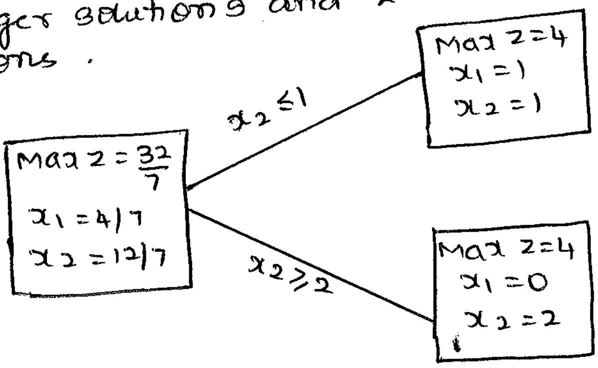
$$\boxed{\begin{array}{l} \text{Max } z = 4 \\ x_1 = 0 \\ x_2 = 2 \end{array}}$$

Sub problem B, produces a integer soln for x_1 & x_2. No need to branch further

Subproblem C also produces integer soln for x_1, x_2. No need to branch further.

Both subprob B, C are integer solutions and z value are same. So there exists two solutions.

$$\boxed{\begin{array}{l} \text{Max } z = 4, x_1 = 1, x_2 = 1 \\ \quad\quad\quad\quad \text{or} \\ \text{Max } z = 4, x_1 = 0, x_2 = 2 \end{array}}$$



3. Use branch and bound technique to solve the following:

Max $Z = x_1 + 4x_2$

$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_1, x_2 \ge 0$ and are integers.

$7 - \frac{20}{14} = 7 - \frac{10}{7} = \frac{49-10}{7}$

Solution: Solve using Graphical method.

Convert inequality to equality

i) $2x_1 + 4x_2 = 7$

$x_1 = 0 \quad x_2 = 7/4$

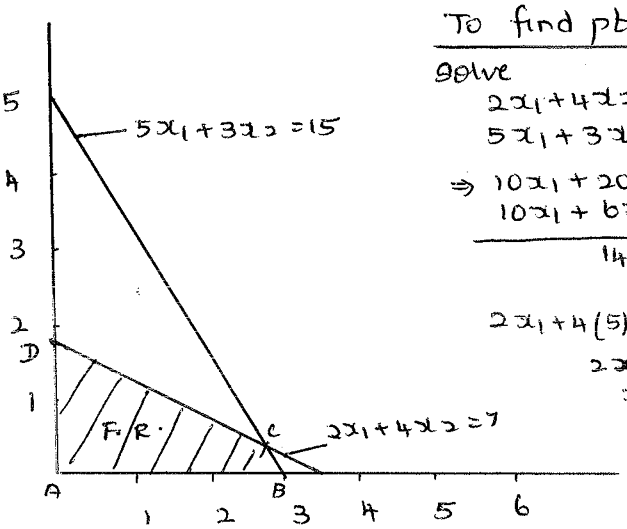
$x_2 = 0 \quad x_1 = 7/2$

2) $5x_1 + 3x_2 = 15$

$x_1 = 0 \quad x_2 = 5$

$x_2 = 0 \quad x_1 = 3$.

A(0,0)	Z = 0
B(3,0)	Z = 3
C $\left(\frac{39}{14}, \frac{5}{14}\right)$	Z = 4.21
D (0, 7/4)	Z = 7



To find pt. C

Solve

$2x_1 + 4x_2 = 7$

$5x_1 + 3x_2 = 15$

$\Rightarrow 10x_1 + 20x_2 = 35$

$10x_1 + 6x_2 = 30$

$14x_2 = 5$

$x_2 = 5/14$

$2x_1 + 4(5/14) = 7$

$2x_1 = 39/7$

$x_1 = 39/14$

The optimum solution to this problem is

Max $z = 7$, $x_1 = 0$. $x_2 = 7/4$

Since the value of x_2 is not an integer, this problem is branched into two subproblems with $x_2 \le 1$ and $x_2 \ge 2$

Subproblem B

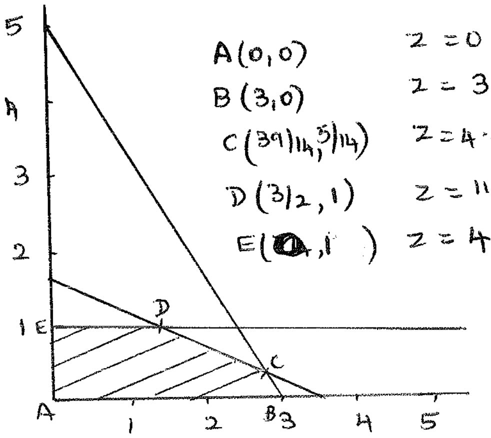
Max $z = x_1 + 4x_2$

$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1, x_2 \ge 0$



A(0,0)	Z = 0
B (3,0)	Z = 3
C(39/14, 5/14)	Z = 4.21
D (3/2, 1)	Z = 11/2
E(0,1)	Z = 4

The optimal solution is

Max $z = 11/2$ $x_1 = 3/2$ $x_2 = 1$

Since x_1 is not an integer it is further branched (prob D, prob E) with $x_1 \le 1$, $x_1 \ge 2$

The new upperbound is $z = 11/2$.

Subproblem C

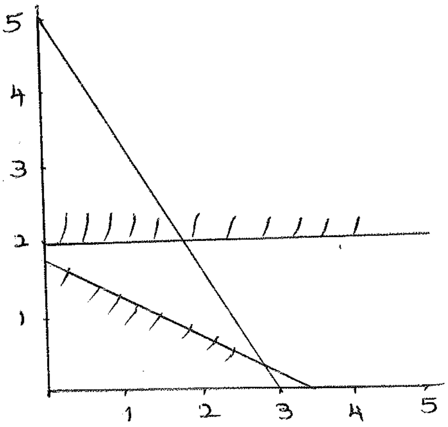
Max $z = x_1 + 4x_2$

$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \ge 2$

$x_1, x_2 \ge 0$.



This problem does not posses a feasible solution.

Infeasible, fathomed

Subproblem B, branched into subproblem D and subprob.E

Subproblem D

Max $z = x_1 + 4x_2$

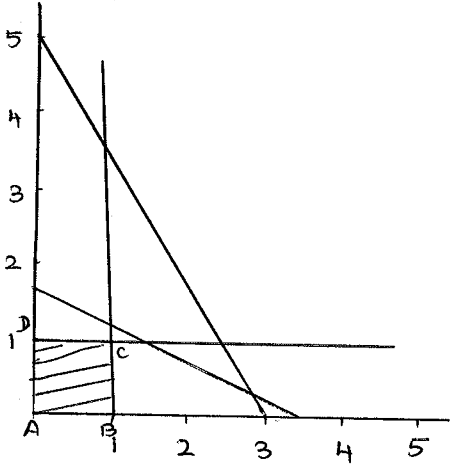
$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \le 1$

$x_1, x_2 \geq 0$



A (0,0)	$z = 0$
B (1,0)	$z = 1$
C (1,1)	$z = 5$
D (0,1)	$z = 4$

The optimal solution is Max $z = 5$, $x_1 = 1$, $x_2 = 1$

As both x_1, & x_2 are integers, no need for further branching.

(Now new upperbound is $z = 5$)

Subproblem E

Max $z = x_1 + 4x_2$

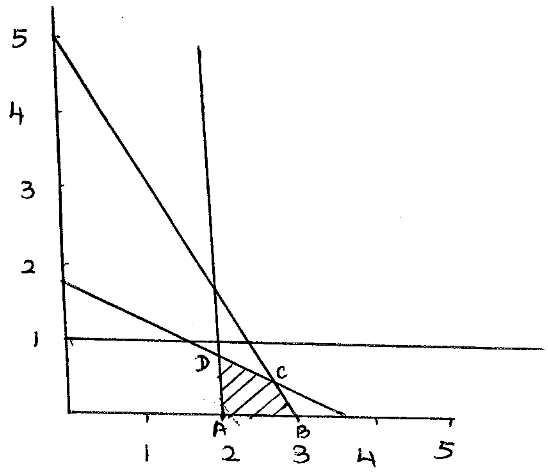
$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \geq 2$

$x_1, x_2 \geq 0$



A (2,0)	$z = 2$
B (3,0)	$z = 3$
C (39/14, 5/14)	$z = 4·21$
D (2, 3/4)	$z = 5$

The optimal solution is Max $z = 5$ $x_1 = 2$ $x_2 = 3/4$

But x_2 is not an integer and the z value is equal to the upperbound $z = 5$, it can be further branched with $x_2 \le 0$ and $x_2 \geq 1$ (Subprob-F & Subprob G)

Subproblem F

Max $z = x_1 + 4x_2$

$2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \geq 2$

$x_2 \le 0$

Subproblem G

Max $z = x_1 + 4x_2$

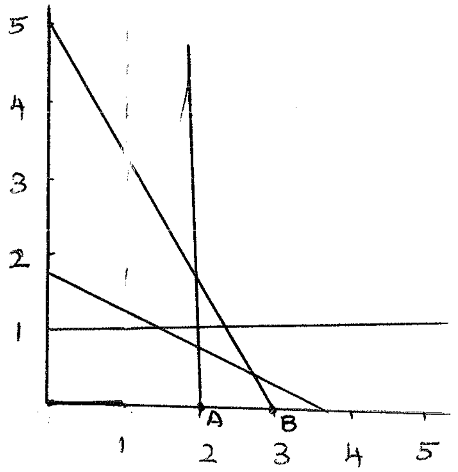
$2x_1 + 4x_2 \le 7$

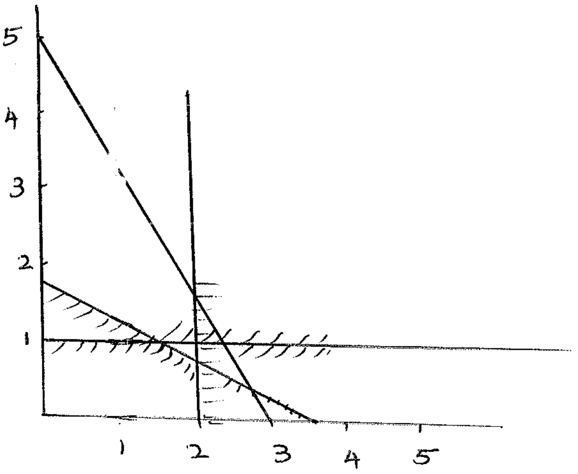
$5x_1 + 3x_2 \le 15$

$x_2 \le 1$

$x_1 \geq 2$

$x_2 \geq 1$





A(2,0) z = 2

B(3,0) z = 3.

Solution is Max z = 3,

$x_1 = 3$ $x_2 = 0$.

x_1 & x_2 are integers.

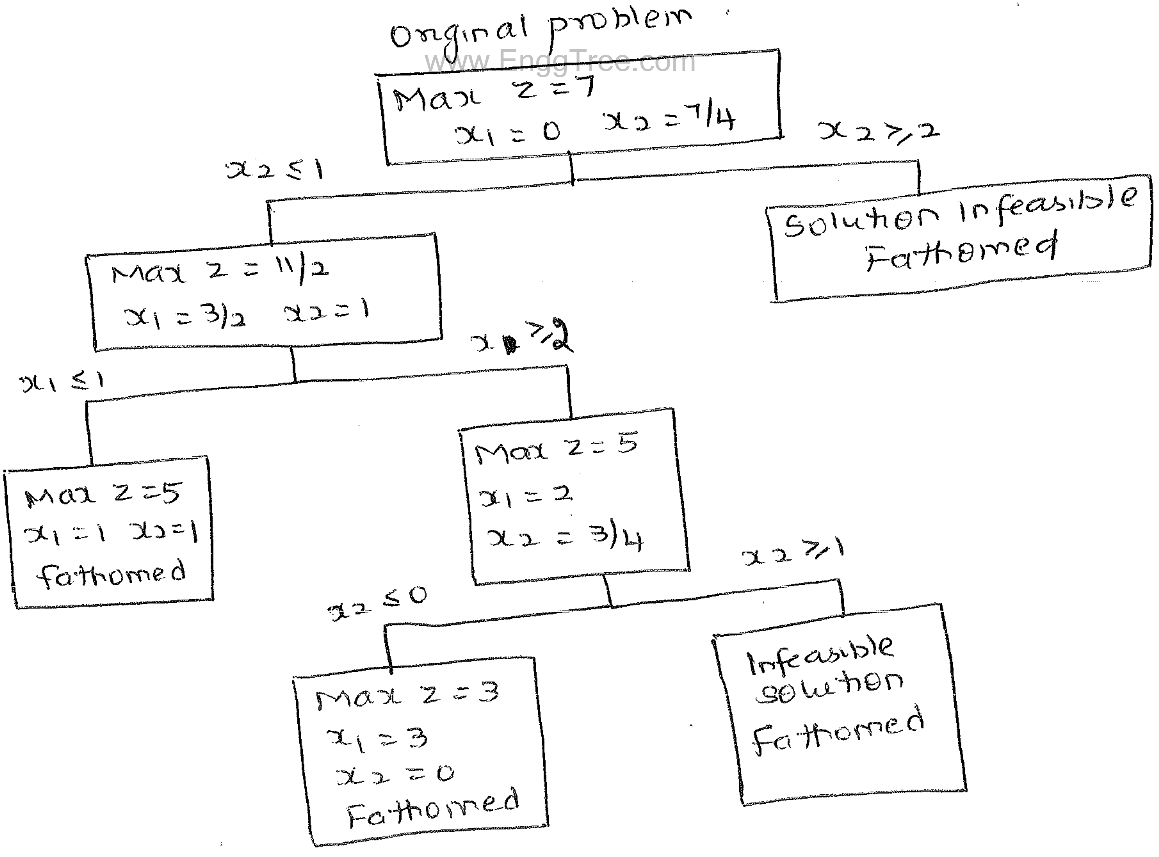
Stop branching.

Solution infeasible.

fathomed.

So, there are two integer solutions with z = 5 and z = 3. Better Max-value is z = 5. So omit the z = 3 solution.

∴ Optimal solution is, $\boxed{\text{Max } z = 5 \quad x_1 = 1 \quad x_2 = 1}$



Best available integer solution is

Max z = 5 $x_1 = 1$ $x_2 = 1$

Dynamic Programming:

The decision making process often involves several decisions to be taken at different times. For example, problems of inventory control, evaluation of investment opportunities, long-term corporate planning, etc. The mathematical technique of optimizing a sequence of interrelated decisions over a period of time is called

Dynamic programming.

The dynamic program, breaks the problem into a series of interrelated decision stages (also called subproblem) where the outcome of a decision at one stage affects the decision at each of the following stages. The word dynamic has been used because time is explicitly taken into consideration.

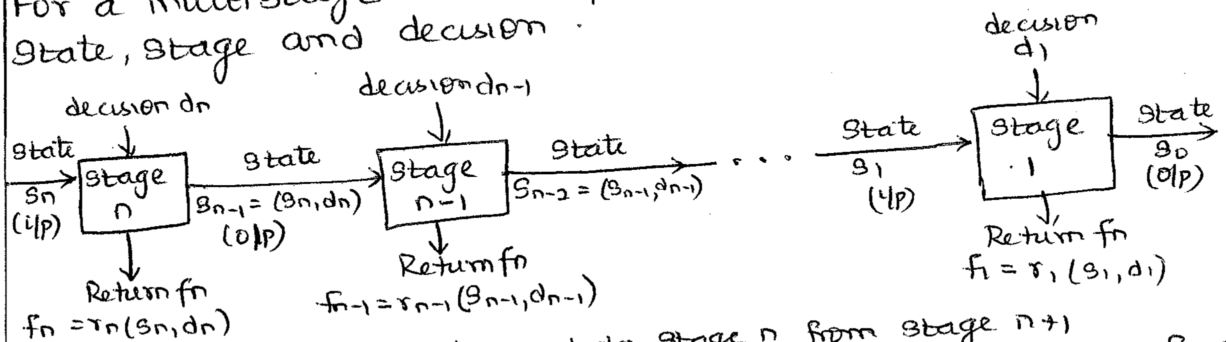
Dynamic programming terminology.

STAGE: The DPP can be decomposed or divided into a sequence of smaller subproblems called stages.

STATE: Each stage in DPP is associated with a certain number of states, which represents various conditions of the decision process at a stage.

RETURN FUNCTION: At each stage, a decision is made that can affect the state of the system at the next stage and help in arriving at the optimal solution at the current stage.

For a multistage decision process, fn. relationship between state, stage and decision.

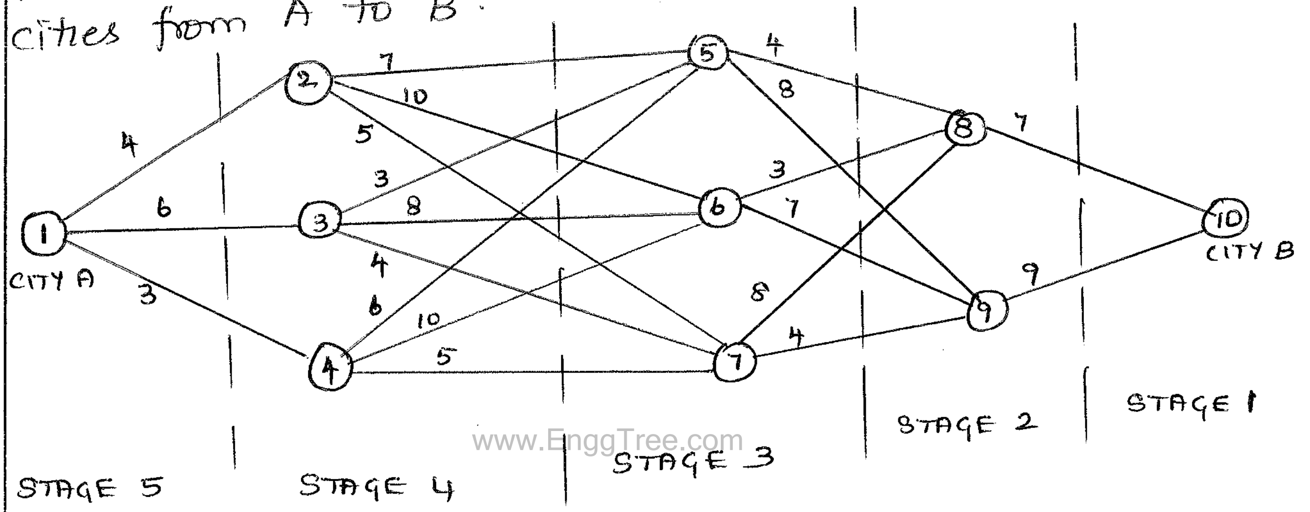


n→ stage no. s_n - state input to stage n from stage n+1

d_n→ decision variable at stage n. $f_n = r_n(s_n, d_n)$ = return fn. for stage n.

Shortest Route problem.

1) A salesman located in a city A decided to travel to city B. He knew the distances of alternative routes from city A to city B. He then drew a highway n/w map. The city of origin A, is city 1. The destination city B is city 10. Other cities through which the salesman will have to pass through are numbered 2 to 9. The arrow representing routes between cities and distances in kilometers are indicated on each route. The salesman's problem is to find the shortest route that covers all the selected cities from A to B.



To solve the problem, we need to define problem stages, decision variables, state variables, return function and transition function.

d_n = decision variables that define immediate destinations.

s_n = state variables describe a specific city at any stage.

D_{s_n,d_n} - distance associated with the state variable, s_n & dec. var d_n for the current stage (n^{th} stage)

$f_n(s_n,d_n)$ = Min. total distance for the last n stages, given that salesman is in state s_n and selects d_n as immediate destination

$f_n^*(s_n)$ = optimal path when the salesman is in state s_n.

The recursive relationship for this problem is

$$f_n^*(s_n) = \min_{d_n}\left\{D_{s_n,d_n} + f_{n-1}^*(d_n)\right\} \quad ; \quad n=1,2,3,4$$

where $f_{n-1}^*(d_n)$ is the optimal distance for the previous stages.

Working backward in stages from city B to city A, we determine the shortest distance to city B (node 10) in stage 1 from state $s_1=8$ (node 8) and state $s_1=9$ (node 9) in stage 2.

Since the distance associated with entering stage 2 from State $s_1=8$ and $s_1=9$ are $D_{8,10}=7$ and $D_{9,10}=9$ respectively.

Decision d_1 →		$f_1(s_1,d_1)=D_{s_1,d_1}$ 10	Min. Distance $f_1^*(s_1)$	optimal decision d_1
States, s_1	8	7	7	8
	9	9	9	9

We move backward to stage 3. Suppose that the salesman is at state $s_2=5$ (node 5) Here he has to decide whether he should go to either $d_2=8$ (node 8) or $d_2=9$ (node 9). for this he must evaluate

$$D_{5,8}+f_1^*(8) = 4+7=11 \quad (\text{to state } s_1=8)$$
$$D_{5,9}+f_1^*(9) = 8+9=17 \quad (\text{to state } s_1=9)$$

The distance function for travelling from state $s_2=5$, is the smallest of these two evaluated above.

$$f_2(s_2) = \min_{d_2=8,9}\{11,17\} = 11 \quad (\text{to state } s_1=8)$$

Similarly, the calculation of distance function for travelling from state $s_2=6$ and $s_2=7$ can be completed as follows.

for state $s_2=6$
$$f_2(6) = \min_{d_2=8,9}\begin{cases} D_{6,8}+f_1^*(8) = 3+7=10 \\ D_{6,9}+f_1^*(9) = 7+9=16 \end{cases}$$
$$=10 \quad (\text{to state } s_1=8)$$

for state $s_2=7$
$$f_2(7) = \min_{d_2=8,9}\begin{cases} D_{7,8}+f_1^*(8) = 8+7=15 \\ D_{7,9}+f_1^*(9) = 4+9=13 \end{cases}$$
$$=13 \quad (\text{to state } s_1=9)$$

decision d_2 →		$f_2(s_2,d_2)=D_{s_2,d_2}+f_1^*(d_2)$ 8	 9	Min-Distance $f_2^*(s_2)$	optimal Decision d_2
State, s_2	5	4+7=11	8+9=17	11	8
	6	3+7=10	7+9=16	10	8
	7	8+7=15	4+9=13	13	9

Continue the same process for stage 4, for states $s_3=2$, $s_3=3$, $s_3=4$ respectively.

decision d_3 →		$f_3(s_3,d_3)=D_{s_3,d_3}+f_2^*(d_3)$ 5	 6	 7	Min-distance $f_3^*(s_3)$	optimal decision d_3
States s_3	2	7+11=18	10+10=20	5+13=18	18	5 or 7
	3	3+11=14	8+10=18	4+13=17	14	5
	4	6+11=17	10+10=20	5+13=18	17	5

Continue the same process for stage 5 for state $S_4 = 1$

decision d_4	$f_4(s_4,d_4) = D_{s_4,d_4} + f_3^*(d_4)$			Min-distance $f_4^*(s_4)$	optimal decision d_4
	2	3	4		
States s_4 1	4+18=22	6+14=20	3+17=20	20	3 or 4

The optimal result at various stages are

Sequence $1 \rightarrow 3 \rightarrow 5 \rightarrow 8 \rightarrow 10$

Distance $6 + 3 + 4 + 7 = 20$.

Alternate solution

Sequence $1 \rightarrow 4 \rightarrow 5 \rightarrow 8 \rightarrow 10$

Distance $3 + 6 + 4 + 7 = 20$.

From the above it is clear that there are two alternative shortest routes for this problem, both having a minimum distance of 20 kilometeres.

CS6704 - RESOURCE MANAGEMENT TECHNIQUES

UNIT IV

CLASSICAL OPTIMIZATION THEORY

Introduction :

The classical optimization methods are used to obtain an optimal solution of certain types of problems that involve continuous and differentiable functions. These methods are analytical in nature and make use of differential calculus to find points of maxima and minima for both unconstrained and constrained continuous objective functions.

UNCONSTRAINED OPTIMIZATION

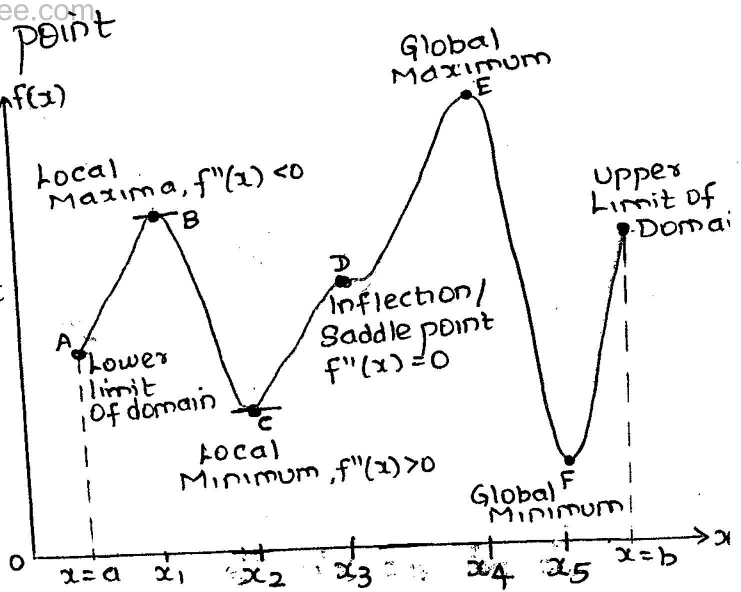
Let us consider a continuous function $y=f(x)$ of single independent variable, x in the domain (a,b). The domain is the range of values of x. The domain limits or end points called Stationary or Critical points.

There are two categories of stationary points

- Inflection point or saddle point
- Extreme points
  - Local or relative
  - Global or absolute

Local extreme points represent the maximum or minimum values of the functions in the given range of values of the variable.

Points $a, x_1, x_2, x_3, x_4, x_5, b$ are all extrema of $f(x)$



Local and Global Optimum

The classical method does not provide a direct method to obtain global value of a function. The global minimum is the minimum value among all local minimum values and global maximum is the maximum values among all local maximum values.

Conditions for Local Minimum and Local Maximum Value:

Theorem 4.1 [Necessary Condition]

A necessary condition for a point x_0 to be the local extrema [local maximum & minimum] of a function $y=f(x)$ defined in the interval $a \leq x \leq b$ is that the first derivative of $f(x)$ exists as a finite number at $x=x_0$ and $f'(x_0)=0$

for $x=x_0$ to be local maximum or minimum value, the sign of $f(x_0+h)-f(x_0)$ and $f(x_0-h)-f(x_0)$ must be the same for all $x=x_0 \pm h$.

If $f(x_0+h)-f(x_0)$ and $f(x_0-h)-f(x_0)$ have the same sign, then $f'(x_0)$ should be zero; otherwise they will have different signs.

Hence the NECESSARY CONDITION for any function $f(x)$ to have local optimum value at any extreme point $x=x_0$, is that its first derivative $f'(x_0)=0$.

Distinction Between Local Minimum & Local Maximum

Examine the direction of change of first derivative, $f'(x_0)$ at $x=x_0$

(i) If the sign of $f'(x_0)$ changes from positive to negative as x increases in the neighbourhood of $x=x_0$, then the value of $f(x)$ will be a local maximum.

(ii) If the sign of $f'(x_0)$ changes from negative to positive as x increases in the neighbourhood of $x=x_0$, then the value of $f(x)$ will be a local minimum.

Theorem 4·2 : SUFFICIENT CONDITION

If at an extreme point $x=x_0$ at $f(x)$, the first $(n-1)$ derivatives of it becomes zero, ie. $f'(x_0) = f''(x_0) \cdots = f^{(n-1)}(x_0)=0$ and $f^{n}(x_0) \neq 0$, then:

(i) local maximum of $f(x)$ occurs at $x=x_0$, if $f^{(n)}(x_0) < 0$, for n even

(ii) local minimum of $f(x)$ occurs at $x=x_0$, if $f^{(n)}(x_0) > 0$ for n even

(iii) point of inflection occurs at $x=x_0$, if $f^{(n)}(x_0) \neq 0$ for n odd.

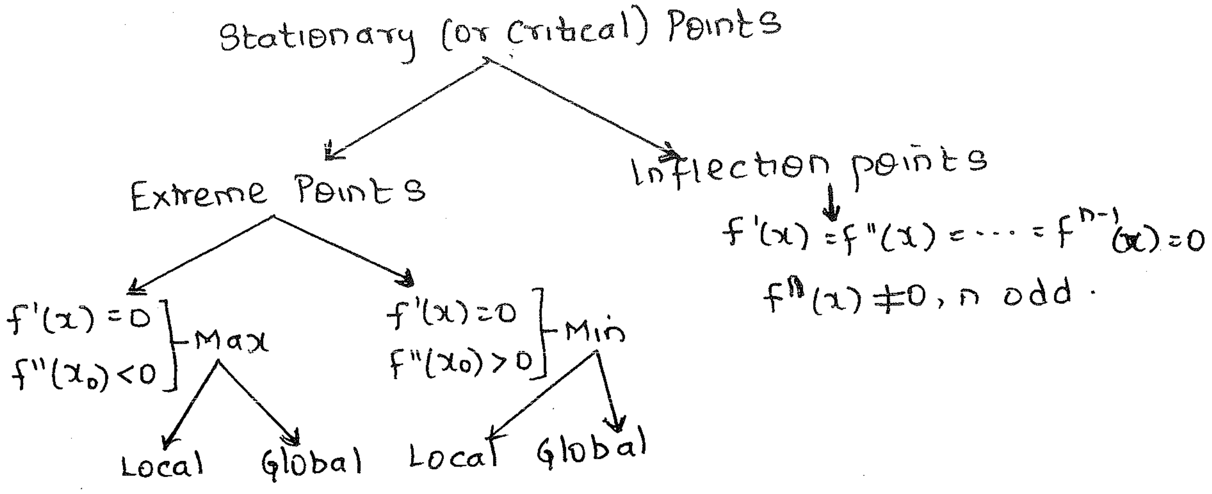
case (i) - If $f''(x_0)>0$, then both $f(x_0+h)-f(x_0)$ and $f(x_0-h)-f(x_0)$ are positive and hence local minimum value of $f(x)$ exists at $x=x_0$

case (ii) If $f''(x_0)<0$, then both $f(x_0+h)-f(x_0)$ and $f(x_0-h)-f(x_0)$ are negative and hence local maximum value of $f(x)$ exists at $x=x_0$

case (iii) If $f''(x_0)=0$, then no information is obtained about the maximum or minimum value of $f(x)$. That is, the function $f(x)$ may have a local maximum, a local minimum, or a point of inflection (or SADDLE POINT). Hence, if $f''(x_0)=0$, then examine successively higher order derivatives of $f(x)$ at $x=x_0$ until a derivative ST $f^{(n)}(x_0) \neq 0$, for $n \geq 2$ is found.

Necessary Condition	Sufficient Condition	Nature of function	Conclusion
$f'(x_0)=0$	$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0)=0$ and $f^{(n)}(x_0) < 0$, n even	Concave	Local Maximum at $x=x_0$
$f'(x_0)=0$	$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0)=0$ and $f^{(n)}(x_0) > 0$, n even	Convex	Local Minimum at $x=x_0$
$f'(x_0)=0$	$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0)=0$ and $f^{(n)}(x_0) \neq 0$, n is odd	-	Point of inflection at $x=x_0$

Determination of Critical Point



Problems.

1. Find the maxima & minima

$f(x) = 4x^4 - x^2 + 5$

$f'(x) = 16x^3 - 2x$

~~Suf~~ Necessary condition

$f'(x) = 0$

$16x^3 - 2x = 0$

$2x(8x^2 - 1) = 0$

$\therefore \quad 2x = 0 \qquad 8x^2 - 1 = 0$

$x = 0 \qquad x = \pm\sqrt{1/8}$

find $f''(x)$

$f''(x) = 48x^2 - 2$

Suffiaent condition

when $x = 0 \quad f''(x) = -2 < 0$ $\quad x = 0$ is max point & max value of $f(x)$ at $x = 0$

At $x = 0$, $f(x) = 4(0) - 0 + 5 = 5$

when $x = \pm\sqrt{1/8}$, $f''(x) = 48\left(1/\sqrt{8}\right)^2 - 2 = 4 > 0$ $\quad x = 1/\sqrt{8}$ is min point

At $x = 1/\sqrt{8}$ $\quad f(x) = 4\left(\frac{1}{\sqrt{8}}\right)^4 - \left(\frac{1}{\sqrt{8}}\right)^2 + 5 = \frac{79}{16}$

Maxima $f(x) = 5 \qquad$ Minima $f(x) = 79/16$

2. Consider the function

$$f(x) = x_1 + 2x_2 + x_1 x_2 - x_1^2 - x_2^2$$

Determine the maximum or minimum point (if any) of the function.

$\dfrac{\partial f(x)}{\partial x_1}$

This is a Multiple Variable Problem

The Necessary & Sufficient conditions are

Necessary condition	Sufficient condition	Conclusion
$\nabla f(x_0) = 0$	$H(x_0)$ is positive definite	Local Minimum at $x = x_0$
$\nabla f(x_0) = 0$	$H(x_0)$ is negative definite	Local Maximum at $x = x_0$
$\nabla f(x_0) = 0$	$H(x_0)$ is indefinite	Point of Inflection at $x = x_0$

$H(X)$ = Hessian Matrix

$$H(X) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & & & \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}$$

(i) $H(x)$ - Positive Definite
If all its leading principal minors of order 1x1 are positive. [local Min]

(ii) $H(x)$ - Negative Definite [local Max]
If ~~all~~ the signs of all even leading principal minors is positive

(iii) if (i) and (ii) not met, then $H(x_0)$ is Indefinite. (may be either max or min or neither)

$f(x) = x_1 + 2x_2 + x_1 x_2 - x_1^2 - x_2^2$

Solution: Necessary condition

$$\frac{\partial f}{\partial x_1} = 1 + x_2 - 2x_1 = 0 \qquad \frac{\partial f}{\partial x_2} = 2 + x_1 - 2x_2 = 0$$

$$(x_1, x_2) = (4/3, 5/3)$$

Solving

$$2 + 2x_2 - 4x_1 = 0$$
$$2 - 2x_2 + x_1 = 0$$
$$4 \quad -3x_1 = 0$$
$$x_1 = 4/3$$
$$2 + 4/3 - 2x_2 = 0$$
$$x_2 = 5/3$$

Sufficient condition: Hessian Matrix

$$H(X) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Principal Minors $A_1 = \left[\frac{\partial^2 f}{\partial x_1^2}\right] = -2$. $\quad \det A_2 = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = 3$

Since the signs of principal minor determinants are alternating matrix $H(x)$ is NEGATIVE DEFINITE and the point $x_0 = (4/3, 5/3)$ is the local MAXIMUM.

$$f(x) = \frac{4}{3} + 2(5/3) + \frac{4}{3} \cdot \frac{5}{3} - \left(\frac{4}{3}\right)^2 - \left(\frac{5}{3}\right)^2 = 7/3$$

3) $f(x,y) = 3x^2+y^2-10$

Necessary condition : As there are 2 variables x, y, find partial derivative with respect to x & y

$p = \frac{\partial f}{\partial x} = 6x$ $q = \frac{\partial f}{\partial y} = 2y$

equate it to zero

$6x = 0$ $2y = 0$

$x = 0$ $y = 0$

Sufficient condition

Find second order derivative

$r = \frac{\partial^2 f}{\partial x^2} = 6$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$

$t = \frac{\partial^2 f}{\partial y^2} = 2$

If $rt-s^2$ is positive then there exist either maxima or minima

Check the sign of r

If r is +ve, then f is min

else if r is -ve, then f is max

$rt-s^2 = 6\times2 - 0 = 12$

Positive. There exist either maxima or minima.

Sign of r is positive, ∴ $f(x,y)$ is the min value.

Sub $x=0$, $y=0$ in $f(x,y)$ to find the minima

$f(x,y) = 3(0)^2+(0)^2-10$

$= -10$

Minima is -10

Alternate Solution using Hessian Matrix

SUFFICIENT CONDITION

$H = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix}$

Sign of principal minor is checked

Principal Minor of H are 6, 2. Positive Definite. It is minimum

$|6| = 6$ $\begin{vmatrix} +6 & 0 \\ 0 & 2 \end{vmatrix} = 12$

$x=0$ $y=0$ $f(x,y) = -10$

Minima is -10.

Newton-Raphson Method

The Newton-Raphson method is an iterative procedure for solving simultaneous non-linear equations. It is actually a part of the gradient methods for optimizing unconstrained functions numerically.

It represents a general method for finding the extrema (Minima or Maxima) of a given function $f(x)$ in an Iterative manner.

for Minima, the first derivative $f'(x)$ must be zero and the second derivative $f''(x)$ must have a positive value; while for maxima $f'(x)$ is again zero and the second derivative $f''(x)$ has a negative value.

The next point x_{n+1} in the iterative series is found

$$x_{n+1} = x_n - \left[f'(x_n) / f''(x_n) \right]$$

The convergence criterion can be

(1) The difference between the functional values of two consecutive iterations $df = f(x_{n+1}) - f(x_n)$

(2) The difference between the values of x itself between two consecutive iterations $dx = x_{n+1} - x_n$

Problems :

Consider the following function

$f(x) = 3x^3 - 10x^2 - 56x + 5$

find the maxima & minima, considering the convergence criterion of $dx = .001$

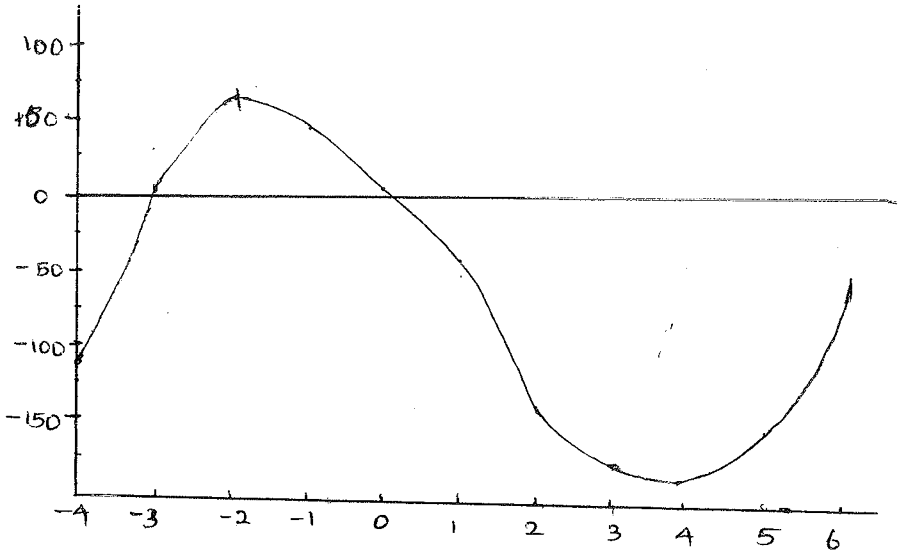
Solution

$f(x) = 3x^3 - 10x^2 - 56x + 5$

$f'(x) = 9x^2 - 20x - 56$

$f''(x) = 18x^{\cancel{2}} - 20$

To determine the starting point, draw the graph for the range $-4 \leq x \leq 6$



X	f(x)
-4	-123
-3	2
-2	69
-1	48
0	5
1	-58
2	-139
3	-172
4	-187
5	-150
6	-43

Let us consider the first starting $x_0 = 0.0$

Iterative process shown below

Iteration	x_n	f(x)	f''(x)	f''(x)	dx
0	0.0	+5.0	-56.0	-20.0	-2.8
1	-2.8	+17.5440	+70.56	-70.4	+1.0023
2	-1.7977	+55.9249	+9.0395	-52.3586	+0.1726
3	-1.6251	+56.7207	+0.2683	-49.251	-0.0054
4	-1.6197	+56.7214	+0.0049	-49.1546	-0.0001

$$dx = x_n - \frac{f'(x)}{f''(x)}$$

$$x_{n+i} = x_n + dx$$

$$dx = 0 - \frac{(-56)}{(-20)}$$

$$= -2.8$$

$$x_{n+1} = 0 - 2.8 = -2.8$$

It locates functional maximum at x = -1.6197. because f'(x) = 0 and f''(x) is Negative

consider the second starting point $x_0 = 2.0$

iteration	x_n	f(x)	f'(x)	f''(x)	dx
0	+2.0	-123.0	-60.0	+16.0	+3.75
1	+5.75	-77.2969	+70.56	+83.5	-1.5157
2	+4.2343	-183.6597	+9.0395	+56.2174	-0.3678
3	+3.8665	-187.6118	+0.2683	+49.5967	-0.0246
4	+3.8419	-187.6268	+0.0049	+49.1551	-0.0001

It locates the functional minimum at x = +3.8419 because f'(x) = 0 and f''(x) is positive.

2. Determine the stationary points of the function

$$g(x) = (3x-2)^2 (2x-3)^2$$

$$g(x) = (9x^2 - 12x + 4)(4x^2 - 12x + 9)$$

$$= 36x^4 - 108x^3 + 81x^2 - 48x^3 + 144x^2 - 108x + 16x^2 - 48x + 36$$

$$= 36x^4 - 156x^3 + 241x^2 - 156x + 36$$

$$g'(x) = 144x^3 - 468x^2 + 482x - 156 = 0$$

$$72x^3 - 234x^2 + 241x - 78 = 0$$

$$g''(x) = 216x^2 - 468x + 241$$

To determine Stationary point

(i) Start from initial point

(ii) find $f'(x), f''(x)$

(iii) $x_{k+1} = \boxed{x_k - \frac{f'(x)}{f''(x)}}$

Disadvantage:
The convergence is not always guaranteed unless the function f is well behaved

Starting with $x_0 = 10$

Iteration	x_n	$g'(x)$	$g''(x)$	$g'(x)/g''(x)$	$x_{n+1} = x_n - \frac{g'(x)}{g''(x)}$
0	10	50932	17161	2.96	7.0321
1	7.0321	15082.7	7631.29	1.97	5.0557
2	5.0557	4463.43	3395.88	1.31	3.7413
3	3.7413	1318.81	1513.51	0.87	2.8699
4	2.8699	338.28	676.97	0.57	2.2964
5	2.2964	113.36	305.35	0.37	1.9215
6	1.9215	32.43	140.57	0.23	1.6944
7	1.6944	8.79	68.16	0.13	1.5654
8	1.5654	2.04	37.71	0.05	1.5113
9	1.5113	0.29	27.06	0.01	1.5004
10	1.5004	0.011	25.08	0.00	1.5000

The method converges to $x = 1.5$.

Actually 3 stationary points exists at $x = 2/3$, $x = 13/12$, $x = 3/2$.
The remaining 2 points can be found by choosing a different starting value.

Lagrangian Method

1. Solve the following problem by using the method of Lagrangian Multiplier.

$$\text{Max } Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$$

Sub. to
$$x_1 + x_2 + x_3 = 15$$
$$2x_1 - x_2 + 2x_3 = 20$$
$$x_1, x_2, x_3 \geq 0$$

$$f(x) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$$
$$g_1(x) = x_1 + x_2 + x_3 - 15$$
$$g_2(x) = 2x_1 - x_2 + 2x_3 - 20$$

Construct the Lagrangian function

no of con

$$L(x, \lambda) = f(x) - \lambda_1 g_1(x) - \lambda_2 g_2(x)$$
$$= 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15) - \lambda_2(2x_1 - x_2 + 2x_3 - 20)$$

Using Necessary condition

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0 \quad \text{——} (1)$$
$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{——} (2)$$
$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 2x_3 - \lambda_1 - 2\lambda_2 = 0 \quad \text{——} (3)$$
$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow -(x_1 + x_2 + x_3 - 15) = 0 \quad \text{——} (4)$$
$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow -(2x_1 - x_2 + 2x_3 - 20) = 0 \quad \text{——} (5)$$

Solving these equations, we get

$$(x_1, x_2, x_3) = \left(\frac{33}{9}, \frac{10}{3}, 8\right)$$
$$\lambda_1 = \frac{40}{9}, \quad \lambda_2 = \frac{52}{9}$$

The optimal solution is $x_1 = \frac{33}{9}$ $x_2 = \frac{10}{3}$ $x_3 = 8$

$$\text{Max } Z = \frac{7380}{81}$$

Solve ④ + ⑤

④ × 2 $\quad -\cancel{2x_1} - 2x_2 - \cancel{2x_3} + 30 =$
$\quad \overset{(+)}{-\cancel{2x_1}} \overset{(-)}{+ x_2} \overset{(+)}{-\cancel{2x_3}} \overset{(-)}{+ 20} =$

$$-3x_2 + 10 = 0$$
$$x_2 = 10/3.$$

Solve ①, ③

$$8x_1 - 4x_2 - \cancel{\lambda_1} - \cancel{2\lambda_2} = 0$$
$$\underset{-}{2x_3} \underset{+}{-\cancel{\lambda_1}} \underset{+}{-\cancel{2\lambda_2}} = 0$$
$$8x_1 - 4x_2 - 2x_3 = 0 \rightarrow ⑥$$

Solve ⑤, ⑥

$$8x_1 - 4x_2 - \cancel{2x_3} = 0$$
$$\underset{+}{-2x_1} \underset{+}{-2x_2} \underset{+}{-\cancel{2x_3}} \underset{-}{+30} = 0 \quad ④ * 2$$
$$10x_1 - 2x_2 = -30$$

$$10x_1 - 2 * 10/3 = -30$$
$$10x_1 = -30 + \frac{20}{3}$$
$$10x_1 = 11''$$

2. Max $z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

Subject to $g(x) = x_1 + x_2 + x_3 = 20$

The Lagrangian function is

$$L(x, \lambda) = f(x) - \lambda\,(g(x))$$
$$= 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda\,(x_1 + x_2 + x_3)$$

The Necessary condition

$\frac{\partial L}{\partial x_1} = 4x_1 + 10 - \lambda = 0$ $\qquad x_1 = \frac{\lambda - 10}{4}$ —— (1)

$\frac{\partial L}{\partial x_2} = 2x_2 + 8 - \lambda = 0$ $\qquad x_2 = \frac{\lambda - 8}{2}$ —— (2)

$\frac{\partial L}{\partial x_3} = 6x_3 + 6 - \lambda = 0$ $\qquad x_3 = \frac{\lambda - 6}{6}$ —— (3)

$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 20) = 0$ $\qquad x_1 + x_2 + x_3 = 20$ —— (4)

Sub (1), (2), (3) in (4)

$$\frac{\lambda - 10}{4} + \frac{\lambda - 8}{2} + \frac{\lambda - 6}{6} = 20$$
$$3\lambda - 30 + 6\lambda - 48 + 2\lambda - 12 = 240$$
$$11\lambda = 330$$
$$\lambda = 30$$

$\therefore x_1 = \frac{\lambda - 10}{4} = \frac{30 - 10}{4} = 5$

$x_2 = \frac{\lambda - 8}{2} = \frac{30 - 8}{2} = 11$

$x_3 = \frac{\lambda - 6}{6} = \frac{30 - 6}{6} = 4$

Max $z = 2(5)^2 + (11)^2 + 3(4)^2 + 10(5) + 8(11) + 6(4) - 100$

$= \boxed{281}$ 371

3. Min $z = x_1^2 + x_2^2 + x_3^2$

Sub. to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

The Lagrangian function is

$$L(x, \lambda) = f(x) - \lambda_1(g_1(x)) - \lambda_2(g_2(x))$$

$$= x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

The Necessary condition is

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 5\lambda_2 = 0 \qquad x_1 = \frac{\lambda_1 + 5\lambda_1}{2} \text{ ——— } (1)$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - 2\lambda_2 = 0 \qquad x_2 = \frac{\lambda_1 + 2\lambda_2}{2} \text{ ——— } (2)$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - \lambda_2 = 0 \qquad x_3 = \frac{3\lambda_1 + \lambda_2}{2} \text{ ——— } (3)$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + 3x_3 - 2) = 0 \qquad x_1 + x_2 + 3x_3 = 2 \text{ —— } (4)$$

$$\frac{\partial L}{\partial \lambda_2} = -(5x_1 + 2x_2 + x_3 - 5) = 0 \qquad 5x_1 + 2x_2 + x_3 = 5 \text{ —— } (5)$$

Sub. ~~to~~ (1) (2) (3) in (4)

$$\frac{\lambda_1 + 5\lambda_2}{2} + \frac{\lambda_1 + 2\lambda_2}{2} + 3 \cdot \frac{3\lambda_1 + \lambda_2}{2} = 2$$

$$\lambda_1 + 5\lambda_2 + \lambda_1 + 2\lambda_2 + 9\lambda_1 + 3\lambda_2 = 4$$

$$11\lambda_1 + 10\lambda_2 = 4 \text{ ——— } (6)$$

Sub (1) (2) (3) in (5)

$$5\,\frac{\lambda_1 + 5\lambda_2}{2} + 2\,\frac{\lambda_1 + 2\lambda_2}{2} + \frac{3\lambda_1 + \lambda_2}{2} = 5$$

$$5\lambda_1 + 25\lambda_2 + 2\lambda_1 + 4\lambda_2 + 3\lambda_1 + \lambda_2 = 10$$

$$10\lambda_1 + 30\lambda_2 = 10$$

$$\lambda_1 + 3\lambda_2 = 1 \text{ ——— } (7)$$

Solve (6), (7)

(6) $11\lambda_1 + 10\lambda_2 = 4$

(7) × 11 $11\lambda_1 + 33\lambda_2 = 11$

$-23\lambda_2 = -7$

$\lambda_2 = 7/23$

Sub in (7) $\lambda_1 + 3(7/23) = 1$

$\lambda_1 = 2/23$

Sub λ_1, λ_2 in (1) (2) (3)

$$x_1 = \frac{\lambda_1 + 5\lambda_2}{2} = \frac{2/23 + 5(7/23)}{2} = \frac{37}{46}$$

$$x_2 = \frac{\lambda_1 + 2\lambda_2}{2} = \frac{2/23 + 2(7/23)}{2} = \frac{16}{46}$$

$$x_3 = \frac{3(\lambda_1) + \lambda_2}{2} = \frac{3(2/23) + (7/23)}{2} = \frac{13}{46}$$

$$\text{Min } Z = x_1^2 + x_2^2 + x_3^2$$

$$= \left(\frac{37}{46}\right)^2 + \left(\frac{16}{46}\right)^2 + \left(\frac{13}{46}\right)^2$$

$$= \frac{1369 + 256 + 169}{2116} = \frac{1794}{2116} = \frac{897}{1058} = \frac{39}{46}$$

4. Max $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

Sub. to. $4x_1 + 3x_2 = 16$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geqslant 0$

The Lagrangian function is

$$L(x,\lambda) = f(x) - \lambda_1 g_1(x) - \lambda_2 g_2(x)$$

$$= 6x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda_1(4x_1 + 3x_2 - 16) - \lambda_2(3x_1 + 5x_2 - 15)$$

The Necessary condition

$$\frac{\partial L}{\partial x_1} = 6 - 2x_1 - 4\lambda_1 - 3\lambda_2 = 0 \qquad x_1 = \frac{4\lambda_1 + 3\lambda_2 - 6}{2} \quad \text{——— (1)}$$

$$\frac{\partial L}{\partial x_2} = 8 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0 \qquad x_2 = \frac{3\lambda_1 + 5\lambda_2 - 8}{2} \quad \text{——— (2)}$$

$\frac{\partial L}{\partial \lambda_1} = -(4x_1 + 3x_2 - 16) = 0$ $\qquad 4x_1 + 3x_2 = 16$ ——— (3)

$\frac{\partial L}{\partial \lambda_2} = -(3x_1 + 5x_2 - 15) = 0$ $\qquad 3x_1 + 5x_2 = 15$ ——— (4)

Solve (3) & (4)

(3) × 3 $\quad 12x_1 + 9x_2 = 48$

(4) × 4 $\quad \underset{-}{12x_1} \underset{-}{+} 20x_2 = \underset{-}{60}$

$$-11x_2 = -12$$

$$x_2 = 12/11$$

$$4(x_1) + 3(12/11) = 16$$

$$4x_1 = 16 - \frac{36}{11}$$

$$= \frac{176 - 36}{11} = \frac{140}{11}$$

$$x_1 = 35/11$$

$X_0 = (x_1, x_2) = (35/11, 12/11)$

$$\text{Max } Z = 6(35/11) + 8(12/11) - (35/11)^2 - (12/11)^2$$

$$= \frac{2310 + 1056 - 1225 - 144}{121}$$

$$\text{Max } Z = \frac{1997}{121}$$

JACOBIAN METHOD [CONSTRAINED DERIVATIVES]

The constrained derivatives are used to develop a closed-form expression for the first partial derivatives of $f(x)$ at all points that satisfy the constraints $g(x)=0$. The corresponding stationary points are identified as the points at which these partial derivatives vanish.

The sufficiency conditions aned then used to check the identity of stationary points.

Consider there are n unknowns and m equations. If $m=n$, then x has no feasible neighbourhood. If $n>m$, it requires further processing

Define $x = (Y, Z)$

Such that $Y = (y_1 \, y_2 \cdots y_m)$ & $Z = (z_1 \, z_2 \cdots z_{n-m})$

The vectors Y called Depedendent and Z called Independent Variables.

$$\nabla f(Y,Z) = (\nabla_Y f, \nabla_Z f)$$

$$\nabla g(Y,Z) = (\nabla_Y g, \nabla_Z g)$$

$$J = \nabla_Y g = \begin{pmatrix} \nabla_Y g_1 \\ \vdots \\ \nabla_Y g_m \end{pmatrix} \qquad C = \nabla_Z g \begin{pmatrix} \nabla_Z g_1 \\ \vdots \\ \nabla_Z g_m \end{pmatrix}$$

$J_{m \times m}$ is called the Jacobian Matrix, $C_{m \times n-m}$ the control matrix

The original set of equations in $\partial f(x)$ and ∂x may be written as

$$\partial f(Y,Z) = \nabla_Y f \partial Y + \nabla_Z f \partial Z$$

&

$$J \partial Y = -C \partial Z$$

$$\partial Y = -J^{-1} C \partial Z$$

Substituting ∂Y in eq. for $\partial f(x)$ gives ∂f as a function of ∂Z

$$\partial f(Y,Z) = (\nabla_Z f - \nabla_Y f J^{-1} C) \partial Z$$

$$\therefore \nabla_c f = \frac{\partial_c f(Y,Z)}{\partial_c Z} = \nabla_Z f - \nabla_Y f J^{-1} C$$

where $\nabla_c f$ is the constrained gradient vector of f w.r.t Z

Sufficient Condition: for X_0 to ber an extrema is that Hessian Mat H evaluated S.T H is +ve definite if X_0 is minimum pt and H is -ve definite if X_0 is maximum.

1) Consider, $f(x) = x_1^2 + 3x_2^2 + 5x_1x_3^2$

Sub. to $g_1(x) = x_1x_3 + 2x_2 + x_2^2 - 11 = 0$

$g_2(x) = x_1^2 + 2x_1x_2 + x_3^2 - 14 = 0$

Given the feasible point $x^0 = (1,2,3)$, discuss the variation in $f(=\partial c f)$ in the feasible neighbourhood of x^0. using $\partial x_2 = 0.01$

Soln: Let $Y = \{x_1\ x_3\}$ $Z = \{x_2\}$

$\therefore$ $\nabla_Y f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_3}\right) = \{2x_1 + 5x_3^2, \ 10x_1x_3\}$

$\nabla_Z f = \frac{\partial f}{\partial x_2} = 6x_2$

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} x_3 & x_1 \\ 2x_1+2x_2 & 2x_3 \end{pmatrix} \qquad C = \begin{pmatrix} \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_1}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x+2 \\ 2x_1 \end{pmatrix}$$

To estimate $\partial c f$ in the feasible region neighbourhood $x^0 = (1,2,3)$ given a small change $\partial x_2 = .01$, in the independent variable.

$$J^{-1}C = \begin{pmatrix} 3 & 1 \\ 6 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 6/12 & -1/12 \\ -6/12 & 3/12 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \simeq \begin{pmatrix} 2.83 \\ -2.50 \end{pmatrix}$$

Incremental value of the constraint. f is given as

$$\partial c f = (\nabla_Z f - \nabla_Y f\ J^{-1}C)\partial Z = \left(6(2) - (47,30)\begin{pmatrix} 2.83 \\ -2.50 \end{pmatrix}\right)\partial x_2$$

$\simeq -46.01\,\partial x_2$

By specifying the value of ∂x_2 for the independent variable x_2, feasible values of the ∂x_1 & ∂x_3 are determined for the dependent variables x_1 & x_3 using the formula

$\partial Y = -J^{-1}C\,\partial Z$

for $\partial x_2 = .01$ $\begin{pmatrix} \partial x_1 \\ \partial x_3 \end{pmatrix} = -J^{-1}C\,\partial x_2 = \begin{pmatrix} -0.0283 \\ 0.0250 \end{pmatrix}$

compare the values of $\partial c f$ with the difference $f(x^0+\partial x) - f(x^0)$

given $\partial x_2 = .01$ $x^0 + \partial x = (1 - .0283, 2 + .01, 3 + .025)$

$(0.9717$ ~~0.9717~~$, 2.01, 3.025)$

This yields $f(x^0) = 58$ $f(x^0 + \partial x) = 57.523$

$f(x^0+\partial x) - f(x^0) = -0.477$

This compares favourably with $\partial c f = -46.01\,\partial x_2 = -0.4601$.

The difference between 2 values is the result of linear approximat

2) Consider the problem

$$\text{Minimize } f(x) = x_1^2 + x_2^2 + x_3^2$$

Subject to $g_1(x) = x_1 + x_2 + 3x_3 - 2 = 0$

$g_2(x) = 5x_1 + 2x_2 + x_3 - 5 = 0$

Determine the constrained extreme points using constrained derivatives or Jacobian Method.

Step 1 : Determine Independent & Dependent variable set

No. of unknowns $= n = 3$ No. of equations $= m = 2$

$n > m$ No. of Independent var $= n - m = 1$

$\therefore$ No. of dependent variables $= 2$

Let $Y = \{x_1, x_2\}$ dependent $Z = \{x_3\}$ Independent.

Step 2 : find $\nabla_Y f$ and $\nabla_Z f$

$$\nabla_Y f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = (2x_1, 2x_2) \qquad \nabla_Z f = \left(\frac{\partial f}{\partial x_3}\right) = 2x_3$$

Step 3 : find the Jacobian Matrix & control Matrix.

Jacobian Matrix – co-efficients of dependent variables in constraints

control Matrix – co-efficient of Independent variables in constraints

$$J = \begin{pmatrix} 1 & 1 \\ 5 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$J^{-1} = \frac{1}{-3}\begin{pmatrix} 2 & -1 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{pmatrix}$$

Step 4 : find $\nabla_c f$

$$\nabla_c f = \frac{\partial_c f}{\partial_c x_3} = \nabla_Z f - \nabla_Y f \cdot J^{-1} \cdot C$$

$$= 2x_3 - (2x_1, 2x_2)\begin{pmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 2x_3 - (2x_1, 2x_2)\begin{pmatrix} -5/3 \\ 14/3 \end{pmatrix}$$

$$= \frac{10}{3}x_1 - \frac{28}{3}x_2 + 2x_3$$

Step 5: Determine the stationary points

$\nabla_c f = 0 \qquad \frac{10}{3}x_1 - \frac{28}{3}x_2 + 2x_3 = 0$

$\qquad\qquad\qquad 10x_1 - 28x_2 + 6x_3 = 0$

$g_1(x) = 0 \qquad x_1 + x_2 + 3x_3 - 2 = 0$

$g_2(x) = 0 \qquad 5x_1 + 2x_2 + x_3 - 5 = 0$

Solve

$$\begin{pmatrix} 10 & -28 & 6 \\ 1 & 1 & 3 \\ 5 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

Solution is $X^0 \approx (\cdot 81, \cdot 35, \cdot 28)$

Step 6: Verification

Sufficiency condition

$$\frac{\partial_c^2 f}{\partial_c x_3^2} = \frac{10}{3}\left(\frac{dx_1}{dx_3}\right) - \frac{28}{3}\left(\frac{dx_2}{dx_3}\right) + 2$$

$$= \begin{pmatrix} \frac{10}{3} & \frac{-28}{3} \end{pmatrix}\begin{pmatrix} \frac{dx_1}{dx_3} \\ \frac{dx_2}{dx_3} \end{pmatrix} + 2$$

from the Jacobian method

$$\begin{pmatrix} \frac{dx_1}{dx_3} \\ \frac{dx_2}{dx_3} \end{pmatrix} = -J^{-1}\cdot C = -\begin{pmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/3 \\ -14/3 \end{pmatrix}$$

$$\therefore \frac{\partial_c^2 f}{\partial_c x_3^2} = \begin{pmatrix} \frac{10}{3} & \frac{-28}{3} \end{pmatrix}\begin{pmatrix} 5/3 \\ -14/3 \end{pmatrix} + 2$$

$$= \frac{50}{9} + \frac{392}{9} + 2 = \frac{460}{9} > 0$$

Hence X^0 is the minimum point.

KUHN-TUCKER CONDITIONS [KARUSH-KUHN TUCKER METHOD - KKT]

The kuhn-Tucker conditions provides the basic theory for inequality constraints in Non-Linear Programming. It is an extension of Lagrangean Method.

Necessary Condition

optimize $z = f(x)$

Subject to $g_i(x) \leq 0$ for $i = 1, 2, \cdots, m$

Add the non-negativity slack variable S_i^2

$\therefore$ optimize $z = f(x)$

Subject to $g_i(x) + S_i^2 = 0$ $\quad i = 1, 2, \cdots, m$

The new problem is the constrained multi variable optimization problem with equality constraints with $n+m$ variables. Can be solved by using the Lagrangian Multiplier Method

Lagrangian function,

$$L(x, S, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i [g_i(x) + S_i^2]$$

Necessary condition for extreme point to be local min/max are

Option 1

$$\frac{\partial L}{\partial x_j} = \frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0$$

$$\frac{\partial L}{\partial \lambda_i} = -[g_i(x) + S_i^2] = 0$$

$$\frac{\partial L}{\partial S_i} = -2x_j \lambda_i = 0$$

$\lambda_1 S_1 = 0$

$\lambda_2 S_2 = 0$

Four cases are

i) $\lambda_1 = \lambda_2 = 0$

ii) $\lambda_1 = 0$ $\quad S_2 = 0$

(iii) $S_1 = 0$ $\quad \lambda_2 = 0$

(iv) $S_1 = 0$ $\quad S_2 = 0$

$\lambda_1 = 0$

$\lambda_1 = 0, \quad S_2 = 0$

$S_1 = 0 \quad \lambda_2 = 0$

$S_1 = 0 \quad S_2 = 0$

Option 2

$$\frac{\partial L}{\partial x_j} = 0$$

$\lambda_i g_i(x) = 0$

$g_i(x) \leq 0$

$\lambda_i \geq 0$

Four cases

(i) $\lambda_1 = 0$ $\quad \lambda_2 = 0$

(ii) $\lambda_1 = 0$ $\quad \lambda_2 \neq 0$

(iii) $\lambda_1 \neq 0$ $\quad \lambda_2 = 0$

(iv) $\lambda_1 \neq 0$ $\quad \lambda_2 \neq 0$

1. find the optimum value of the objective function ~~when~~ using Kuhn-Tucker Condition.

$$\text{Max } z = 10x_1 - x_1^2 + 10x_2 - x_2^2$$

Sub. to $x_1 + x_2 \leq 14$

$-x_1 + x_2 \leq 6$

$x_1, x_2 \geq 0$

Here the constraints are

$g_1(x) = x_1 + x_2 + S_1^2 - 14 = 0$

$g_2(x) = -x_1 + x_2 + S_2^2 - 6 = 0$

The Lagrangian function is formulated as

$$L(x, S, \lambda) = 10x_1 - x_1^2 + 10x_2 - x_2^2 - \lambda_1(x_1 + x_2 + S_1^2 - 14) - \lambda_2(-x_1 + x_2 + S_2^2 - 6)$$

The Kuhn Tucker condition - Necessary condition

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0 \quad \text{——— (1)}$$

$$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0 \quad \text{——— (2)}$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + S_1^2 - 14) = 0 \quad \text{——— (3)}$$

$$\frac{\partial L}{\partial \lambda_2} = -(-x_1 + x_2 + S_2^2 - 6) = 0 \quad \text{——— (4)}$$

$$\frac{\partial L}{\partial S_1} = -2\lambda_1 S_1 = 0 \quad \text{——— (5)}$$

$$\frac{\partial L}{\partial S_2} = -2\lambda_2 S_2 = 0 \quad \text{——— (6)}$$

Four Cases are

(i) $\lambda_1 = \lambda_2 = 0$

(ii) $\lambda_1 = S_2 = 0$

(iii) $S_1 = \lambda_2 = 0$

(iv) $S_1 = S_2 = 0$

$g_1(x) = x_1 + x_2 + S_1^2 - 14 = 0$

$5 + 5 + S_1^2 - 14 = 0$

$S_1^2 = 4$

$g_2(x) = \boxed{S^2 = 6}$

case (i) $\lambda_1 = 0 \quad \lambda_2 = 0$

in eq (1) $10 - 2x_1 = 0$, $x_1 = 5$ In. eq (2) $10 - 2x_2 = 0$, $x_2 = 5$

Max $z = 50$ This is an unconstrained solution.

Since both S_1^2 & S_2^2 are positive, the solution is feasible. As the solution, so obtained, is unconstrained therefore in order to find whether or not the solution is maximum we test the Hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\det A_1 = \left| \frac{\partial^2 z}{\partial x_1^2} \right| = -2$ $\det A_2 = |H| = 4$

Since signs of the principal minors of H are alternating, matrix H is negative definite and the point $x = (5,5)$ gives the local maximum.

2) Max $z = 10x_1 - x_1^2 + 10x_2 - x_2^2$

Sub. to $x_1 + x_2 \leq 8$

$-x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0$

Constraints are

$g_1(x) = x_1 + x_2 - 8 = 0$

$g_2(x) = -x_1 + x_2 - 5 = 0$

The Lagrangian function is

$L(x, \lambda, s) = (10x_1 - x_1^2 + 10x_2 - x_2^2) - \lambda_1 (x_1 + x_2 + s_1^2 - 8) - \lambda_2 (-x_1 + x_2 + s_2^2 - 5)$

The Kuhn Tucker Necessary Conditions are

$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$ —— (1)

$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$ —— (2) $10 - 8 - 0 - \lambda_2 = 0$

$\lambda_2 = -2$

$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 8) = 0$ —— (3)

$\frac{\partial L}{\partial \lambda_2} = -(-x_1 + x_2 + s_2^2 - 5) = 0$ —— (4)

$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0$ —— (5)

$\frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0$ —— (6)

case (i) $\lambda_1 = 0$ $\lambda_2 = 0$

In (1) $10 - 2x_1 = 0$
$x_1 = 5$

In (2) $10 - 2x_2 = 0$
$x_2 = 5$

Sub (x_1) (x_2) in (3)
$5 + 5 + S_1^2 - 8 = 0$
$S_1^2 = -2$

Sub x_1, x_2 in (4)
$-5 + 5 + S_2^2 - 5 = 0$
$S_2^2 = 5$

The unconstrained solution is

$x_1 = 5$ $x_2 = 5$ $S_1^2 = -2$ $S_2^2 = 5$ Max $z = 50$

Since $S_1^2 = -ve$ this solution is infeasible.

case (ii) $S_1 = 0$ $\lambda_2 = 0$

$\lambda_2 = 0$ in eq (1) $10 - 2x_1 - \lambda_1 = 0$

$\lambda_2 = 0$ in eq (2) $10 - 2x_2 - \lambda_1 = 0$

$$\begin{array}{r} 10 - 2x_1 - \lambda_1 = 0 \\ \underline{\overset{-}{10} \overset{+}{-} 2x_2 \overset{+}{-} \lambda_1 = 0} \\ -2x_1 + 2x_2 = 0 \end{array} \quad \text{——— (7)}$$

$S_1 = 0$ in eq(3) $x_1 + x_2 - 8 = 0$

$x_1 + x_2 = 8$ ——— (8)

Solving (7), (8)

$$\begin{array}{r} -2x_1 + 2x_2 = 0 \\ \underline{2x_1 + 2x_2 = 16} \\ x_2 = 4 \end{array}$$

Sub x_1, x_2 in (4)
$-4 + 4 + S_2^2 - 5 = 0$
$S_2^2 = 5$

$x_1 + x_2 = 8$
$x_1 = 4$

The solution is $x_1 = 4$, $x_2 = 4$, $S_1^2 = 0$, $S_2^2 = 5$, $\lambda_1 = 2$
Max $z = 48$

2

This solution satisfies both the constraints & the conditions $\underset{2 \neq 0}{\lambda_1} S_1 = \lambda_2 S_2 = 0$ are also satisfied.

$\therefore$ $x = (4, 4)$ gives the maximum of the objective function Z.

3 · Max $z = 10x_1 - x_1^2 + 10x_2 - x_2^2$

Sub. to. $x_1 + x_2 \le 9$

$x_1 - x_2 \geq 6$ convert to $\le$ $-x_1 + x_2 \le -6$

$x_1, x_2 \geq 0$

constraints are

$$g_1(x) = x_1 + x_2 + s_1^2 - 9 = 0$$

$$g_2(x) = -x_1 + x_2 + s_2^2 + 6 = 0$$

The Lagrangian function is

$$L(x,s,\lambda) = 10x_1 - x_1^2 + 10x_2 - x_2^2 - \lambda_1(x_1 + x_2 + s_1^2 - 9) - \lambda_2(-x_1 + x_2 + s_2^2 + 6)$$

The Kuhn-Tucker necessary conditions are

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0 \text{ —— } (1)$$

$$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0 \text{ —— } (2)$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 9) = 0 \text{ —— } (3)$$

$$\frac{\partial L}{\partial \lambda_2} = -(-x_1 + x_2 + s_2^2 + 6) = 0 \text{ —— } (4)$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0 \text{ —— } (5)$$

$$\frac{\partial L}{\partial s_2} = -2\lambda_2 s_2 = 0 \text{ —— } (6)$$

case (i) $\lambda_1 = 0$ $\lambda_2 = 0$

in eq(1) $10 - 2x_1 = 0$, $x_1 = 5$ in eq(2) $10 - 2x_2 = 0$, $x_2 = 5$

in eq(3) $5 + 5 + s_1^2 - 9 = 0$, $s_1^2 = -1$ in eq(4) $-5 + 5 + s_2^2 + 6 = 0$, $s_2^2 = -6$

The solution is $x_1 = 5$ $x_2 = 5$ $s_1^2 = -1$ $s_2^2 = -6$ Max $z = 50$.

Since both the s_1^2, s_2^2 are negative the solution is infeasible.

case (ii) $S_2 = 0$ $\lambda_1 = 0$

In eq(1) $10 - 2x_1 + \lambda_2 = 0$ —— (7) In eq(2) $10 - 2x_2 - \lambda_2 = 0$ —— (8)

Solving (7),(8)

$$\begin{array}{l} 10 - 2x_1 + \cancel{\lambda_2} = 0 \\ 10 - 2x_2 - \cancel{\lambda_2} = 0 \\ \hline 20 - 2x_1 - 2x_2 = 0 \end{array} \quad \text{—— (9)}$$

$S_2 = 0$

In eq (4) $-x_1 + x_2 + 6 = 0$ —— (10)

Solving (9) (10),

$$\begin{array}{l} -2x_1 - 2x_2 = -20 \\ -2x_1 + 2x_2 = -12 \\ \hline -4x_1 \qquad = -32 \\ \qquad x_1 = 8 \end{array}$$

Sub $x_1 = 8$ in (10)

$-8 + x_2 = 6$

$x_2 = 2$

Sub $x_1 = 8$ $x_2 = 2$ in eq (7)

$10 - 16 + \lambda_2 = 0$

$\lambda_2 = 6$

Sub $x_1 = 8$, $x_2 = 2$ in (3) $8 + 2 + S_1^2 - 9 = 0$

$S_1^2 = -1$

The solution is $x_1 = 8$, $x_2 = 2$, $S_1^2 = -1$, $\lambda_2 = 6$ Max $z =$ ~~50~~ 32.

Since S_1^2 is negative, It is not a feasible solution

Case (iii) $S_1 = 0$ $S_2 = 0$

Sub in (3) $x_1 + x_2 = 9$

(4) $-x_1 + x_2 = -6$

$2x_2 = 3$

$x_2 = 3/2$

$x_1 + \frac{3}{2} = 9$

$x_1 = 15/2$

(1) $10 - 2x_1 - \lambda_1 + \lambda_2 = 0$

(2) $10 - 2x_2 - \lambda_1 - \lambda_2 = 0$

$20 - 2x_1 - 2x_2 - 2\lambda_1 = 0$

$20 - 2\left(\frac{15}{2}\right) - 2\left(\frac{3}{2}\right) - 2\lambda_1 = 0$

$20 - 15 - 3 - 2\lambda_1 = 0$

$\lambda_1 = 1$

Sub in (1) $10 - 2\left(\frac{15}{2}\right) - 1 + \lambda_2 = 0$

$\lambda_2 = 6$

The Solution is $x_1 = 15/2$ $x_2 = 3/2$ $\lambda_1 = 1$ $\lambda_2 = 6$

Max $z = 31.50$

This solution does not violate any of the conditions. therefore the point $x = (7.5, 1.5)$ gives the maximum of the objective function z.

4. Determine x_1, x_2 so as to

Maximize $Z = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

Subject to $x_2 \le 8$

$x_1 + x_2 \le 10$

$x_1, x_2 \geq 0$

Note: Solving the problem using Kuhn-Tucker Necessary Cond. Option 2.

$\frac{\partial L}{\partial x_j} = 0$

$\lambda_i g_i(x) = 0$

$g_i(x) \le 0$

$\lambda_i \geq 0$

Here $f(x_1, x_2) = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

$g_1(x_1, x_2) = x_2 - 8 \le 0$

$g_2(x_1, x_2) = x_1 + x_2 - 10 \le 0$.

The Lagrangian function

$L(x, s, \lambda) = f(x) - \lambda_1[g_1(x) + S_1^2] - \lambda_2[g_2(x) + S_2^2]$

$= 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2 - \lambda_1(x_2 - 8 + S_1^2) - \lambda_2(x_1 + x_2 - 10 +$

The Kuhn Tucker Necessary condition

condition (i) $\frac{\partial L}{\partial x_j} = 0$

$\frac{\partial L}{\partial x_1} = 12 + 2x_2 - 4x_1 - \lambda_2 = 0$ ——— (1)

$\frac{\partial L}{\partial x_2} = 21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0$ ——— (2)

condition (ii)

$\lambda_i g_i(x) = 0$

$\lambda_1(x_2 - 8) = 0$ ——— (3)

$\lambda_2(x_1 + x_2 - 10) = 0$ ——— (4)

condition (iii) $g_i(x) \le 0$

$x_2 - 8 \le 0$ ——— (5)

$x_1 + x_2 - 10 \le 0$ ——— (6)

condition (iv) $\lambda_i \geq 0$

$\lambda_1 \geq 0$ ——— (7)

$\lambda_2 \geq 0$ ——— (8)

case (i) $\lambda_1 = 0 \quad \lambda_2 = 0$

(1) $12 + 2x_2 - 4x_1 = 0$

(2) $21 + 2x_1 - 4x_2 = 0$

$$\begin{array}{r} 24 + 4x_2 - 8x_1 = 0 \\ 21 - 4x_2 + 2x_1 = 0 \\ \hline 45 \qquad -6x_1 = 0 \end{array}$$

$x_1 = \frac{45}{6} = 15/2$

$12 + 2x_2 - 4(15/2) = 0$

$x_2 = 9$

Substitute $x_1 = 15/2$, $x_2 = 9$ in all 4 conditions

This solution violates eq. (5) and eq (6).

So discarded.

case (ii) $\lambda_1 \neq 0$ $\lambda_2 \neq 0$

from (3) $x_2 - 8 = 0$
$x_2 = 8$

from (4) $x_1 + x_2 - 10 = 0$
$x_1 + 8 - 10 = 0$
$x_1 = 2$

Substitute $x_1 = 2$, $x_2 = 8$ in ~~all~~ 1st conditions. we get $\lambda_1 = -27$, $\lambda_2 = 20$

This solution violates eq. (7) and ∴ discarded.

case (iii) $\lambda_1 \neq 0$ $\lambda_2 = 0$

from (3) $x_2 - 8 = 0$ $x_2 = 8$

from (1) $2x_2 - 4x_1 = -12$
$2(8) - 4x_1 = -12$
$x_1 = 7$

from (2) $21 + 2x_1 - 4x_2 - \lambda_1 - \lambda_2 = 0$

$x_1 = 7$
$x_2 = 8$
$\lambda_2 = 0$

$21 + 2(7) - 4(8) - \lambda_1 = 0$
$-\lambda_1 = -3$
$\lambda_1 = 3$

Substitute $x_1 = 7$ $x_2 = 8$ in 6th eq. It is violated.

So it is discarded

case (iv) $\lambda_1 = 0$ $\lambda_2 \neq 0$

from (4) $x_1 + x_2 = 10$

from (1) $2x_2 - 4x_1 - \lambda_2 = -12$

from (2) $\mp 4x_2 \pm 2x_1 \mp \lambda_2 = \mp 21$

$$6x_2 - 6x_1 = 9$$

([illegible]) × 2

$$6x_2 + 6x_1 = 60$$

$$12x_2 = 69$$
$$x_2 = 69/12 = 23/4$$

Sub $x_2 = 23/4$ in
$x_1 + x_2 = 10$
$x_1 + \frac{23}{4} = 10$
$x_1 = 10 - \frac{23}{4}$
$x_1 = 17/4$

Sub $x_1 ; x_2$ in
$2x_2 - 4x_1 - \lambda_2 = -12$
$2(23/4) - 4(17/4) - \lambda_2 = -12$
$\lambda_2 = 12 + \frac{46}{4} - \frac{51}{4} = 13/4$

Substitute $x_1 = 17/4$, $x_2 = 23/4$ $\lambda_1 = 0$
$\lambda_2 = 13/4$ Max. $Z = 1734/16$

The values does not violate any condition.

optimum solution Max. $Z = 867/8$
$x_1 = 17/4$ $x_2 = 23/4$

UNIT V - PROJECT MANAGEMENT

5.1

PERT & CPM

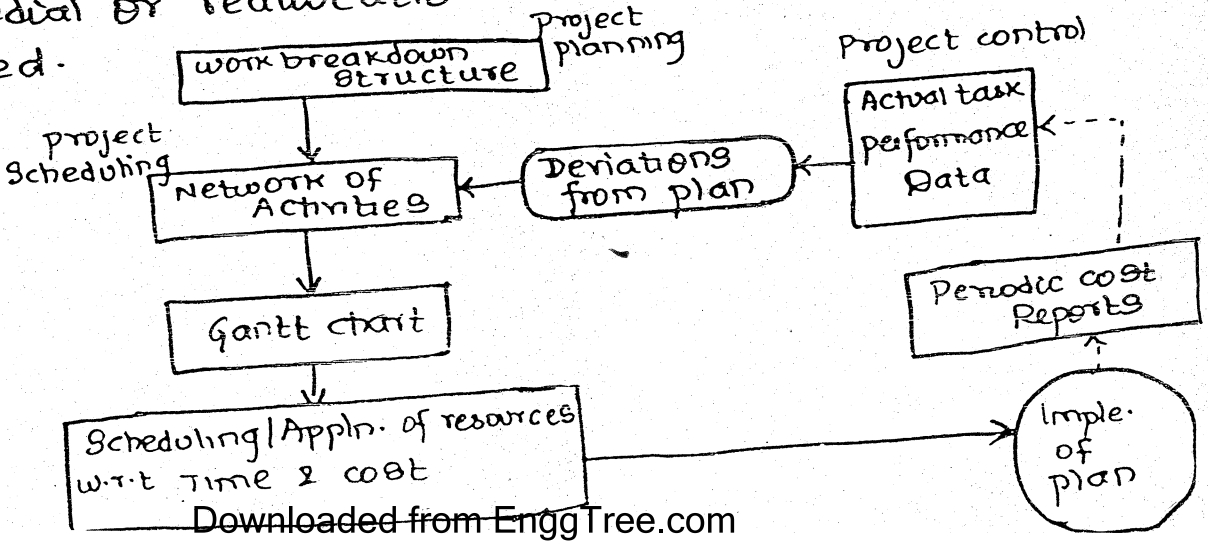
Project Management is to schedule activities associated with any project in an efficient manner so as to complete the project on or before a specified time limit & at the minimum cost.

- Network Analysis, Network Planning & Scheduling Techniques are used for planning, scheduling & controlling large & complex projects.
- A network is a graphical presentation of arrows & nodes for showing the logical sequence of various activities to be performed to achieve project objectives.
- Two Techniques
  - PERT [Program Evaluation and Review Technique]
  - CPM [Critical Path Method]

Phases of Project Management

1. Project Planning Phase :- Identify various activities, determines requirement of resources, assign responsibilities, Allocate resources, Estimate cost & time, etc.

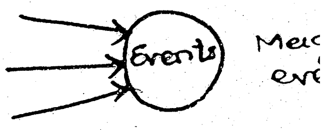
2. Scheduling Phase :- Identify people responsible for task, Estimate the expected duration of each activity, develop a network diagram, calculate project duration, identify critical path etc.
3. Project control phase :- Evaluation of the actual progress against the plan. If significant difference is found, then remedial or reallocation of resources measures are adopted.



PERT/CPM Network components and precedence Relationships

- 2 major components : Events , Activities

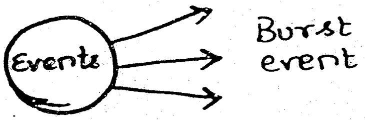
EVENTS : represents project milestones like start/completion of an activity or activities.



Events represented by circles.

Further classified into two

(1) Merge Event - represents the joint completion of more than one activity



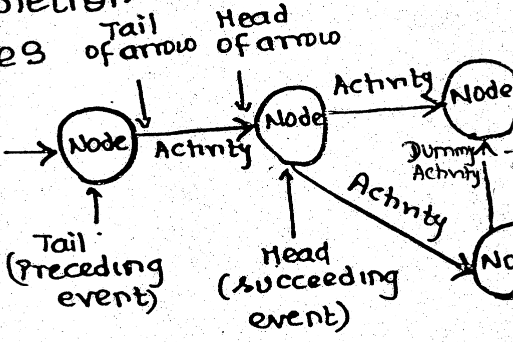
(2) Burst Event - represents the initiation of more than one activity.

- Events identified by numbers. Event number should be higher than the one allocated to its predecessor event.

ACTIVITIES : represents project operations. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the activity.

arrow (i,j) between 2 events, tail event i represents the start & head event j represents the completion.

Further classified into three categories



(1) Predecessor Activity : An activity which must be completed before 1 or more other activities start.

(2) Successor Activity : An activity which starts immediately after one or more of other activities are completed.

(3) Dummy Activity : An activity which does not consume either any resource and/or time.

PRECEDENCE NETWORK : Two types

// Activity-on-Node (AON) Network : Each node represents a specific task while arcs represent the ordering between tasks.

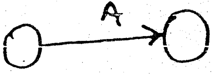
Lack of dummy activities.

Activity-on-Arrow (AoA) Network : Each end of activity arrow is a node. Nodes represent points in time or instants.

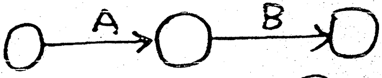
Rules for Network construction

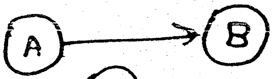
AOA AON

1) Activity A

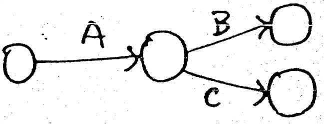


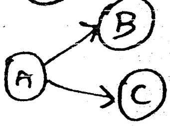
2) B must follow A



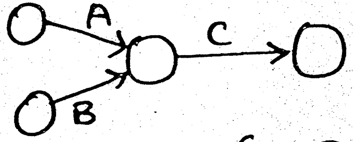


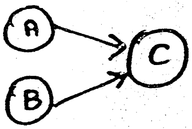
3) B & C must follow A



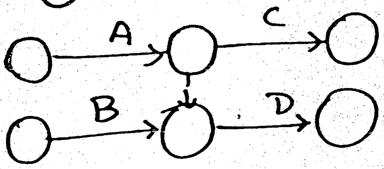


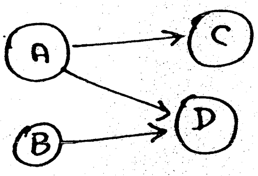
4) C must follow A & B





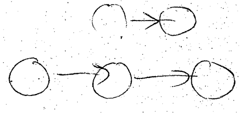
5) C must follow A & D must follow A & B

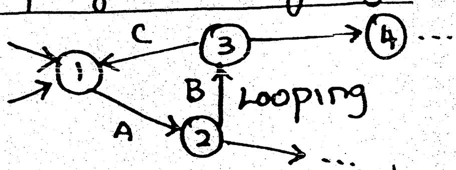


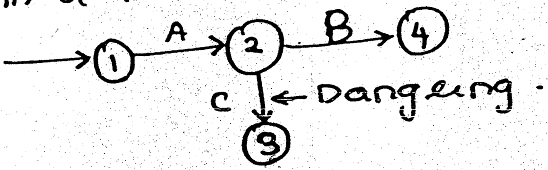


Errors & Dummies in N/W

Looping & Dangling - Faults in a network.





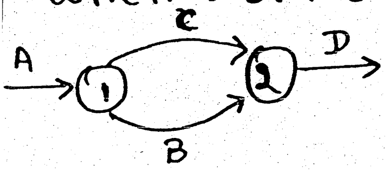


case of endless loop - Looping

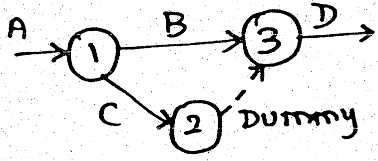
Case of disconnect activity before completion of all activities.

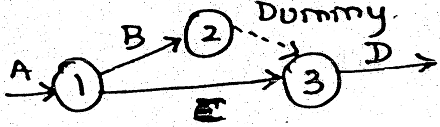
Dummy (or Redundant activity) - Use of dummy activity in 2 cases

- when 2 or more parallel activities have the same head & tail) events.

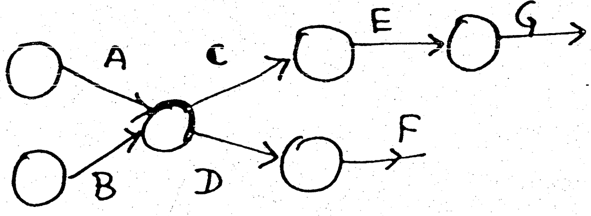


Parallel Activity

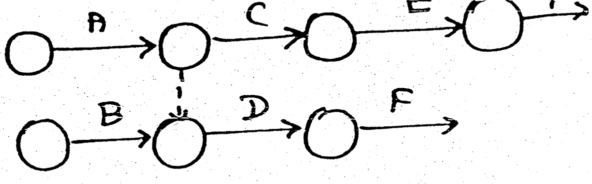




- when 2 chains of activities have a common event, yet are completely or partly independent of each other. also called Logical Dummy Activity.



Dependent Events



Independent events.

FORD - FULKERSON RULE

Nodes may be numbered using ford & fulkerson's

1) Number the start node which has no predecessor activity, as 1

2) Delete all the activities emanating from this nod

3) Number all the resulting start nodes without predecessor as 2,3,...

4) Delete all the activities originating from start of node 2,3,... in step 3.

5) Number all the resulting new start nodes with any predecessor next to the last number used in step (3).

6) Repeat the process until the terminal node with any successor activity is reached and number th terminal node suitably.

Problem 1: If there are 5 activities P, Q, R, S, T s that P, Q, R have no immediate predecessors b S & T have immediate predecessors P, Q an Q, R respectively. Represent by a network.

Problem 2: Prepare a network arrow diagram

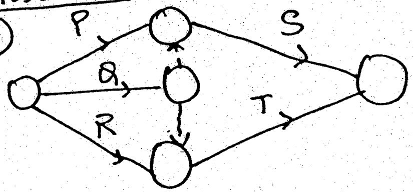
Activity	Name of the Activity	Predecessor Activity
1-2	A	–
1-3	B	–
1-4	C	–
2-5	D	A
3-5	E	B
4-6	F	C
5-7	G	D, E
3-6	H	B
6-7	I	H, F

Problem 3

Act :	A	B	C	D
pred :	-	-	-	A, B

Answers

①



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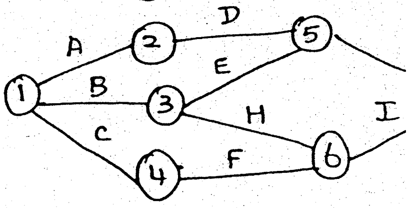

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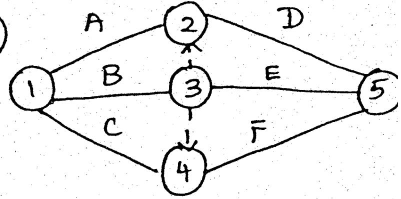


4. Construct the network diagram.

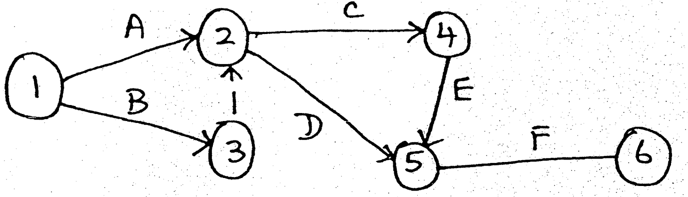
Activity & the precedence relationships given below

A< C ,D ; B <C,D ; C<E; D,E< F

Soln

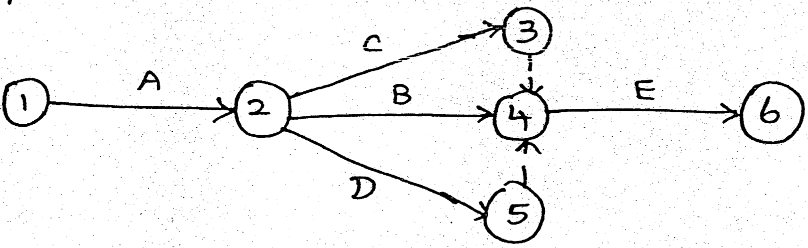
Activity :	A	B	C	D	E	F
Predecessor :	–	–	A,B	A,B	C	D, E

A<C,D
C, D predecessor is A
B < C,D
C,D predecessor is B
C< E
E predecessor is C
D,E<F
F predecessor is D,E



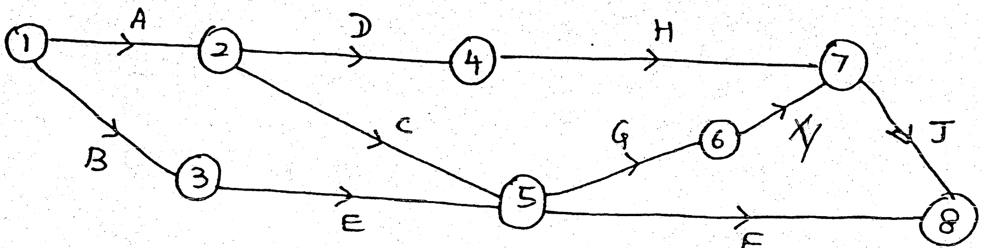
5)

Activity	A	B	C	D	E
Immediate predecessor	–	A	A	A	B,C,D



) A<C,D ; B<E; C,E<F ,G; D<H; G<I; H,I < J

Activity	A	B	C	D	E	F	G	H	I	J
Immediate Predecessor	–	–	A	A	B	C,E	C,E	D	G	H,I



CRITICAL PATH ANALYSIS

- Used to estimate the total project duration & to assign Starting & finishing time to all activities involved in the proj
- Factors prepared for project Scheduling
  - * Total completion time of the project
  - * Earlier & latest start time of each activity.
  - * Critical activities & critical path
  - * float for each activity

Notations

E_i - Earliest occurrence time of an event, i.

L_i - Latest allowable time of an event, i.

ES_{ij} - Early starting time of an activity (i,j)

LS_{ij} - Latest starting time of an activity (i,j)

EF_{ij} - Early finishing time of an activity (i,j)

LF_{ij} - Late finishing time of an activity (i,j)

Network diagram should have only 1 initial & end even

Forward & Backward pass used to calculate earliest occurrence & latest allowable time.

FORWARD PASS METHOD (For Earliest Event Time)

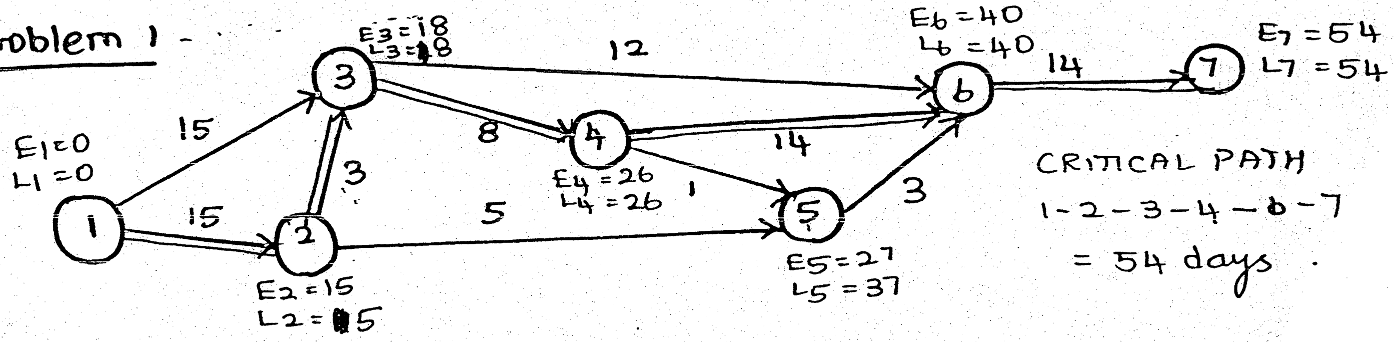
1. Set earliest occurrence of initial event 1 to zero. $E_1 = 0$ f
2. Calculate earliest start time for each activity that beg at event i. = Earliest occurrent time of event, i (tail) event.
   $ES_{ij} = E_i$, for all activities (i,j) starting at event i
3. Calculate earliest finish time of each activity that begin at event i. = Earliest start time plus duration.
   $EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij}$ for all activities (i,j) begg. at ever
4. Proceed to next event, say j; $j > i$
5. Calculate earliest occurrence time for the event j.
   = Max. of earliest finish times of all act. ending into that eve
   $E_j = \text{Max}\{EF_{ij}\} = \text{Max}\{E_i + t_{ij}\}$ for all immediate predecess
6. If $j = N$ (Final Event), then earliest finish time for the project is earliest occurrence time E_N for final event.
   $E_N = \text{Max}\{EF_{ij}\} = \text{Max}\{E_{N-1} + t_{ij}\}$ for all terminal activ

BACKWARD PASS METHOD (for Latest Allowable Event Time)

- Calculations begin from the final event N.

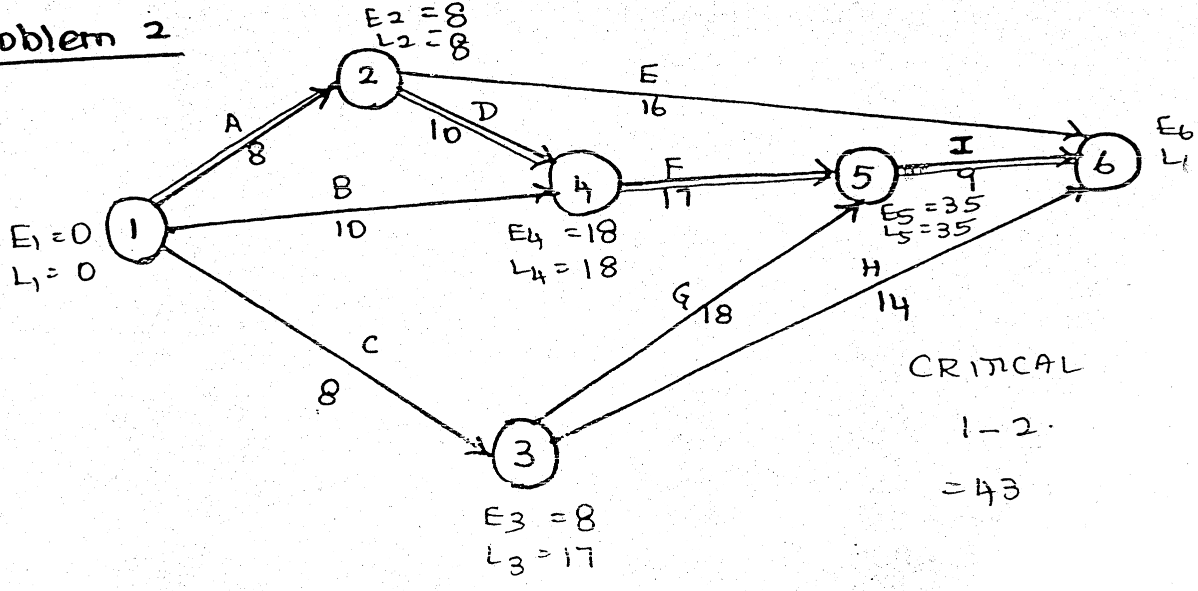
1. Set the latest occurrence time of last event, N equal to its earliest occurrence time. $L_N = E_N$ for $j = N$
2. Calculate the latest finish time of each activity which ends at event j. = latest occurrence time of final event
   $LF_{ij} = L_j$, for all activities (i, j) ending at event j
3. Calculate the latest start time of all activities ending at j. = Subtract the duration of act. from latest finish time of the act
   $LF_{ij} = L_j$ and $LS_{ij} = LF_{ij} - t_{ij} = L_j - t_{ij}$ for all act (i, j) ending at j
4. Proceed backward to the event in the seq, that decreases j by
5. Calculate the latest occurrence time of i (i<j). Calculate the latest occurrence time of event. = minimum of the latest start times of all activities from the event.
   $L_i = Min\{LS_{ij}\} = Min\{L_j - t_{ij}\}$, for all immediate successor act.
6. If j = 1 (initial event), then latest start time for proj. ie. latest occurrence time L_1 is
   $L_1 = Min\{LS_{ij}\} = Min\{L_{j-1} - t_{ij}\}$ for all imm. succ. act.

Problem 1



CRITICAL PATH
1-2-3-4-6-7
= 54 days.

Problem 2



CRITICAL
1-2.
= 43

FLOAT (SLACK) OF AN ACTIVITY & EVENT

- float (slack) or free time is the length of time in which a non-critical activity and/or an event can be delayed or extended without delaying the total project completion Time.

Slack of an Event : Difference between its latest occurre time (L_i) and its earliest occurrence time (E_i)

Event float = $L_i - E_i$

(ie) is a measure of how long an event can be delayed without increasing the project completion time.

if $L = E$, CRITICAL EVENTS

if $L \neq E$, float can be negative ($L<E$) or positive ($L>$

Slack of an Activity : Amount of activity time that can b delayed or increased without delaying project completion ti

-Difference between latest finish time & earliest finish time fo the activity.

TOTAL FLOAT : Length of time by which an activity can be delayed until all preceding activities are completed a their earliest possible time & all successor activities c be delayed until the latest permissible time

Total float of an activity (i-j) = $(LF)_{ij} - (EF)_{ij}$
or $(LS)_{ij} - (ES)_{ij}$

Free Float : It is that portion of the total float which can be used for rescheduling that activity without affecting the succeeding activity

Free Float of an activity (i-j) = Total float of (i-j) − (L−E) of event j

Total Float ≤ free float

Independent float : It is the amount of time by which the activity can be rescheduled without affecting the preceeding or succeeding activities of that activity.

Independent float of an activity (i-j) = Free Float of (i-j) − (L−E) of event

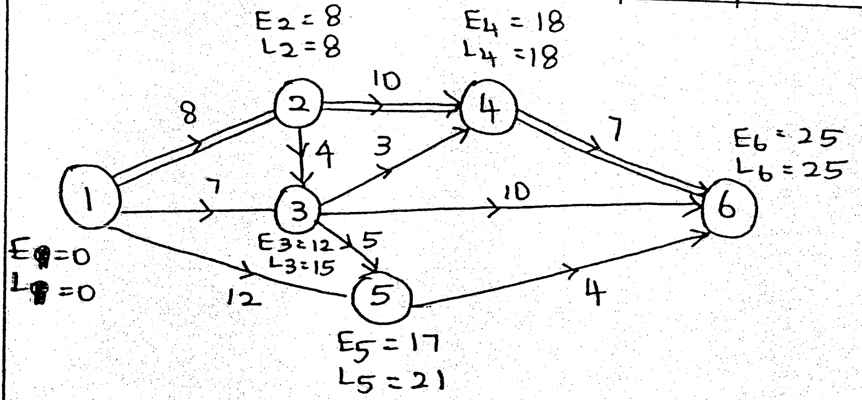
Interfering float or Interference float of an activity (i-j) is nothing but the slack of head event j

Interfering float of i-j = Total float of (i-j) − Free float of (i

CPM - CRITICAL PATH METHOD Problems.

1. Calculate the free Float, total and independent float for the project management whose activities are given

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6	5-6
Duration (weeks)	8	7	12	4	10	3	5	10	7	4



Forward Pass : Calculation of Earliest start of event.

$E_1 = 0$ (initialize)

$E_2 = E_1 + 8 = 0 + 8 = 8$

$E_3 = Max(E_1 + 7, E_2 + 4) = Max(0+7, 8+4) = 12$

$E_5 = Max(E_1 + 12, E_3 + 5) = Max(0+12, 12+5) = 17$

$E_4 = Max(E_2 + 10, E_3 + 3) = Max(8+10, 12+3) = 18$

$E_6 = Max(E_3 + 10, E_4 + 7, E_5 + 4) = Max(12+10, 18+7, 17+4) = 25$

Backward Pass : Calculation of latest start of event

Initialize $L_6 = E_6 = 25$

$L_5 = L_6 - 4 = 25 - 4 = 21$

$L_4 = L_6 - 7 = 25 - 7 = 18$

$L_3 = Min(L_6 - 10, L_4 - 3, L_5 - 5) = Min(25-10, 18-3, 21-5) = 15$

$L_2 = Min(L_4 - 10, L_3 - 4) = Min(18-10, 15-4) = 8$

$L_1 = Min(L_5 - 12, L_3 - 7, L_2 - 8) = Min(21-12, 15-7, 8-8) = 0$

CRITICAL Path: Path connecting the 1st initial node to the very last terminal node of longest duration in any project duration.

For critical activities E, L values will be same. The current critical activity(i-j)'s $E_j = L_j$ $\overset{E_i = L_i}{\wedge}$ and also $E_j = E_i +$ duration of (i-j)

CRITICAL PATH = 1-2-4-6 = 25 days

Example
Critical activity (2-4)
$E_2 = L_2 = 8$
$E_4 = L_4 = 18$
Also $E_4 = E_2 + [illegible]0 = 8+1$
$= 18$

Activity	Duration	Earliest		Latest		Floats		
		Start ES	Finish EF	Start LS	Finish LF	Total Float	Free float	Independent float
1-2	8	0	8	0	8	0	0	0
1-3	7	0	7	8	15	8	5	5
1-5	12	0	12	9	21	9	5	5
2-3	4	8	12	11	15	3	0	0
2-4	10	8	18	8	18	0	0	0
3-4	3	12	15	15	18	3	3	0
3-5	5	12	17	16	21	4	0	0
3-6	10	12	22	15	25	3	3	0
4-6	7	18	25	18	25	0	0	0
5-6	4	17	21	21	25	4	4	0

Total Float = (LF − EF)

TF of 1-2 = 8 − 8 = 0 TF of 1-3 = 15 − 8 = 7

TF of 1-5 = 21 − 12 = 9 TF of 2-3 = 15 − 12 = 3

...

Free float of (i-j) = Total Float of (i-j) − (L−E) of event j

FF of (1-2) = TF of 1-2 − $(L_2 - E_2)$ of 2 = 0 − 0 = 0

FF of (1-3) = TF of 1-3 − $(L_3 - E_3)$ = 8 − 3 = 5

FF of (1-5) = TF of 1-5 − $(L_5 - E_5)$ = 9 − 4 = 5

FF of (2-3) = TF of 2-3 − $(L_3 - E_3)$ = 3 − 3 = 0

⋮

Independent float of (i-j) = free float of (i-j) − (L − E) of event i

IF of (1-2) = FF of (1-2) − $(L_1 - E_1)$ = 0 − 0 = 0

IF of (1-3) = FF of (1-3) − $(L_1 - E_1)$ = 5 − 0 = 0

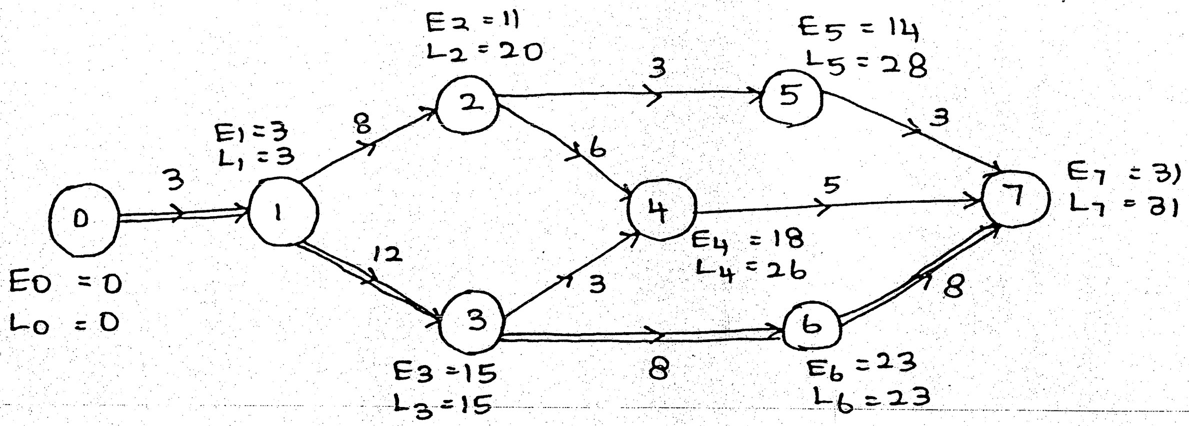
IF of (1-5) = FF of (1-5) − $(L_1 - E_1)$ = 5 − 0 = 0

IF of (2-3) = FF of (2-3) − $(L_2 - E_2)$ = 0 − 0 = 0

⋮

2) Construct the network for the project & compute the total, free and Independent float of each activity, also the Critical path & project duration

Activity	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7
Duration (weeks)	3	8	12	6	3	3	8	5	3	8



CRITICAL PATH = 0-1-3-6-7 = 31 weeks

Activity	Duration weeks	Earliest Start ES	Earliest Finish EF	Latest Start LS	Latest Finish LF	Total Float	Free Float	Independent Float
0-1	3	0	3	0	3	0	0	0
1-2	8	3	11	12	20	9	0	0
1-3	12	3	15	3	15	0	0	0
2-4	6	11	17	20	26	9	1	-8 ≃ 0
2-5	3	11	14	25	28	14	0	-9 ≃ 0
3-4	3	15	18	23	26	8	0	0
3-6	8	15	23	15	23	0	0	0
4-7	5	18	23	26	31	8	8	0
5-7	3	14	17	28	31	14	14	0
6-7	8	23	31	23	31	0	0	0

3. A project consists of a series of tasks labelled A, B, - - -, H, I with the following relationship. (w < x, y means x & y cannot start until w is completed; x, y < w means w cannot start until both x, y are completed) with this notation construct the network diagram.

A < D, E B, D < F C < G B < H F, G < I.

Find also the minimum time of completion of the project, when the time (in days) of completion of each task.

Task	A	B	C	D	E	F	G	H	I
Time	23	8	20	16	24	18	19	4	10

Solution : i) A < D, E   A is predecessor of D, E

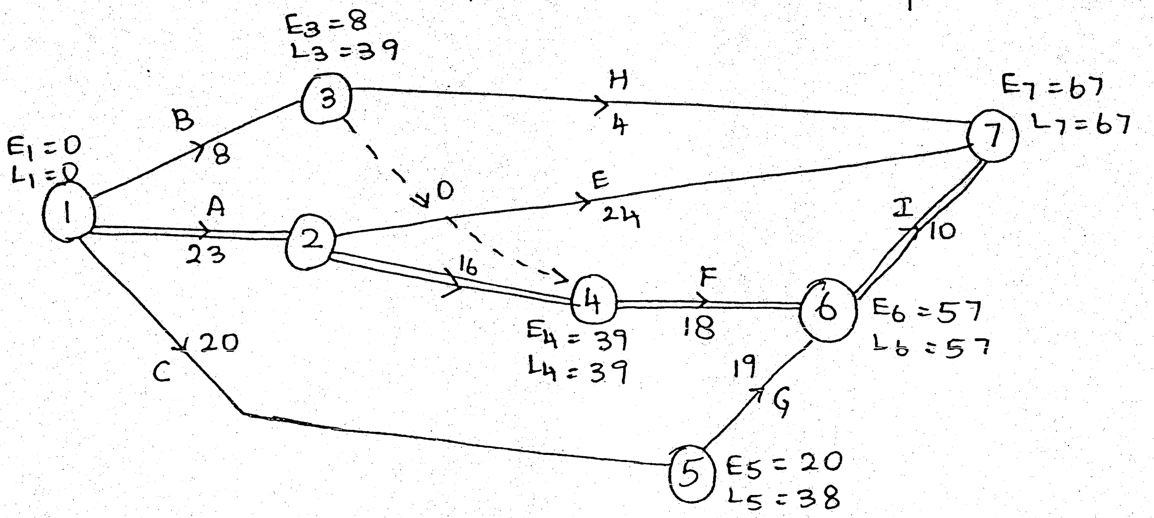
(ii) B, D < F   B, D is predecessor of F

(iii) C < G   C predecessor of G

(iv) B < H   B predecess of H

(v) F, G < I   F, G predecessor of

	Activity	Predecessor	Duration
1-2	A	–	23
1-3	B	–	8
1-4	C	–	20
2-5	D	A	16
2-6	E	A	24
3-6	F	B, D	18
4-7	G	C	19
	H	B	4
	I	F, G	10

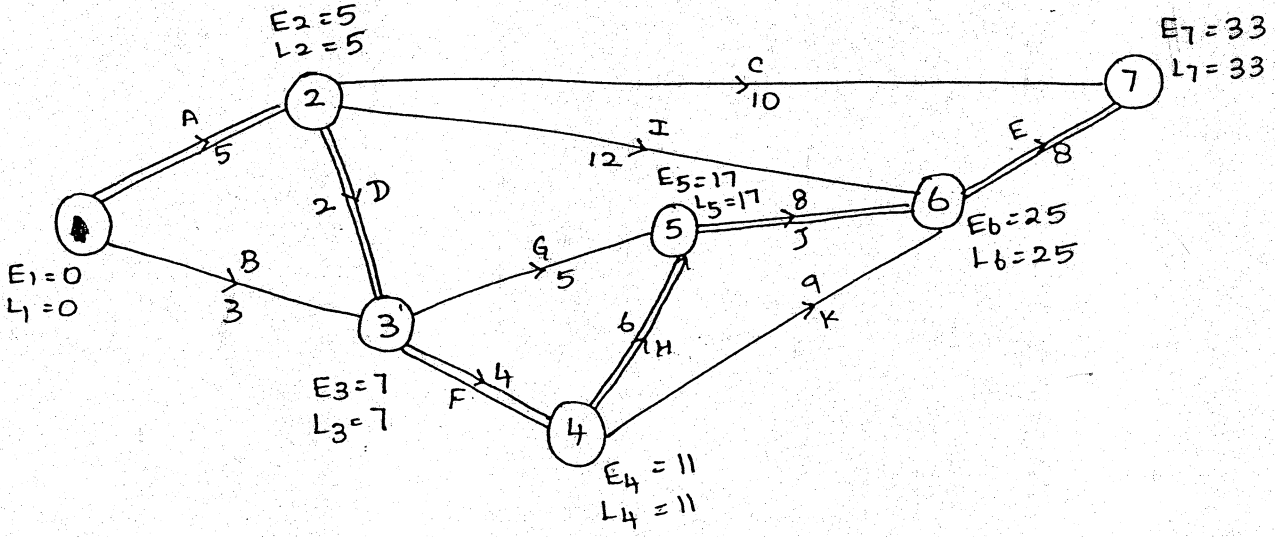


CRITICAL PATH = 1-2-4-6-7 = 67 days.

4. The following refers to a project network

	Activity	Precedence Activity	Duration
1-2	A	–	5
1-3	B	–	3
2-4	C	A	10
2-5	D	A	2
	E	I, J, K	8
	F	B, D	4
	G	B, D	5
	H	F	6
2-6	I	A	12
	J	G, H	8
	K	F	9

Draw the network for the project. Find the critical path



CRITICAL PATH = 1-2-3-4-5-6-7

= 33 days

Using CPM find the critical path and the minimum time for completion of the project whose details are given below.

Activities A and B can start simultaneously each taking 15 days. Activity C can start after 7 days and activity D after 5 days of starting the activity A.

Activity D can start after 4 days of starting activity C and 7 days of starting activity B.

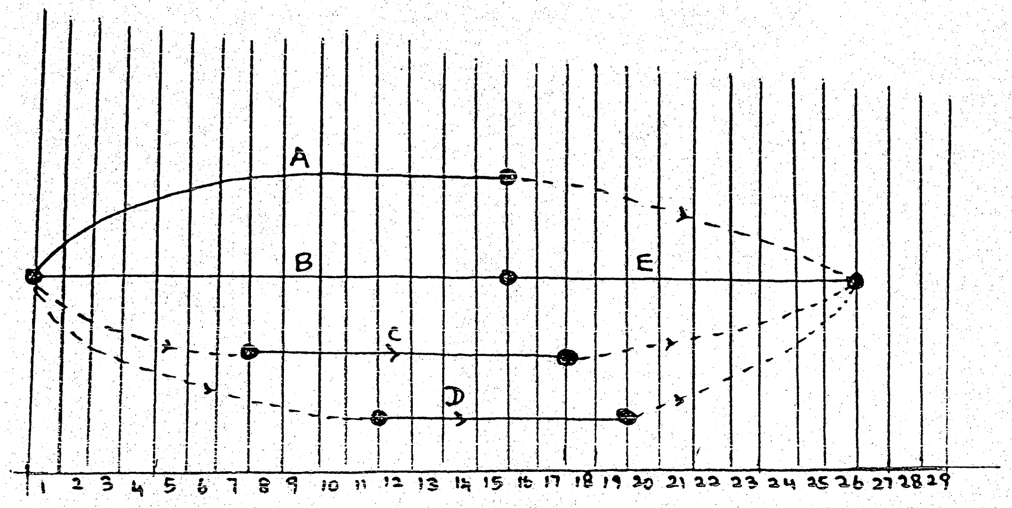
Activity E can start after activity A is one third finished and activity B is completely finished.

Activities C, D, E can take 10, 8 & 11 days respectively.

Solution

Duration	Activity		Starting time
15 15	A B	- Starts simultaneously each 15 days	0 0
10	C	- can start after 7 days of start of act. A = (0+7=7)	7
8	D	- can start after 5 days of start of act. A (0+5=5) - can start after 4 day of act C. (7+4=11) - can start after 7 days of start of act B = (0+7=7)	max = (5,11,7) = 11
11	E	- can start after activity A is 1/3 finished = $15 \times \frac{1}{3} = 5$ can start after activity B completed finished = 15 max (5,15) = 15	

Activity	A	B	C	D	E
Duration	15	15	10	8	11
Starting time	0	0	7	11	15



CRITICAL PATH - B-E completion time = 26 Days.

Programming Evaluation Review Technique : PERT

This technique, unlike CPM, takes into account the uncertainty of project durations into account. PERT calculations depend upon the following three time estimates.

Optimistic (least) time estimate : (to (or) a) is the duration of any activity when everything goes on very well during the project (i.e.,) labourers are available and come in time, machines are working properly, money is available whenever needed, there is no scarcity of raw material needed, etc.,.

Pessimistic (greatest) time estimate : (tp (or) b) is the duration of any activity when almost every thing goes against our will and a lot of difficulties is faced while doing a project.

Most likely time estimate : (tm (or) m) is the duration of any activity when sometimes things go on very well, sometimes things go on very bad while doing the projec

Two main assumptions made in PERT calculations are

(*). The activity durations are independent (i.e.,) the tim required to complete an activity will have no bearing the completion times of any other activity of the pro

(*). The activity durations follow β distribution.

β-distribution is a probability distribution with density function $K(t-a)^{\alpha}(b-t)^{\beta}$ with mean, $t_e = \frac{1}{3}\left[2t_m + \frac{1}{2}(t_o - t_p)\right]$ and the standard deviation $\sigma_t = \frac{t_p - t_o}{6}$.

PERT PROCEDURE :-

(1). Draw the project network.

(2). compute Expected duration of each activity $t_e = \frac{t_o + 4t_m + t_p}{6}$.

(3). Compute the Expected Variance, $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$ of each activity.

(4). compute earliest start, earliest finish, latest start, latest finish and total float.

(5). Determine critical path and identify critical activities.

(6). compute the expected standard deviation of the project length σ_c & calculate standard normal deviate $\frac{T_S - T_E}{\sigma_c}$

T_S = Scheduled time to complete project.

T_E = Normal expected project duration.

σ_c = Expected standard deviation of project length.

(7). compute expected variance σ_c^2.

(8). Using (6) one can estimate the probability of completing the project within a specified time, using normal curve.

PERT PROBLEMS

1. The following table indicates the details of a project. The durations are in days. 'a' refers to optimistic time, 'm' ref to most likely time and 'b' refers to pessimistic time du

	Activity	1-2	1-3	1-4	2-4	2-5	3-5	4-5
to	a	2	3	4	8	6	2	2
tm	m	4	4	5	9	8	3	5
tp	b	5	6	6	11	12	4	7

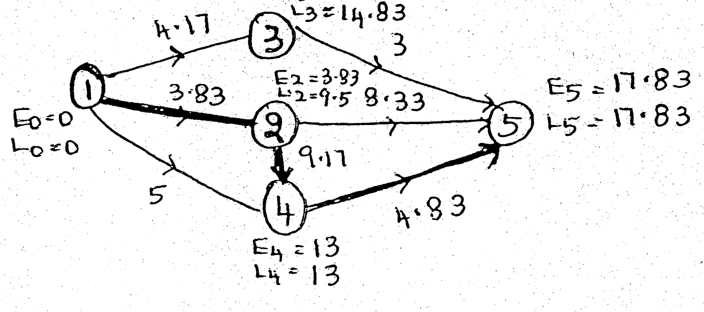
a) Draw the network

b) find the critical path

c) Determine the expected standard deviation of the completion time

Solution: Expected Duration $t_e = \frac{t_o + 4t_m + t_p}{6}$ Expected Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$

Activity	a	m	b	Expected duration te	Expected variance σ^2
1-2	2	4	5	3·83	1/4
1-3	3	4	6	4·17	1/4
1-4	4	5	6	5	1/9
2-4	8	9	11	9·17	1/4
2-5	6	8	12	8·33	1
3-5	2	3	4	3	1/9
4-5	2	5	7	4·83	25/36



CRITICAL PATH = 1-2-4-5

= 17·83 days

Expected variance of project completion σ_c^2 $= \frac{1}{4} + \frac{1}{4} + \frac{25}{36}$

Expected std-deviation of the completion time σ_c $= \sqrt{\frac{43}{36}} = 1·09$

A project consists of the following activities and time estimates

Activity	1-2	[illegible]-3	1-4	2-5	2-6	3-6	4-7	5-7	6-7
Least time days	3	2	6	2	5	3	3	1	2
Greatest time days	15	14	30	8	17	15	27	7	8
Most likely time days	6	5	12	5	11	6	9	4	5

a) Draw the network

b) what is the probability that the project will be completed in 27 days?

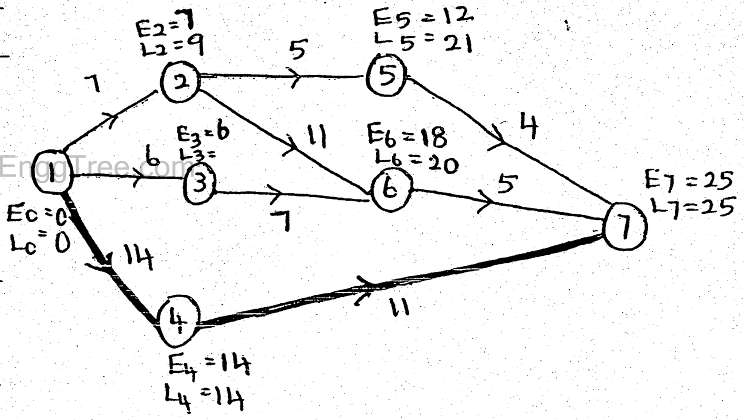
c) what is the chance that due date is 90% met.

Least time = Optimistic time = to

Greatest time = Pessimistic time = tp

Most Likely time = tm.

Activity	to	tp	tm	Expected time te	expected variance σ^2
1-2	3	15	6	7	4
[illegible]-3	2	14	5	6	4
1-4	6	30	12	14	16
2-5	2	8	5	5	1
2-6	5	17	11	11	4
3-6	3	15	6	7	4
4-7	3	27	9	11	16
5-7	1	7	4	4	1
6-7	2	8	5	5	1



a) CRITICAL PATH = 1-4-7 = 25 days

Expected var. of project length σ_c^2 = Sum of the expected variances of all critical activities = 16+16 = 32

Expected std-deviation of project length = $\sqrt{\sigma_c^2}$ = 5·656

b) Probability that project will be completed in 27 days

$$Z = \frac{TS - TE}{\sigma c} = \frac{27-25}{5·656} = 0·35$$

$$P(Ts \le 27) = P(Z \le 0·35) = 63·7\%$$

(From table (Nor. dist)
or using calculator) ✓

c) Percentage = 90%. Find the z value using normal distribution table for ·9 = ·5 + 0·4 · From table ·3997 is 1·28

z = 1·28

$$1·28 = \frac{TS - TE}{\sigma c} \Rightarrow 1·28 = \frac{TS - 25}{5·656} \quad TS = 1·28 \times 5·656 + 25 = 32·24$$

32·24 days Approximately needed for 90% chance.

Resource Levelling

These programs attempt to reduce the peak resourc requirements and smooth out period to period assignme without changing the constraint on project duration.

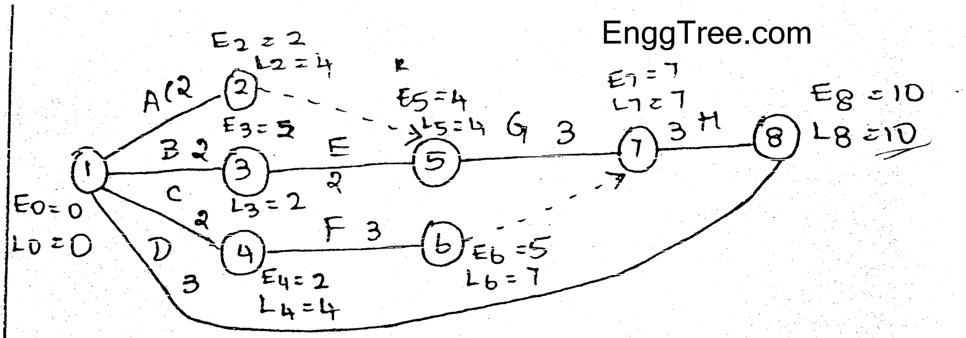
Using the resource requirements data of the earl start schedule, the program attempts to reduce peak resource requirements by shifting jobs with slack to non-peak periods. Resource limits are not specif but peak requirements are levelled as much as possib without delaying the specified due date.

Steps:

1) Draw the early start schedule graph.
2) Draw the corresponding manpower chart
3) Identify the activities with slack
4) Adjust the activities identified in step (3) & adjust th to level the peak resource requirements.

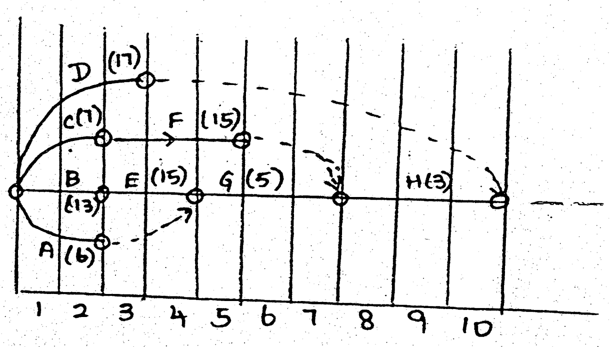
1) Draw the early start schedule graph of a project given below. The man power requirement for each activ is indicated in the paranthesis. Using resource levelling programming reduce the peak resource requirements.

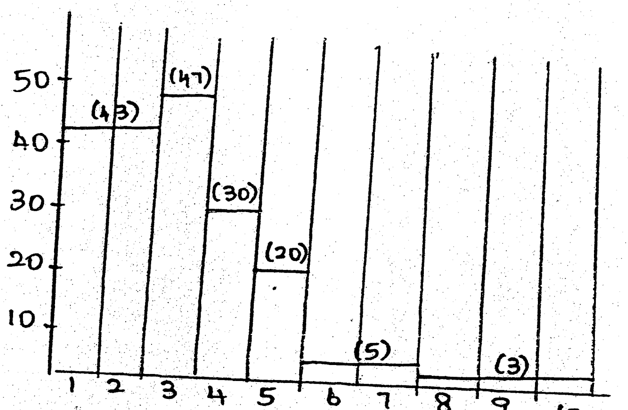
Activity	A	B	C	D	E	F	G	H
Predecessor	–	–	–	–	B	C	A, E	F, G
Duration	2	2	2	3	2	3	3	3
Manpower Requirement	(6)	(13)	(7)	(17)	(15)	(15)	(5)	(3)



Schedule Graph

Manpower chart





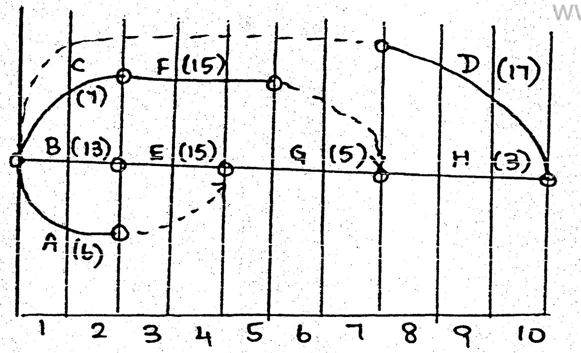
Manpowerchart in weeks.

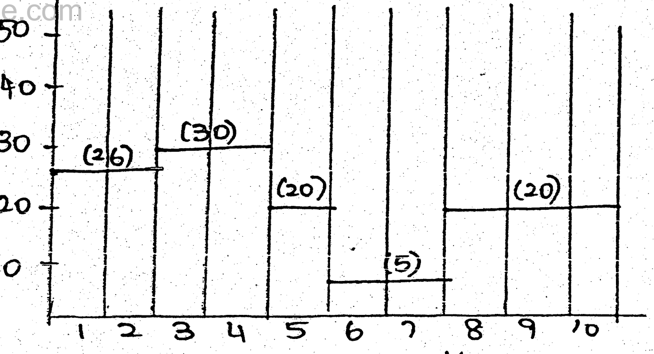
week 3 requires 47 men

week 8,9,10 requires only 3 men

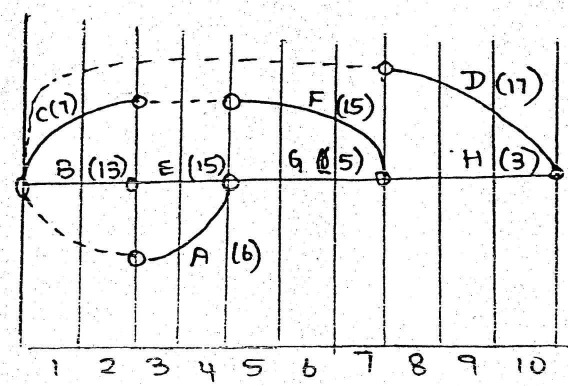
B-E-G-H - critical path

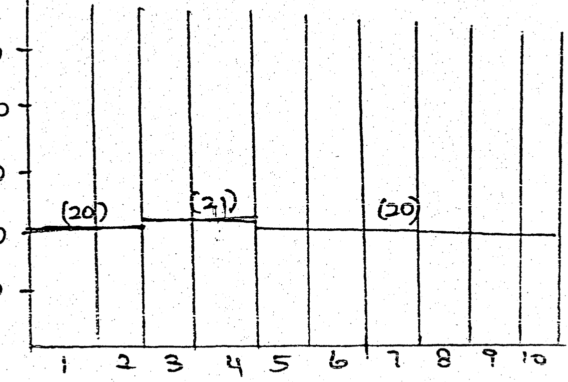
Since D has max. slack and peak resource requirements, it can be delayed by 7 weeks.





Max. manpower requirement is 30 men during 3rd & 4th week. Shift start of activity F by 2 weeks & also delay start of activity A by 2 weeks.



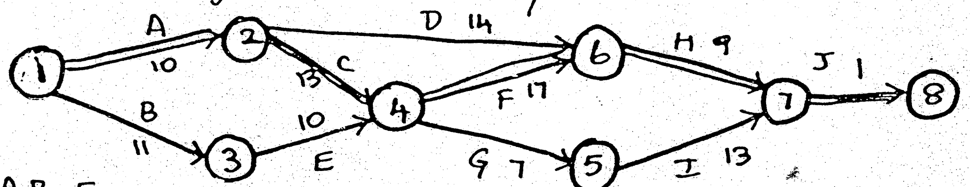


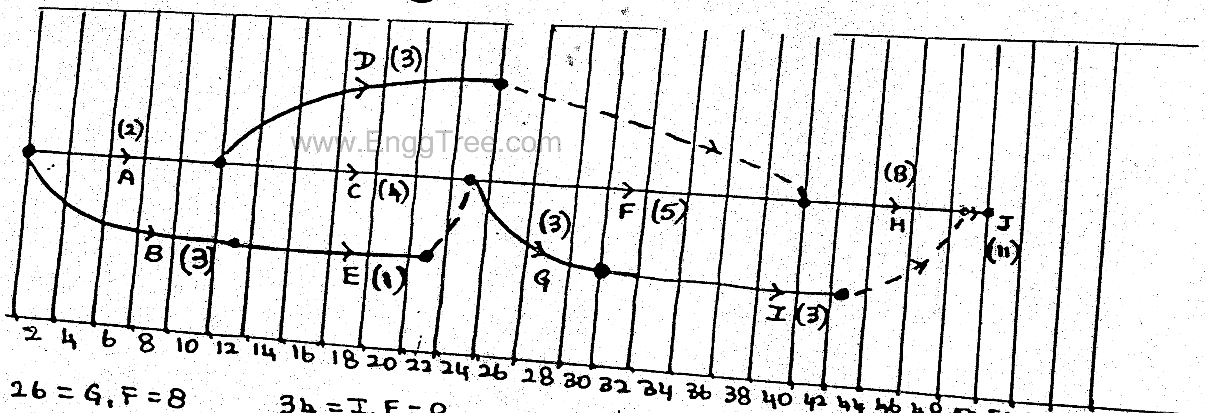
Levelling further is impossible. ∴ 20 men required on 1st, 2nd, 4th to 10th week. 21 men required during 3rd & 4th week.

② Manpower required for each activity of a project given belo

Activity	Normal Time (days)	Manpower req per day	Activity	Normal Time (days)	Manpower per day
A 1-2	10	2	G 4-5	7	3
B 1-3	11	3	F 4-6	17	5
C 2-4	13	4	I 5-7	13	3
D 2-6	14	3	H 6-7	9	8
E 3-4	10	1	J 7-8	1	11

Contractor stipulates that first 26 days, only 4 to 5 men & during the remaining days 8 to 11 men only are avail find whether it is possible to rearrange the activity for levelling the man-power resources.





2 - A, B - 5
4 - A, B = 5
6 - A, B = 5
8 - A, B = 5
10 - A, B = 5
11 - B, C, D = 10
12 - E, C, D = 8
14 - E, C, D = 8
16 = E, C, D = 8
18 = E, C, D = 8
20 = E, C, D = 8
21 = E, C, D = 8
22 = C, D = 7
23 = C, D = 7
24 = G, F, D = 11

26 = G, F = 8
28 = G, F = 8
30 = G, F = 8
32 = I, F = 8

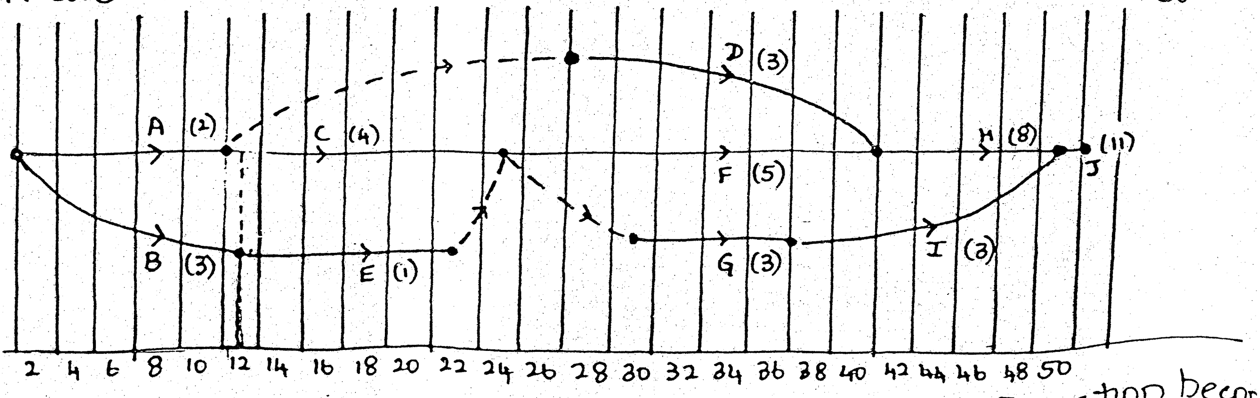
34 = I, F = 8
36 = I, F = 8
38 = I, F = 8
40 = I, F = 8

42 = I, H = 11
43 = I, H = 11
44 = H - 8
46 = H - 8

48 = 8
49 = 8
50 = 11

Project Duration = 50 days.

The final schedule graph is shown below. The activity D, I, have problem on 11th day where only 4 or 5 men are available, 7 men are needed. On all other days there is adequate manpo avail

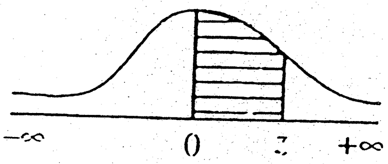


One solution is ⇒ Delay C by one day or pay overtime. ∴ Duration becor

APPENDIX

Table 1 : Area under A standard Normal distribution

An entry in the table is the area under the entire curve which is between $z = 0$ and a positive value of z as shown in the figure. Area for negative values of z are obtained by symmetry.



1	.00	.01	.02	.03	0.4	0.5	.06	0.7	.08	.09
00	.0000	.0040	.0080	.0120	.0160	0.019	0.239	0.279	0.319	0.359
01	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
02	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
03	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
04	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
05	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
06	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
07	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2853
08	.2881	.2901	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
09	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
10	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4392	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990