Unit - I.
MATRICES.

Matrix

A system of mn numbers (elements) arranged in a rectangular arrangement along m rows and n columns and bounded by the brackets [ ] or ( ) is called an m by n matrix, which is written as m×n matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Eg: $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 2 \end{pmatrix}$ is 3×4 matrix.

The order of a matrix is denoted by the number of its rows and columns.

Row & Column matrix:-

Row matrix:- A matrix having a single row is called row matrix

Eg: $[1 \; 2 \; 3]$

Column matrix:- A matrix having a single column is called column matrix. Eg: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Square matrix:

A matrix having n rows and n columns is called a square matrix of order n

Eg: $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 7 & 8 & 1 \end{pmatrix}$.

Trace of the matrix

let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. In this matrix 1, 5, 9 is called the diagonal of the matrix A. This diagonal is called the leading diagonal (or) main (or) Principal diagonal.

The sum of the diagonal elements of a square matrix A is called the trace of A.

Null or Zero matrix

In a matrix, if all the elements are zeros, then the matrix is called a null or zero matrix.

Eg: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Diagonal matrix.

In a square matrix, all the elements except elements in the main diagonal are zeros, then the matrix is called a diagonal matrix.

eg:- $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

~~Square~~ Scalar matrix :

A Square matrix in which all the elements of its leading diagonal are equal and the other elements are zeros is called a scalar matrix.

Eg:- $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Unit matrix or Identity matrix.

A diagonal matrix of order n which has unity for all its diagonal elements and zeros for other elements is called a unit matrix or an identity matrix of order n and is denoted by I_n.

Eg:- $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Upper triangular matrix.

A square matrix in which all the elements below the leading diagonal are zeros is called upper triangular matrix.

Eg:- $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{pmatrix}$

Lower triangular matrix.

A square matrix in which all the elements above the leading diagonal are zeros, is called lower triangular matrix.

Eg:- $A = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{pmatrix}$.

Transpose of a matrix.

The matrix got from any given matrix A, by interchanging its rows and columns is called the transpose of A and denoted by A' or A^T.

Eg: $A = \begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 1 \end{pmatrix}$

Symmetric matrix.

A matrix is symmetric if $A = A^T$.

Skew symmetric matrix.

A matrix is skew symmetric if $A = -A^T$.

Singular matrix.

If $|A| = 0$, then the square matrix A is said to be singular.

Non-Singular matrix.

If $|A| \neq 0$, then the square matrix A is said to be non singular.

Determinant of a matrix A, ie) |A|.

for 2x2 matrix, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1 \times 4) - (2 \times 3)$$
$$= 4 - 6$$
$$= -2$$

for 3x3 matrix, $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(6-2) - 0(3-2) + (-1)(2-4)$$
$$= 1(4) - 0 - 1(-2)$$
$$= 4 + 2 = 6.$$

Inverse of a matrix or Reciprocal matrix.

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A.$$

adj A,

for 2x2, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\operatorname{adj}(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

for 3×3 matrix.

Eg: $\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = A$

Adj (A) =

Cofactor of 2 $= +\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = (0-2) = -2$

Cofactor of −1 $= -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = -(0-2) = 2$

Cofactor of 0 $= +\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0-1 = -1$

Cofactor of 0 $= -\begin{vmatrix} -1 & 0 \\ 1 & 0 \end{vmatrix} = -(0-0) = 0$

Cofactor of 1 $= +\begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} = 0-0 = 0$

Cofactor of 2 $= -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -(2+1) = -3$

Cofactor of 1 $= +\begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} = (-2-0) = -2$

Cofactor of 1 $= -\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 0-4$.

Cofactor of 0 $= +\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2-0 = 2$

Adj (A) $= \begin{pmatrix} -2 & 2 & -1 \\ 0 & 0 & -3 \\ -2 & -4 & 2 \end{pmatrix}^T = \begin{pmatrix} -2 & 0 & -2 \\ 2 & 0 & -4 \\ -1 & -3 & 2 \end{pmatrix}$

Homework:

1) find $|A|$, if $A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$.

2) find the adjoint of the matrix

$$\begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 2 \\ 1 & 5 & 7 \end{pmatrix}$$

Equality of matrices.

Two matrices A and B are said to equal if and only if.

(i) they are of the same order and

(ii) each element of A is equal to the corresponding element of B.

Addition and Subtraction of matrices.

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$.

then $A+B = \begin{pmatrix} 1+0 & 2+1 \\ 3+4 & 4+2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 7 & 6 \end{pmatrix}$.

$A-B = \begin{pmatrix} 1-0 & 2-1 \\ 3-4 & 4-2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$.

Multiplication of a matrix by a scalar.

The product of a matrix A by a scalar K is a matrix whose each element is K times to the corresponding elements of A.

Thus $KA = 5\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 2\times5 & 3\times5 \\ 4\times5 & 5\times5 \end{pmatrix}$

$= \begin{pmatrix} 10 & 15 \\ 20 & 25 \end{pmatrix}$.

Multiplication of matrices

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 3 \end{pmatrix}_{3\times 3}$, $B = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 1 & 2 \end{pmatrix}_{3\times 2}$

$$= \begin{pmatrix} 1(3)+2(4)+3(1) & 1(2)+2(1)+3(2) \\ 4(3)+5(4)+6(1) & 4(2)+5(1)+6(2) \\ 2(3)+1(4)+3(1) & 2(2)+1(1)+3(2) \end{pmatrix}$$

$$= \begin{pmatrix} 3+8+3 & 2+2+6 \\ 12+20+6 & 8+5+12 \\ 6+4+3 & 4+1+6 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 10 \\ 38 & 25 \\ 13 & 11 \end{pmatrix}$$

Note:

$A^2 = A \cdot A$, $A^3 = A^2 \cdot A$.

Orthogonal matrix.

If the product of the matrix A and the transpose matrix A^T is identity matrix, then A is called an Orthogonal matrix.

Unitary matrix

A square matrix is said to be unitary if $(\bar{A})^T A = I$

Hermitian matrix.

A square matrix A is said to be Hermitian if $A = (\bar{A})^T$

Skew Hermitian matrix.

A square matrix A is said to be Skew Hermitian if $(\overline{A})^T = -A$.

Characteristic polynomial

Characteristic polynomial:-

Let A be a square matrix of order n. The determinant $|A-\lambda I|$ is a polynomial of degree n in λ and is called Characteristic polynomial of A.

Characteristic equation:-

The equation $|A-\lambda I| = 0$ is called characteristic equation.

Eg:-1 Find the Characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = A$.

Solution:

The Characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

$$\left|\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0.$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 0 = 0$$

$$\Rightarrow (2 - \lambda - 3\lambda + \lambda^2) = 0$$

$$\Rightarrow (\lambda^2 - 3\lambda + 2) = 0.$$

The Characteristic equation of A is $\lambda^2 - 3\lambda + 2 = 0$.

<u>Note</u>:- If A is a square matrix of order 3, then its characteristic equation can be written as

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

S_1 = trace of A
= sum of the diagonal elements of A

S_2 = Sum of the minors of main diagonal elements.

$S_3 = |A|$.

for, 2x2 matrix, $\lambda^2 - S_1\lambda + S_2$, S_1 = trace of A, $S_2 = |A|$.

<u>Eg:-2</u> Find the characteristic equation of $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$.

<u>Solution</u>

let $A = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

The Characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

S_1 = Sum of the main diagonal elements

$= 2 + 1 + (-4)$

$= -1$

S_2 = Sum of the minors of main diagonal elements

$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$

$= (-4-6) + (-8+5) + (2+9)$

$= -10 - 3 + 11$

$= -2.$

$S_3 = |A| = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix}$

$= 2(-4-6) - (-3)(-12+15) + 1(6+5)$

$= 2(-10) + 3(3) + 1(11)$

$= -20 + 9 + 11$

$= -20 + 20$

$= 0$

The Characteristic equation of A is

$\lambda^3 - (-1)\lambda^2 + (-2)\lambda + 0 = 0.$

$$\lambda^3 + \lambda^2 - 2\lambda = 0.$$

Homework:-

1) find the characteristic equation of the matrix $\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

2) find the characteristic equation of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

$$\lambda^3 + \lambda^2 - 2\lambda = 0.$$

Homework:-

1) find the characteristic equation of the matrix $\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

2) find the characteristic equation of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

Linearly dependent set of vectors.

The vectors $X_1, X_2, \ldots X_m$ are said to be linearly dependent, if the scalars $\lambda_1, \lambda_2, \ldots \lambda_m$ (not all zero) exist, such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_m X_m = 0$$

Linearly independent set of vectors.

The m vectors are said to be linearly independent.

ie) every relation -

$$\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_m X_m = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0, \ldots \lambda_m = 0.$$

Note:-1 If m vectors are linearly dependent then atleast one of them may be expressed as a linear combination of the others.

Note:-2 The rows and columns of A will be linearly dependent if $|A| = 0$.

Note:-3 The rows and Columns of A will be linearly independent if $|A| \neq 0$

Eigen Values and Eigenvectors of a Real matrix

Eigen value :-

Let $A = [a_{ij}]$ be a square matrix. The Characteristic equation of A is $|A - \lambda I| = 0$.

The roots of the Characteristic equation are Called Eigenvalues of A.

Eigen vector :-

Let $A = [a_{ij}]$ be a square matrix of order n.

If there exists a non zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that $AX = \lambda X$, then the vector X is called an Eigenvector of A Corresponding to the Eigenvalue λ.

Now, we see the problems based on Non-symmetric matrices with Repeated Eigenvalues.

Eg:-1 Find the eigen values and eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$

Solution

Let $A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$.

Step:-1 To find the Characteristic equation.

The Characteristic equation of A is $|A - \lambda I| = 0$.

$$\left| \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$$

$$\Rightarrow (-1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 - 3) = 0$$

$$\lambda^2 - 4 = 0.$$

Step:-2 To solve the Characteristic equation.

$$\lambda^2 - 4 = 0$$

$$\lambda^2 = 4$$

$$\lambda = \pm 2$$

$\therefore$ The Eigenvalues are $-2, +2$.

Step: 3 To find the Eigenvectors.

To find the Eigenvectors,

Solve $(A-\lambda I)X=0$.

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ --- } (I)$$

Case 1) If $\lambda = -2$, then (I) becomes,

$$\begin{bmatrix} 1-(-2) & 1 \\ 3 & -1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie) $3x_1 + x_2 = 0$

$3x_1 + x_2 = 0$

$\Rightarrow 3x_1 = -x_2$

$\Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{-3}$

Hence the corresponding eigenvector is $X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

Case :-2 If $\lambda = 2$, then (I) becomes.

$$\begin{pmatrix} 1-2 & 1 \\ 3 & -1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$-x_1 + x_2 = 0$

$3x_1 - 3x_2 = 0.$

We get only one equation

$x_1 - x_2 = 0.$

$x_1 = x_2$

(ie) $\frac{x_1}{1} = \frac{x_2}{1}$

Hence the corresponding eigen vector is $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Note:- * The Eigenvalues are Unique.

* The Eigenvector Corresponding to the Eigenvalue is Not unique

Eg:-2 Find the Eigenvalues and Eigenvectors of $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$.

Solution:-

let $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$

Step:-1 To find the Characteristic equation.

The Characteristic equation of A is $|A-\lambda I|=0$.

ie) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

Where

S_1 = Sum of the main diagonal elements $= 1+2+3 = 6$.

S_2 = Sum of the minors of main diagonal elements

$$= \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-2) + (3+2) + (2-0)$$

$$= 4+5+2 = 11$$

$$S_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(6-2) - 0(3-2) + (-1)(2-4)$$

$$= 1(4) - 1(-2)$$

$$= 4+2 = 6.$$

~~Step-2 To find~~

Therefore, the characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$.

If $\lambda = 1$,

then $(1)^3 - 6(1)^2 + 11(1) - 6 = 0$

$$1 - 6 + 11 - 6 = 0$$

$$12 - 12 = 0.$$

$\therefore \lambda = 1$ is a root.

By Synthetic division.

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

Other roots are given by,

$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda-3)(\lambda-2) = 0$$
$$\lambda = 2, \quad \lambda = 3$$

Hence the eigenvalues of the given matrix are 1, 2, 3.

Step:-3 To find the Eigenvectors.

To find the Eigenvectors

Solve, $(A - \lambda I) = 0$.

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} (1-\lambda)x_1 + 0x_2 - x_3 &= 0 \\ x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right] \text{--- } (I)$$

Case (i) If $\lambda = 1$ then (I) becomes

$$-x_3 = 0 \quad \text{--- (1)}$$
$$x_1 + x_2 + x_3 = 0 \quad \text{--- (2)}$$
$$2x_1 + 2x_2 + 2x_3 = 0. \quad \text{--- (3)}$$

here, (2) & (3) are same

Choose (1) & (2)

(1) $\Rightarrow$ $0x_1 + 0x_2 - x_3 = 0$

(2) $\Rightarrow$ $x_1 + x_2 + x_3 = 0$

	x_1		x_2		x_3	
0		-1		0		0
1		1		1		1

$$\frac{x_1}{0+1} = \frac{x_2}{-1+0} = \frac{x_3}{0-0}.$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$\therefore$ Eigenvector corresponding to the Eigen value, $\lambda = 1$, is

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Case (ii) If $\lambda = 2$, then (I) becomes

$$-x_1 + 0x_2 - x_3 = 0 \quad \text{--- (4)}$$
$$x_1 + 0x_2 + x_3 = 0 \quad \text{--- (5)}$$
$$2x_1 + 2x_2 + x_3 = 0 \quad \text{--- (6)}$$

Choose ⑤ & ⑥ since ④ & ⑤ are same.

⑤ $\Rightarrow x_1 + 0x_2 + x_3 = 0$

⑥ $\Rightarrow 2x_1 + 2x_2 + x_3 = 0.$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 0 & & 1 & & 1 & & 0 \\ 2 & & 1 & & 2 & & 2 \end{array}$$

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

The Eigenvector Corresponding to the Eigenvalue $\lambda = 2$ is.

$$X_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

(Case 3) If $\lambda = 3$, then the equation (I) becomes,

$-2x_1 + 0x_2 - x_3 = 0$ —— ⑦

$x_1 + (-1)x_2 + x_3 = 0$ —— ⑧

$2x_1 + 2x_2 + 0x_3 = 0.$ —— ⑨

Solving ⑧ & ⑨ we get,

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -1 & & 1 & & 1 & & -1 \\ 2 & & 0 & & 2 & & 2 \end{array}$$

$$\frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{4}$$

ie) $\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$.

Hence, the corresponding eigenvector is $X_3 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

eigen value $= 1, 2, 3$

eigen vector. $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $X_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ $= X_3 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

II) Problems based on Non-symmetric matrices with repeated eigen values.

Eg:-3 Find all the eigen values and eigenvectors of the matrix

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Soln: let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

Step:-1 To find the characteristic equation.

The characteristic equation of A is $|A - \lambda I| = 0$.

ie) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where S_1 = sum of the main diagonal elements

$= -2 + 1 + 0 = -1$

S_2 = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= (0-12) + (0-3) + (-2-4)$

$= -12 - 3 - 6 = -21$

$S_3 = |A|$

$= -2(0-12) - 2((0-6) + (-3)(-4+1)$

$= 24 + 12 + 9 = 45.$

Therefore, the characteristic equation.

$\lambda^3 - (-1)\lambda^2 + (-21)\lambda - 45 = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$

Step:- 2 To solve the characteristic equation

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

If $\lambda = 1$, then $\lambda^3 + \lambda^2 - 21\lambda - 45 \neq 0$

If $\lambda = -1$, then $\lambda^3 + \lambda^2 - 21\lambda - 45 \neq 0$

If $\lambda = 2$, then $\lambda^3 + \lambda^2 - 21\lambda - 45 \neq 0$

If $\lambda = -2$, then $\lambda^3 + \lambda^2 - 21\lambda - 45 \neq 0$

If $\lambda = 3$, then $\lambda^3 + \lambda^2 - 21\lambda - 45 \neq 0$

If $\lambda = -3$ then $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$

Hence $\lambda = -3$ is a root.

By synthetic division,

$$\begin{array}{c|cccc} -3 & 1 & 1 & -21 & -15 \\ & 0 & -3 & 6 & 15 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$\therefore \lambda^2 - 2\lambda - 15 = 0$

$\lambda^2 - 2\lambda - 15 = 0$

$\Rightarrow (\lambda - 5)(\lambda + 3) = 0$

ie) $\lambda = -3, -3, 5$

Therefore, the Eigen values are $-3, -3, 5$.

Step:-3 To find the Eigenvectors.

Solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} (-2-\lambda) & 2 & -3 \\ 2 & (1-\lambda) & -6 \\ -1 & -2 & (0-\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (-2-\lambda)x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + (1-\lambda)x_2 - 6x_3 &= 0 \\ -1x_1 - 2x_2 - \lambda x_3 &= 0 \end{aligned}\right\} \quad \text{— (I)}$$

Case i) If $\lambda = -3$, then (I) becomes,

$x_1 + 2x_2 - 3x_3 = 0$ — (1)

$2x_1 + 4x_2 - 6x_3 = 0$ — (2)

$-x_1 - 2x_2 + 3x_3 = 0$. — (3)

Here (1) & (2) and (3) are same equations.

$$x_1 + 2x_2 - 3x_3 = 0.$$

Put $x_1 = 0$. we get.

$$2x_2 = 3x_3$$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

Hence, the corresponding eigenvector is $X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

Put $x_2 = 0$, we get.

$$x_1 - 3x_3 = 0.$$

$$x_1 = 3x_3$$

$$\therefore \frac{x_1}{3} = \frac{x_3}{1}$$

Hence the corresponding eigenvector is $X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

Case ii) If $\lambda = 5$, then (I) becomes.

$-\cancel{5}\,1$

$$-7x_1 + 2x_2 - 3x_3 = 0 \quad \text{---} (4)$$

$$2x_1 - 4x_2 - 6x_3 = 0 \quad \text{---} (5)$$

$$-x_1 - 2x_2 - 5x_3 = 0 \quad \text{---} (6)$$

Solving (4) & (5) by cross rule multiplication, we get.

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 2 & -3 & -7 & 2 \\ -4 & -6 & 2 & -4 \end{array}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector is $X_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

Eigen values of the given matrix A are $-3, -3, 5$

Eigen vector corresponding to the eigenvalue $\lambda = -3$ is $X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

$X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

If $\lambda = 5$ is $X_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

Note:- Non Symmetric matrix

If a square matrix A is non symmetric then $A \neq A^T$

* In a non symmetric matrix, the eigen values are non-repeated then we get a linearly independent sets of Eigen vectors.

* In a non symmetric matrix the eigen values are repeated, then we may or may not be possible to get a linearly independent Eigenvectors.

* If we form a linearly independent sets of Eigenvectors, then diagonalisation is possible through similarity transformation.

Symmetric matrix.

* In a symmetric matrix, the Eigen values are non-repeated, then we get a linearly independent and pairwise orthogonal sets of Eigenvectors.

* In a symmetric matrix, the Eigenvalues are repeated, the we may or may not be possible to get linearly independent and pairwise orthogonal sets of Eigenvectors.

* If we form a linearly independent and pairwise orthogonal sets of eigenvectors, then diagonalisation is possible through orthogonal transformation.

Homework:-

1. Find the eigenvalues and Eigenvectors of $\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$

Homework:-

1. Find the eigenvalues and Eigenvectors of $\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$

Problem based on symmetric matrices with non-repeated eigenvalues.

Eg:-4 find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 7 & -2 & 0 \\ -2 & +6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$.

Soln:-

let $A = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

Step:-1 To find the characteristic equation

The characteristic equation of the given matrix is $|A - \lambda I| = 0$.

i.e) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = Sum of its leading diagonal elements $= 7+6+5 = 18$

S_2 = Sum of the minors of its leading diagonal elements

$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$

$= (30-4) + (35-0) + (42-4)$

$= 26 + 35 + 38 = 99$

$S_3 = |A|$

$$= \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$$

$$= 7(30-4) + 2(-10-0) + 0(0)$$

$$= 7(26) - 20 = 182 - 20 = 162.$$

The Characteristic equation of A is,

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

Step: 2 To solve the characteristic equation

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

If $\lambda = 1$, then $1 - 18 + 99 - 162 \neq 0$

If $\lambda = -1$, then $-1 - 18 - 99 - 162 \neq 0$

If $\lambda = 2$, then $8 - 72 + 198 - 162 \neq 0$

If $\lambda = -2$, then $-8 - 72 - 198 - 162 \neq 0$

If $\lambda = 3$, then $27 - 162 + 297 - 162 = 0$

$\therefore \lambda = 3$ is a root

By synthetic division

$$\begin{array}{r|rrrr} 3 & 1 & -18 & 99 & -162 \\ & 0 & 3 & -45 & +162 \\ \hline & 1 & -15 & -54 & 0 \end{array}$$

other roots are given by,

$$\lambda^2 - 15\lambda + 54 = 0$$

$$(\lambda - 9)(\lambda - 6) = 0$$

$\lambda = 9, \ \lambda = 6.$

Hence the eigenvalues are 3, 6, 9.

Step:-3 To get eigenvectors,

Solve $(A-\lambda I)X = 0$.

$$\begin{pmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

~~Case i) when $\lambda=3$, we get,~~

$$\left.\begin{aligned} (7-\lambda)x_1 - 2x_2 + 0x_3 &= 0 \quad -1 \\ -2x_1 + (6-\lambda)x_2 - 2x_3 &= 0 \quad -2 \\ 0x_1 - 2x_2 + (5-\lambda)x_3 &= 0 \quad -3 \end{aligned}\right\} -\text{I}$$

Case (i) when $\lambda=3$, we get,

$4x_1 - 2x_2 + 0x_3 = 0$ — ①

$-2x_1 + 3x_2 - 2x_3 = 0.$ — ②

$0x_1 - 2x_2 - 2x_3 = 0.$ — ③

Solving ② & ③ by rule of cross multiplication, we get.

$$\frac{x_1}{6-4} = \frac{x_2}{0+4} = \frac{x_3}{4-0}$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 3 & -2 & -2 & 3 \\ -2 & 2 & 0 & -2 \end{array}$$

$$\frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}.$$

Hence the corresponding Eigenvector is $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Case ii) when $\lambda = 6$, we get.

$x_1 - 2x_2 + 0x_3 = 0$ —— (4)

$-2x_1 + 0x_2 - 2x_3 = 0$ —— (5)

$0x_1 - 2x_2 - x_3 = 0$ —— (6).

Solving (5) & (6) we get,

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 0 & -2 & -2 & 0 & \\ -2 & -1 & 0 & -2 \end{array}$$

$$\frac{x_1}{0-4} = \frac{x_2}{0-2} = \frac{x_3}{4-0}$$

$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence, the corresponding Eigenvector is $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

Case (iii) When $\lambda = 9$, we get

$-2x_1 - 2x_2 + 0x_3 = 0$ —— (7)

$-2x_1 - 3x_2 - 2x_3 = 0$ —— (8)

$0x_1 - 2x_2 - 4x_3 = 0$ —— (9).

Solving (8) & (9), we get

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 \\ -3 & -2 & -2 & -3 \\ -2 & -4 & 0 & -2 \end{array}$$

$$\frac{x_1}{12-4} = \frac{x_2}{0-8} = \frac{x_3}{4-0}.$$

$$\frac{x_1}{8} = \frac{x_2}{-8} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}.$$

Hence, the corresponding eigenvector is $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Problems based on symmetric matrices with repeated eigenvalues.

Eg:-5 Find the Eigenvalues and Eigenvectors of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Determine the algebraic and geometric multiplicity.

Soln:- let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Step:-1 To find the characteristic equation.

The characteristic equation of the given matrix is $|A - \lambda I| = 0$.

ie) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 0$

$S_2 = (0-1) + (0-1) + (0-1) = -3$

$S_3 = |A| = 2$

The characteristic equation is

$$\lambda^3 - 3\lambda - 2 = 0.$$

Step:-2 To Solve the Characteristic equation.

$$\lambda^3 - 3\lambda - 2 = 0.$$

If $\lambda = 1$, then $\lambda^3 - 3\lambda - 2 \neq 0$.

If $\lambda = -1$, then $\lambda^3 - 3\lambda - 2$

$$= -1 + 3 - 2 = 0.$$

$\therefore \lambda = -1$ is a root.

By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

Other roots are given by,

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda + 1)(\lambda - 2) = 0$$

$$\lambda = -1, 2$$

$\lambda = -1, -1, 2$ are the eigenvalues

Step:-3 To find the Eigen ~~values~~ vectors.

To find the Eigenvectors:

Solve $(A - \lambda I)X = 0$.

$$\text{ie)} \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left.\begin{aligned} -\lambda x_1 + x_2 + x_3 &= 0 \\ x_1 - \lambda x_2 + x_3 &= 0 \\ x_1 + x_2 - \lambda x_3 &= 0. \end{aligned}\right\} \text{— (I)}$$

Case (i) If $\lambda = 2$, then (I) becomes

$-2x_1 + x_2 + x_3 = 0$ — (1)

$x_1 - 2x_2 + x_3 = 0$ — (2)

$x_1 + x_2 - 2x_3 = 0$ — (3).

Solving (1) & (2) by rule of cross multiplication, we get,

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & -2 & 1 & \\ 1 & 1 & -2 & \end{array}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}.$$

(ie) $\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

Hence, the corresponding eigenvector is $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Case (ii) If $\lambda = -1$, then (I) becomes.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$x_1 + x_2 + x_3 = 0$ — (4)

$x_1 + x_2 + x_3 = 0$ — (5)

$x_1 + x_2 + x_3 = 0.$ — (6).

(4), (5), (6) represents the same eqn

$\therefore x_1 + x_2 + x_3 = 0.$

$$x_2 = -x_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

Hence the corresponding Eigenvector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

To find: Pairwise orthogonal vector.

let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ as X_3 is orthogonal to X_1 and X_2.

Since the given matrix is symmetric

$$(1, 1, 1)\begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0 \Rightarrow l + m + n = 0 \quad (7)$$

$$(0 \;\; 1 \;\; -1)\begin{pmatrix} l \\ m \\ n \end{pmatrix} \Rightarrow 0l + m - n = 0 \quad (8)$$

Solving (7) & (8) we get

l	m	n
1	1	1
1	-1	0

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$$

Hence, the Corresponding eigenvector

is $X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$.

The algebraic multiplicity of $\lambda = -1$

is two.

The geometric multiplicity of

Eigen value $\lambda = -1$ is two.

Home work:-

1. (Symmetric matrices with repeated eigen values) find the eigen values and eigenvectors of the matrix.

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Properties of Eigenvalues and Eigenvectors

Property :- 1 The sum of the eigenvalues of a matrix is the sum of the elements of the principal (main) diagonal. (or)

(i) The sum of the eigenvalues of a matrix is equal to the trace of the matrix.

(ii) Product of the eigenvalues is equal to the determinant of the matrix.

Proof :- Let A be a square matrix of order n.

The characteristic equation of A is $|A-\lambda I|=0$.

$$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - \dots + (-1)^n S_n = 0 \quad \text{--- (1)}$$

where S_1 = Sum of the diagonal elements of A.

S_2 = Sum of the minors of the main diagonal elements

$\vdots$

$S_n = |A|$.

We know the roots of the characteristic equation are called eigenvalues of the given matrix.

Solving ①, we get n roots

let the n roots be $\lambda_1, \lambda_2 \dots \lambda_n$.

We know already,

$\lambda^n -$ (Sum of the roots) λ^{n-1}
+ (Sum of the product of the roots taken two at a time) $\lambda^{n-2} - \dots$
$+ (-1)^n$ (product of the roots) $= 0$ — ②

Sum of the roots $= S_1$ by ① & ②.

$\lambda_1 + \lambda_2 + \dots + \lambda_n = S_1$

Sum of the Eigen values
= Sum of the main diagonal elements.

Product of the roots $= S_n$
by ① + ②.

$\lambda_1 \lambda_2 \dots \lambda_n =$ det of A

Product of the Eigen values $= |A|$

Property :- 2

A square matrix A and its transpose A^T have the same eigen values
(or) A square matrix A and its transpose have the same characteristic values.

Proof : let A be a square matrix of order n.

The characteristic equation of A and A^T are,

$|A-\lambda I|=0$ —①

$|A^T-\lambda I|=0$ —②.

Since the determinant value is unaltered by the interchange of rows and columns,

$|A|=|A^T|$.

$\therefore$ (1) & (2) are identical.

$\therefore$ The Eigenvalues of A and A^T are the same.

Property:-5 The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

(or) The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof:-

Let us consider the triangular matrix.

$$A=\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Characteristic equation of A is $|A-\lambda I|=0$.

$$\begin{vmatrix} a_{11}-\lambda & 0 & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0.$$

On expansion it gives

$$(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)=0$$

$$\therefore \lambda = a_{11}, a_{22}, a_{33}$$

Which are diagonal elements of the matrix.

Property:-4 If λ is an Eigenvalue of a matrix A, then $\frac{1}{\lambda}$ $(\lambda \neq 0)$ is the Eigenvalue of A^{-1}.

Proof: If X be the Eigenvector corresponding to λ,

then $AX = \lambda X$.

Premultiplying both sides by A^{-1} we get

$$A^{-1}AX = A^{-1}\lambda X$$

$$IX = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\div \lambda \Rightarrow \frac{1}{\lambda}X = A^{-1}X$$

$$A^{-1}X = \frac{1}{\lambda}X.$$

This being of the same form as (i), shows that $\frac{1}{\lambda}$ is an Eigenvalue of the inverse matrix A^{-1}

Property: 5 If λ is an Eigenvalue of an orthogonal matrix, then $1/\lambda$ is also its Eigenvalue.

Proof: Let A be an orthogonal matrix. Given λ is an Eigenvalue of A.

$\Rightarrow 1/\lambda$ is an Eigenvalue of A^{-1}

Since $A^T = A^{-1}$,

$1/\lambda$ is an Eigenvalue of A^T,

Both the matrices A and A^T have the same Eigenvalues, since the determinants $|A-\lambda I|$ and $|A^T - \lambda I|$ are the same.

Hence, $1/\lambda$ is also an Eigenvalue of A.

Property :- 6 If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the Eigenvalues of a matrix A, then A^m has the Eigenvalues.

$\lambda_1^m, \lambda_2^m \ldots \lambda_n^m$

Proof: let λ_i be the Eigenvalue of A and x_i the corresponding ~~eigenvalues~~ eigenvectors.

Then $Ax_i = \lambda_i x_i$ —— (1)

we have $A^2 x_i = A(Ax_i)$

$= A(\lambda_i x_i)$

$= \lambda_i (A_i x_i)$

$= \lambda_i (\lambda_i x_i)$

$= \lambda_i^2 x_i$

Similarly, $A^3 x_i = \lambda_i^3 x_i$

In general, $A^m x_i = \lambda_i^m x_i$ — (2)

Hence λ_i^m is an Eigenvalue of A^m.

The corresponding Eigenvector is the same x_i

Note:- If λ is the Eigenvalue of the matrix A then λ^2 is the Eigenvalue of A^2.

Property:-7 The Eigenvalues of a real symmetric matrix are real numbers.

Proof:- let λ be an Eigenvalue (may be complex) of the real symmetric matrix A.

let the corresponding Eigenvector be X. let A' denote the transpose of A.

We have $AX = \lambda X$.

Pre multiplying this equation by $1 \times n$ matrix $\overline{X}'$ where the bar denotes that all elements of $\overline{X}'$

are the complex conjugate of those of x', we get

$$\overline{x'}AX = \lambda \overline{x'}X \quad \text{--- (1)}$$

Taking the conjugate complex of this we get,

$$x'\overline{A}\,\overline{x} = \overline{\lambda}\,x'\overline{x} \quad \text{or}$$

$$x'A\overline{x} = \overline{\lambda}\,x'\overline{x}.$$

Since $\overline{A} = A$, for A is real.

Taking the transpose on both sides we get,

$$(x'A\overline{x})' = (\lambda x'\overline{x})'$$

(ie) $\overline{x}'A'x = \overline{\lambda}\,\overline{x}'x$

(ie) $\overline{x'}AX = \overline{\lambda}\,\overline{x'}X.$

Since $A' = A$ for A is symmetric.

But, from (1),

$$\overline{x'}AX = \lambda \overline{x'}X$$

Hence $\lambda \overline{x'}x = \overline{\lambda}\,\overline{x'}\,x$.

Since $\overline{x'}x$ is an 1×1 matrix whose only element is a positive value

$$\lambda = \overline{\lambda}.$$

(ie) λ is real.

Property:-8 The Eigenvectors Corresponding to distinct Eigenvalues of a real symmetric matrix are orthogonal.

Proof: For a real Symmetric matrix A, the Eigenvalues are real.

let X_1, X_2 be Eigenvectors Corresponding to two distinct eigen values λ_1, λ_2 (λ_1, λ_2 are real).

$$AX_1 = \lambda_1 X_1 \quad \text{---} (1)$$

$$AX_2 = \lambda_2 X_2 \quad \text{---} (2)$$

Premultiplying (1) by X_2' we get.

$$X_2' A X_1 = X_2' \lambda_1 X_1$$

$$= \lambda_1 X_2' X_1$$

Premultiplying (2) by X_1', we get.

$$X_1' A X_2 = \lambda_2 X_1' X_2 \quad \text{---} (3)$$

But $(X_2' A X_1)' = (\lambda_1 X_2' X_1)'$

$$X_1' A' X_2 = \lambda_1 X_1' X_2$$

ie) $X_1' A X_2 = \lambda X_1' X_2$ --- (4)

from (3) & (4).

$$\lambda_1 X_1' X_2 = \lambda_2 X_1' X_2$$

ie) $(\lambda_1 - \lambda_2) X_1' X_2 = 0$

$\lambda_1 \neq \lambda_2$, $X_1' X_2 = 0$

$\therefore X_1, X_2$ are orthogonal.

Property :- 9 The similar matrices have same Eigenvalues.

Proof: let A, B be two similar matrices.

Then, there exists a non-singular matrix P such that $B = P^{-1}AP$

$$B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1}AP - P^{-1}\lambda I P$$
$$= P^{-1}(A - \lambda I)P$$

$$|B - \lambda I| = |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= |A - \lambda I|\,|P^{-1}P|$$
$$= |A - \lambda I|\,|I|$$
$$= |A - \lambda I|$$

$\therefore$ Therefore A, B have the same characteristic polynomial and hence characteristic roots.

$\therefore$ They have same Eigenvalue

Property:- 10 If a real symmetric matrix of order 2 has equal Eigenvalues, then the matrix is a scalar matrix.

Proof:

Note:-1. A real symmetric matrix of order n can always be diagonalised.

Note:-2 If any diagonalized matrix with their diagonal elements are equal, then the matrix is a scalar matrix.

let A be the real symmetric matrix

Given: A can always be diagonalized,

let λ_1 and λ_2 be their eigenvalues then we get diagonalized matrix

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

given $\lambda_1 = \lambda_2$.

$\therefore$ we get $= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$.

By Note 2: The given matrix is a scalar matrix.

Property :-11 The Eigenvector X of a matrix A is not unique.

Proof: let λ be the Eigen value of A, then the corresponding Eigenvector X such that $AX = \lambda X$.

Multiply both sides by non zero scalar K,

$K(AX) = K(\lambda X)$

$\Rightarrow A(KX) = \lambda(KX)$

i) an Eigenvector is determined by a multiplicative scalar.

ii) Eigenvector is not unique

Property :- 12 If $\lambda_1, \lambda_2, \dots \lambda_n$ be distinct Eigenvalues of an $n \times n$ matrix then the corresponding Eigenvectors $X_1, X_2, \dots X_n$ form a linearly independent set.

Proof :- let $\lambda_1, \lambda_2, \dots \lambda_m$ $(m \leq n)$ be the distinct Eigenvalues of a square matrix of order n.

let $X_1, X_2, \dots X_m$ be their corresponding Eigenvectors we have to prove

$\sum_{i=1}^{m} \alpha_i X_i = 0$ implies each $\alpha_i = 0$, $i = 1, 2, \dots m$

Multiplying $\sum_{i=1}^{m} \alpha_i X_i = 0$ by $(A - \lambda_1 I)$

we get

$$(A - \lambda_1 I)\alpha_1 X_1 = \alpha_1 (AX_1 - \lambda_1 X_1)$$
$$= \alpha_1 (0) = 0.$$

When $\sum_{i=1}^{m} \alpha_i X_i = 0$ is multiplied by

$(A-\lambda_1 I)(A-\lambda_2 I)\ldots(A-\lambda_{i-1} I)$
$(A-\lambda_i I)(A-\lambda_{i+1} I)\ldots(A-\lambda_m I) = 0.$

we get,

$\alpha_i (\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\ldots(\lambda_i - \lambda_{i-1})$
$(\lambda_i - \lambda_{i+1})\ldots(\lambda_i - \lambda_m) = 0.$

Since λ's are distinct, $\alpha_i = 0$.
Since i is arbitrary, each $\alpha_i = 0$,
$i = 1, 2, \ldots m$

$\sum_{i=1}^{m} \alpha_i X_i = 0$ implies each $\alpha_i = 0$, $i = 1, 2 \ldots m$

Hence $X_1, X_2, \ldots X_m$ are linearly independent.

Property :- 13 If two or more Eigenvalues are equal it may or may not be possible to get linearly independent Eigenvectors corresponding to the equal roots.

Property :- 14 Two eigenvectors X_1 and X_2 are called orthogonal vectors if $X_1^T X_2 = 0$.

Property:- 15 If A and B are $n \times n$ matrices and B is a non singular matrix, then A and $B^{-1}AB$ have same eigenvalues.

Proof:- Characteristic polynomial of $B^{-1}AB$

$$= |B^{-1}AB - \lambda I|$$
$$= |B^{-1}AB - B^{-1}(\lambda I)B|$$
$$= |B^{-1}(A-\lambda I)B|$$
$$= |B^{-1}|\,|A-\lambda I|\,|B|$$
$$= |B^{-1}||B|\,|A-\lambda I|$$
$$= |I|\,|A-\lambda I|$$
$$= |A-\lambda I|$$

= Characteristic polynomial of A.

Hence A and $B^{-1}AB$ have same Eigenvalues.

Problems Based on Properties

1. Problems based on property -1

Example:-1 find the sum and product of the Eigenvalues of the matrix. $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

Solution:-

Sum of the Eigenvalues
= Sum of the diagonal elements
$= (-1) + (-1) + (-1)$
$= -3$.

Product of the Eigen Values.

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1(1-1) - 1(-1-1) + 1(1+1)$
$= -1(0) - 1(-2) + 1(2)$.
$= 0 + 2 + 2 = 4$.

__Example:-2__ The product of two eigenvalues of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. find the third Eigenvalue.

__Soln:__ let the Eigenvalues of the matrix A be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2 = 16$

We know that, $\lambda_1\lambda_2\lambda_3 = |A|$.

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 6(9-1) + 2(-6+2) + 2(2-6)$
$= 6(8) + 2(-4) + 2(-4)$
$= 48 - 8 - 8 = 32$.

$\lambda_1 \lambda_2 \lambda_3 = 32$

$16 \lambda_3 = 32$

$\lambda_3 = \frac{32}{16} = 2$

Example:-3 Two of the Eigenvalues of $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ are 2 and 8. Find the third Eigenvalue.

Soln:- We know that, the sum of the Eigenvalues

= Sum of the main diagonal elements.

$= 6+3+3$

$= 12$

Given $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

We get $\lambda_1 + \lambda_2 + \lambda_3 = 12$

$2+8+\lambda_3 = 12$

$\lambda_3 = 12-10$

$\lambda_3 = 2$

$\therefore$ The third eigenvalue $= 2$

Problems based on property :-2

Example:-1 If 2, 2, 3 are the eigenvalues of $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ find the eigenvalues of A^T.

Soln:- A Square matrix A and its transpose A^T have the same eigenvalues.

Hence Eigenvalues of A^T are 2, 2, 3.

Problems based on property:-3

Example:-1 Find the eigenvalues of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Soln: Clearly given matrix A is an upper triangular matrix. Then by Property, the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Hence, the Eigenvalues are 2, 2, 2.

Problems based on property:-4

Example:-1 Two of the Eigenvalues of $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3 and 6. Find the eigenvalues of A^{-1}

Soln:- Sum of the Eigenvalues = Sum of the main diagonal elements

$$= 3+5+3 = 11 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 11$$

Given that $\lambda_1 = 3$, $\lambda_2 = 6$.

$$3 + 6 + \lambda_3 = 11$$

$$\lambda_3 = 11 - 9 = 2.$$

Problems based on property :-5

Example:-1 The eigenvalues of the given orthogonal matrix.

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \text{ are } \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$$

Show that $\frac{\sqrt{2}}{1+i}$, $\frac{\sqrt{2}}{1-i}$ are also Eigenvalues of A.

Soln: Given $\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}$ are Eigenvalues of A

$$\frac{1}{\left(\frac{1+i}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{1+i};$$

$$\frac{1}{\left(\frac{1-i}{\sqrt{2}}\right)} = \frac{\sqrt{2}}{1-i}$$ are also eigenvalues of A.

Problems based on property :-6

Example:-1 Find the Eigenvalues of A^3 given $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{pmatrix} = A$.

Soln: Given $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{pmatrix}$

Clearly given A is an upper triangular matrix.

Hence the eigenvalues are 1, 2, 3.

By the property:-6, the eigenvalues of the matrix A^3 are $1^3, 2^3, 3^3$.

ie) 1, 8, 27.

Problems based on property:-7.

Example:-1 Show that the Eigenvalues of the real symmetric matrix $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$ are real.

Soln:- Given $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$ is a real symmetric matrix.

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0.$$

$$(-2-\lambda)(1-\lambda) - 4 = 0.$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda = -3, 2.$$

The Eigenvalues are $-3, 2$ (real).

General Problems:-

Example:-1 If $2, -1, -3$ are the Eigen values of the matrix A, then find the Eigenvalues of the matrix $A^2 - 2I$.

Soln:- The Eigenvalues of A are $2, -1, -3$.

The Eigenvalues of A^2 are $4, 1, 9$.

The Eigenvalues of $-2I$ are,

$-2, -2, -2$.

$\therefore$ The Eigenvalues of A^2-2I are

$4-2=2, \; 1-2=-1, \; 9-2=7$.

Example:-2 Find the Eigenvalues of $3A+2I$, where $A = \begin{pmatrix} 5 & 4 \\ 0 & 2 \end{pmatrix}$.

Soln: The Eigenvalues of A are 5 and 2.

The Eigenvalues of $3A$ are

15 and 6.

The Eigenvalues of $2I$ are

2 and 2.

The Eigenvalues of $3A+2I$ are,

$15+2=17, \; 6+2=8$.

Cayley Hamilton Theorem

Cayley Hamilton Theorem

Every Square matrix satisfies its own Characteristic equation.

Uses of Cayley Hamilton Theorem

To Calculate

(i) The positive integral powers of A

and (ii) the inverse of a non-singular square matrix A.

Problems based on Cayley Hamilton theorem.

Example:-1 Show that the matrix $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ satisfies its own characteristic equation.

Soln: let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

The characteristic equation of the given matrix is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) + 4 = 0$$

$$1 - 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0.$$

To Prove :- $A^2 - 2A + 5I = 0$.

$A^2 = A.A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$

$-2A = -2\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix}$

$+5I = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$A^2 - 2A + 5I$

$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\therefore$ The given matrix satisfies its own characteristic equation.

<u>Example :- 2</u> Verify Cayley-Hamilton theorem find A^4 and A^{-1} where

$A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

<u>Soln:-</u> The characteristic equation of A is $|A - \lambda I| = 0$.

(i) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 = 2+2+2 = 6.$

$S_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= (4-1) + (4-2) - (4-1)$

$= 3+2+3 = 8.$

$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2)$

$= 2(3) + 1(-1) + 2(-1)$

$= 6-1-2 = 3.$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\therefore \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0.$

By Cayley Hamilton theorem.

$A^3 - 6A^2 + 8A - 3I = 0.$ —— (1)

<u>Verification:</u>

$A^2 = A.A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 4+1+2 & -2-2-2 & 4+1+4 \\ -2-2-1 & 1+4+1 & -2-2-2 \\ 2+1+2 & -1-2-2 & 2+1+4 \end{pmatrix}$

$= \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$

$A^3 = A \times A^2$

$$= \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 14+5+10 & -12-6-10 & 18+6+14 \\ -7-10-5 & 6+12+5 & -9-12-7 \\ 7+5+10 & -6-6-10 & 9+6+14 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$$

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} + \begin{pmatrix} -42 & 36 & -54 \\ 30 & -36 & 36 \\ -30 & 30 & -42 \end{pmatrix}$$

$$+ \begin{pmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} + \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

$\therefore$ Hence Cayley Hamilton theorem is verified.

<u>To find A^{-1}:-</u>

$① \Rightarrow A^3 - 6A^2 + 8A - 3I = 0.$

$\Rightarrow A^3 = 6A^2 - 8A + 3I$

Multiply both sides by A, we get

$A^4 = 6A^3 - 8A^2 + 3A$

$= 6(6A^2 - 8A + 3I) - 8A^2 + 3A$

$= 36A^2 - 48A + 18I - 8A^2 + 3A$

$A^4 = 28A^2 - 45A + 18I$

$$= 28\begin{pmatrix} 7 & -6 & 9 \\ 5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 45\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 18\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{pmatrix} + \begin{pmatrix} -90 & 45 & -90 \\ 45 & -90 & 45 \\ -45 & 45 & -90 \end{pmatrix} + \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

$$= \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

To find A^{-1}:

(1) $\times A^{-1}$ we get

$A^2 - 6A + 8I - 3A^{-1} = 0.$

$$3A^{-1} = A^2 - 6A + 8I$$

$$3A^{-1} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + \begin{pmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$3A^{-1} = \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

Homework:-

1. Using Cayley Hamilton theorem find A^{-1} where $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

Example:-3 Use Cayley Hamilton theorem find the value of $A^8-5A^7+7A^6-3A^5+8A^4-5A^3+8A^2-2A+I$ if the matrix

$$A=\begin{pmatrix}2&1&1\\0&1&0\\1&1&2\end{pmatrix}$$

let $A^8-5A^7+7A^6-3A^5+8A^4-5A^3+8A^2-2A+I$ be $f(A)$

Soln:-

The characteristic equation of A is $|A-\lambda I|=0$.

ie) $\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$.

$S_1=2+1+2=5$

$$S_2=\begin{vmatrix}1&0\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}+\begin{vmatrix}2&1\\0&1\end{vmatrix}$$

$$=2+3+2=7.$$

$S_3=|A|=3$

The characteristic equation is

$$\lambda^3-5\lambda^2+7\lambda-3=0.$$

By Cayley Hamilton theorem.

$$A^3-5A^2+7A-3I=0 \quad \text{①}$$

$$
\begin{array}{r|l}
 & \lambda^5 + 8\lambda + 35 \\
\hline
\lambda^3 - 5\lambda^2 + 7\lambda - 3 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 + 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\
 & \lambda^8 - 5\lambda^7 + 7\lambda^6 - 3\lambda^5 \\
 & (-) \quad (+) \quad (-) \quad (+) \\
\hline
 & 8\lambda^4 - 5\lambda^3 + 8\lambda^2 - 2\lambda + 1 \\
 & 8\lambda^4 - 40\lambda^3 + 56\lambda^2 - 24\lambda \\
 & (-) \quad (+) \quad (-) \quad (+) \\
\hline
 & 35\lambda^3 - 48\lambda^2 + 22\lambda + 1 \\
 & 35\lambda^3 - 175\lambda^2 + 245\lambda - 105 \\
 & (-) \quad (+) \quad (-) \quad (+) \\
\hline
 & 127\lambda^2 - 233\lambda + 106
\end{array}
$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= (A^3 - 5A^2 + 7A - 3I)(A^5 + 8A + 35I) + 127A^2 - 233A + 106I$$

$$= 0 + 127A^2 - 233A + 106I$$

$$\therefore f(A) = 127\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} - 233\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + 106\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{pmatrix}.$$

Orthogonal matrices

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$ it follows that a matrix is orthogonal if $A^T = A^{-1}$.

Example :- 1 Check whether the matrix $P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Soln: Condition for orthogonal matrix is $PP^T = P^TP = I$

$$P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad P^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

$$PP^T = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Similarly $P^TP = I$. $\therefore$ the matrix P is orthogonal.

Working Rule for diagonalisation (orthogonal transformation)

Step:-1. To find the characteristic equation.

Step:-2 To solve the characteristic equation.

Step:-3 To find the Eigenvectors

Step:-4. If the eigenvectors are orthogonal, then form a normalized modal matrix N

Step:-5 find N^T

Step:-6 Calculate AN.

Step:-7. Calculate $D = N^T A N$.

Problem based on orthogonal transformation of a symmetric matrix to diagonal form.

Example:-1

Diagonalise the matrix

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$ and hence find A^4.

Solution. let $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Step:-1 To find the characteristic equation.

The characteristic equation of A is $|A - \lambda I| = 0$.

i.e) $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where $S_1 = 18$

$S_2 = 45$

$S_3 = 0$.

The Characteristic equation is

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

Step:-2 To solve the characteristic equation

$\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

$\lambda(\lambda^2 - 18\lambda + 45) = 0$.

$\lambda(\lambda - 15)(\lambda - 3) = 0$.

$\lambda = 0, 3, 15$

Step:-3 To find the Eigenvectors.

To find the Eigenvectors,

Solve $(A - \lambda I)X = 0$.

$$\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (8-\lambda)x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + (7-\lambda)x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \rightarrow (I)$$

Case (i)

When $\lambda = 0$, equation (I) becomes,

$8x_1 - 6x_2 + 2x_3 = 0$ — (1)

$-6x_1 + 7x_2 - 4x_3 = 0$ — (2)

$2x_1 - 4x_2 + 3x_3 = 0$ — (3).

Solving (1) & (2) we get,

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

Hence, the corresponding eigenvector $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Case ii) When $\lambda = 3$, the equation (I) becomes.

$5x_1 - 6x_2 + 2x_3 = 0$. — (4)

$-6x_1 + 4x_2 - 4x_3 = 0$. — (5)

$2x_1 - 4x_2 + 0x_3 = 0$. — (6)

Solving (5) & (6) we get,

$$\frac{x_1}{0-16} = \frac{x_2}{-8-0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}.$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence, the eigenvector corresponding to the Eigenvalue $\lambda = 3$ is

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) When $\lambda = 15$ then (I) becomes

$$\begin{cases} -7x_1 - 6x_2 + 2x_3 = 0 & \text{--- (7)} \\ -6x_1 \cancel{+8} -8x_2 - 4x_3 = 0 & \text{--- (8)} \\ 2x_1 - 4x_2 - 12x_3 = 0 & \text{--- (9)} \end{cases}$$

Solving (8) & (9) we get:

$$\frac{x_1}{96-16} = \frac{x_2}{-8-72} = \frac{x_3}{24+16}$$

$$\frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

The set of Eigenvectors are

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

$X_1^T X_2$, $X_1^T X_3$, $X_2^T X_3$, $X_2^T X_3$

are 0.

Hence the Eigenvectors are orthogonal to each other.

Step:-4 To form the Normalised matrix N.

$$N = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step:-5 Find N^T

$$N^T = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Step:-6 Calculate AN.

$$AN = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 8-12+4 & 16-6-4 & 16+12+2 \\ -6+14-8 & -12+7+8 & -12-14-4 \\ 2-8+6 & 4-4-6 & 4+8+3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 6 & 30 \\ 0 & 3 & -30 \\ 0 & -6 & 15 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{pmatrix}$$

Step:-7 : Calculate N^TAN

$$N^TAN = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 10 \\ 0 & 1 & -10 \\ 0 & -2 & 5 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 2+2-4 & 10-20+10 \\ 0 & 4+1+4 & 20-10-10 \\ 0 & 4-2-2 & 20+20+5 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

The diagonal elements are the eigenvalues of A.

Step:-8 To find A^4.

$$D = N^TAN$$

$$A^4 = NDN^T$$

$$= \frac{1}{3}\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 15^4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 50625 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0 & 162 & 101250 \\ 0 & 81 & -101250 \\ 0 & -162 & 50625 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 0+324+202500 & 0+162-202500 & 0-324+101250 \\ 0+162-202500 & 0+81+202500 & 0-162-101250 \\ 0-324+101250 & 0-162-101250 & 0+324+50625 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 202824 & -202338 & 100926 \\ -202338 & 202581 & -101412 \\ 100926 & -101412 & 50949 \end{pmatrix}$$

$$= \begin{pmatrix} 22536 & -22482 & 11214 \\ -22482 & 22509 & -11268 \\ 11214 & -11268 & 5661 \end{pmatrix}$$

Reduction of quadratic form to Canonical form by orthogonal transformation.

- Nature of quadratic form.

Quadratic form

A homogeneous polynomial of the second degree in any number of variables is called a quadratic form.

Example:

$$2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 + 5x_1x_3 - 6x_2x_3$$

is a quadratic form in three variables.

Note:

The matrix Corresponding to the quadratic form is

$$\begin{bmatrix} \text{Coeff, } x_1^2 & \frac{1}{2}\text{ Coeff, } x_1x_2 & \frac{1}{2}\text{ Coeff, } x_1x_3 \\ \frac{1}{2}\text{ coeff, } x_2x_1 & \text{coeff, } x_2^2 & \frac{1}{2}\text{ coeff } x_2x_3 \\ \frac{1}{2}\text{ coeff } x_3x_1 & \frac{1}{2}\text{ coeff, } x_3x_2 & \text{coeff, } x_3^2 \end{bmatrix}$$

Example:-1 Write the matrix of the quadratic form

$$2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3.$$

Soln:-

$$Q = \begin{bmatrix} \text{Coeff. } x_1^2 & \frac{1}{2}\text{Coeff. } x_1x_2 & \frac{1}{2}\text{Coeff. } x_1x_3 \\ \frac{1}{2}\text{Coeff. } x_2x_1 & \text{Coeff. } x_2^2 & \frac{1}{2}\text{Coeff. } x_2x_3 \\ \frac{1}{2}\text{Coeff. } x_3x_1 & \frac{1}{2}\text{Coeff. } x_3x_2 & \text{Coeff. } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$$

Transformations:

let $X'AX$ be a quadratic form where A is the matrix of the quadratic form

let $X = PY$ be a non-singular linear transformation. (P is non-singular)

Then we have

$$X'AX = (PY)'A(PY)$$
$$= (Y'P')A(PY)$$
$$= Y'(P'AP)Y$$
$$= Y'CY \quad \text{where } C = P'AP.$$

Here, $Y'CY$ is also a quadratic form in variables $y_1, y_2, \dots y_n$

$Y'CY$ is the linear transform of the quadratic form $X'AX$ under

the linear transformation
$X = PY$ and $C = P'AP$.

Note:- 1
The matrix C is symmetric

Note:-2
Since $R(A) = R(C)$, the two matrices A and C are congruent matrices.

Canonical form:
A quadratic form $X'AX$ in n unknowns $x_1, x_2, \dots x_n$ can be reduced (via a non-singular transformation) to the canonical form $d_1 y_1^2 + d_2 y_2^2 + \dots + d_n y_n^2$, where $y_1, y_2 \dots y_n$ are the new unknowns. Some of the coefficients $d_1, d_2, \dots d_n$ may of course be zeros.

Fundamental theorem on quadratic forms.
Any quadratic form may be reduced to canonical form by means of a non-singular transformation.

Note:- This form is also called diagonalization of the quadratic form or to express the quadratic form as 'sum of squares'.

Nature of the quadratic form.

Let $Q = X^T A X$ be a quadratic form in n variables $x_1, x_2, \dots x_n$. If the rank of A is r, then the canonical form of Q consists only 'r' square terms.

Index of the Q.F:-

The number of positive square terms in the canonical form is called the index of the quadratic form.

$\therefore$ The number of positive square term is index $= s$.

Signature of the Q.F:-

The difference of number of positive and negative square terms $= s-(r-s) = 2s-r$ is called the signature of the quadratic form (or)

$\left(\text{Number of (+)ve terms in the C.F}\right) - \left(\text{Number of (−)ve terms in the C.F}\right)$

The quadratic form $Q = X^T A X$ in n variables is said to be

(i) Positive definite if $r = n$ and $s = n$

(or) if all the Eigenvalues of A are positive numbers.

(ii) Negative definite if $r = n$ and $s = 0$.

(or) if all the Eigenvalues of A are negative numbers.

(iii) Positive semi-definite if $r < n$ and $s = r$

(or) if all the Eigenvalues of $A \geq 0$ and atleast one Eigenvalue is zero.

(iv) Negative semi-definite if $r < n$ and $s = 0$.

(or) if all the Eigen values of $A \leq 0$ and atleast one Eigenvalue is zero.

(v) Indefinite in all other cases.

(or) if A has both positive and negative Eigenvalues.

Test for Nature of a Quadratic form through principal minors.

let $A = [a_{ij}]$ be the matrix of the quadratic form in n variables $x_1, x_2, \ldots x_n$. Then A is a square symmetric matrix of order n.

let $D_1 = |a_{11}| = a_{11}$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\vdots$

$D_n = |A|$

$D_1, D_2, \ldots D_n$ are the principal minors of A.

* The Q.F is positive definite if $D_1, D_2 \ldots D_n$ are all positive

i.e) $D_n > 0$ for all n.

* The Q.F is Negative definite if $D_1, D_3, D_5 \ldots$ are all negative and $D_2, D_4, D_6 \ldots$ are all positive for all n.

i.e) $(-1)^n D_n > 0$

* The Q.F is positive semidefinite if $D_n \geq 0$ and atleast one $D_i = 0$.
* The Q.F is Negative semidefinite if $(-1)^i D_n \geq 0$ and atleast one $D_i = 0$.
* The Q.F is indefinite in all the other cases.

Example:-1 Determine the nature of the following quadratic form.
$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2$

Sol: $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 0x_3^2$ is already in Canonical form.

The C.F (Canonical form) Contains two positive terms and one zero term.

∴ The Q.F is positive semi-definite

Example:-2 Give the nature of a quadratic form whose matrix is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Sol:- The Eigenvalues of the given matrix are $-1, -1, -2$.

All the Eigenvalues are negative numbers.

The nature of the quadratic form is negative definite.

Example:-3 What is the nature of the quadratic form ~~in four~~ of $x^2+y^2+z^2$ in four variables.

Soln:-

$$f(x,y,z,t) = x^2+y^2+z^2+0t^2.$$

it is already in Canonical form

The C.F contains three positive terms and one zero term.

$\therefore$ The Q.F is positive semi-definite.

Example:-4 Prove that the Q.F $x^2+2y^2+3z^2+2xy+2yz-2xz$ is indefinite.

Soln:-

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

$$D_1 = |1| = 1 \ (+ve)$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1 \ (+ve)$$

$$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

$= 1(6-1) - (3+1) - 1(1+2)$

$= 5 - 4 - 3 = -2$ (-ve)

$\therefore$ The Q.F is indefinite.

Example:-5 Find the index and signature of the Q.F $x_1^2 + 2x_2^2 - 3x_3^2$

Sol:- $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 3x_3^2$

it is already in the canonical form.

Index = Number of positive terms in the C.F

$= 2$.

Signature = Number of positive terms in the C.F − Number of negative terms in the C.F

$= 2 - 1 = 1$.

Example:-6 Identify the nature, index and signature of the quadratic form $2x_1x_2 + 2x_2x_3 + 2x_3x_1$

Sol:-

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The Eigenvalues are $-1, -1, 2$.

Eigen values are both positive and negative.

$\therefore$ the Q.F is indefinite.

Index = Number of positive terms in the C.F = 1

Signature = Number of positive terms in the C.F − Number of negative terms in the C.F

$= 1 - 2 = -1$.

Example:-1 Reduce the quadratic form $x^2+y^2+z^2-2xy-2yz-2zx$ to Canonical form through an orthogonal transformation.

Solution:

Step:-1 The matrix of the Q.F is

$$A = \begin{pmatrix} \text{Coeff } x^2 & \frac{1}{2}\text{Coeff } xy & \frac{1}{2}\text{Coeff } xz \\ \frac{1}{2}\text{Coeff } yx & \text{Coeff } y^2 & \frac{1}{2}\text{Coeff } yz \\ \frac{1}{2}\text{Coeff } zx & \frac{1}{2}\text{Coeff } zy & \text{Coeff } z^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

Step:-2 To find the Characteristic equation of A

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

where S_1 = Sum of the main diagonal elements

$= 1 + 1 + 1 = 3$

S_2 = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$= (1-1) + (1-1) + (1-1)$

$= 0$

$$S_3 = |A| = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

$= 1(1-1) + 1(-1-1) - 1(1+1).$

$= 1(0) + 1(-2) - 1(2)$

$= -2 - 2$

$= -4.$

The Characteristic equation is

$$\lambda^3 - 3\lambda^2 + 4 = 0.$$

Step:-3 To solve the Characteristic solution.

If

$$
\begin{array}{r|rrrr}
-1 & 1 & -3 & 0 & 4 \\
 & 0 & -1 & 4 & -4 \\
\hline
 & 1 & -4 & 4 & \boxed{0}
\end{array}
$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2, 2$$

Hence the eigenvalues are

$2, 2, -1$.

Step:-4 To find the Eigenvectors.

To get the Eigenvectors,

Solve $(A - \lambda I)X = 0$.

$$\begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

~~Case i) if $\lambda = -1$~~

$$\left.\begin{aligned} (1-\lambda)x_1 - x_2 - x_3 &= 0. \\ -x_1 + (1-\lambda)x_2 - x_3 &= 0. \\ -x_1 - x_2 + (1-\lambda)x_3 &= 0. \end{aligned}\right\} \quad \text{—(I)}$$

Case (i) If $\lambda = -1$ then (I) becomes,

$$2x_1 - x_2 - x_3 = 0 \quad \text{—(1)}$$

$$-x_1 + 2x_2 - x_3 = 0 \quad \text{—(2)}$$

$$-x_1 - x_2 + 2x_3 = 0 \quad \text{—(3)}$$

Solving ① & ② by rule of Cross multiplication, we get.

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ -1 & -1 & 2 & -1 \\ 2 & -1 & -1 & 2 \end{array}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Hence, the corresponding eigenvector is $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Case ii) If $\lambda = 2$, then the equation (I) becomes.

$$\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

$-x_1 - x_2 - x_3 = 0$ — ④

$-x_1 - x_2 - x_3 = 0$ — ⑤

$-x_1 - x_2 - x_3 = 0$ — ⑥

Equation ④, ⑤, and ⑥ are same.

$$-x_1 - x_2 - x_3 = 0.$$

If $x_1 = 0$, then $-x_2 - x_3 = 0.$

$$x_2 = -x_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

Hence the corresponding Eigenvector is $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

To find the third eigenvector orthogonal to X_1 and X_2, since the matrix A is symmetric.

Let $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$X_1^T X_3 = 0 \Rightarrow l+m+n=0$

$(1\ 1\ 1) \begin{pmatrix} l \\ m \\ n \end{pmatrix} \Rightarrow l+m+n=0$ — (7)

$X_2^T X_3 = 0 \Rightarrow (0\ 1\ -1) \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$\Rightarrow 0l + m - n = 0$ — (8).

Solving (7) & (8) by the rule of Cross multiplication, we get.

$$\frac{l}{-1-1} = \frac{m}{0+1} = \frac{n}{1-0}$$

$$\begin{array}{cccccc} & l & m & n & \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{array}$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}.$$

$$X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

Eigenvector $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ $X_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

Normalised form $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ $\begin{pmatrix} 0/\sqrt{2} \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ $\begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

Step:-5 Form normalised matrix N

find N^T

$$N = \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$N^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}$$

Step:-6 find N^TAN.

$$D = N^TAN$$

$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Step: 7 Canonical form

$$(y_1 \quad y_2 \quad y_3) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

$$= -y_1^2 + 2y_2^2 + 2y_3^2.$$

1) If λ is an eigenvalue of A, then Prove that λ^2 is an eigenvalue of A^2.

Proof: Let λ be an eigenvalue of A,

then $(A-\lambda I)X = 0$.

$\Rightarrow AX = \lambda X$

Premultiplying both sides by A,

we get $A(AX) = A(\lambda X)$

$$\Rightarrow A^2X = \lambda(AX)$$
$$= \lambda(\lambda X)$$
$$= \lambda^2 X$$

$$\Rightarrow A^2X = \lambda^2 X$$

$\Rightarrow \lambda^2$ is an eigenvalue of A^2.

2) If λ is an eigenvalue of a matrix A, Show that $K\lambda$ is an eigenvalue of the matrix KA.

Proof: Let X be an eigen vector Corresponding to an eigenvalue λ,

then $AX = \lambda X$

$$(KA)X = (K\lambda)X$$

$\Rightarrow K\lambda$ is an eigenvalue of the matrix KA.

Find the matrix A, if the eigen values and eigen vectors are given.

Eg:-1 The eigen vectors of a real symmetric matrix A corresponding to the eigen values 2, 3, 6 are respectively $(1\ 0\ -1)^T$, $(1, 1\ 1)^T$ and $(-1, 2, -1)^T$. find the matrix A.

Soln:

Given that eigen values of A are 2, 3, 6.

eigen vectors of A are

$(+1\ 0\ -1)^T$, $(1\ 1\ 1)^T$, $(-1\ 2\ -1)^T$

Hence Normalised modal matrix.

$$N = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

To find, A

We know that $D = N^T A N$.

$$\Rightarrow NDN^T = NN^T A NN^T$$
$$= \cancel{A} = IAI$$

$$\therefore A = NDN^T$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 2/\sqrt{2} & 3/\sqrt{3} & -6/\sqrt{6} \\ 0 & 3/\sqrt{3} & 12/\sqrt{6} \\ -2/\sqrt{2} & 3/\sqrt{3} & -6/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{2}+\frac{3}{3}+\frac{6}{6} & \frac{3}{3}-\frac{12}{6} & -\frac{2}{2}+\frac{3}{3}+\frac{6}{6} \\ 0+\frac{3}{3}-\frac{12}{6} & 0+\frac{3}{3}+\frac{24}{6} & 0+\frac{3}{3}-\frac{12}{6} \\ -\frac{2}{2}+\frac{3}{3}+\frac{6}{6} & 0+\frac{3}{3}-\frac{12}{6} & +\frac{2}{2}+\frac{3}{3}+\frac{6}{6} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & +1 \\ -1 & 5 & -1 \\ +1 & -1 & +3 \end{pmatrix}$$

<u>Eg:2</u> The eigenvectors of a real symmetric matrix A corresponding to the eigenvalues 1, 3, 6 are respectively $(2\ -1\ 0)^T$, $(0\ 0\ 1)^T$ and $(1\ 2\ 0)^T$. Find the matrix A.

Soln: we know that

$$D = N^T A N$$

$$A = N D N^T$$

Given that eigen values are 1, 3, 6

eigenvectors are $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

Hence Normalised modal matrix.

$$N = \begin{pmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix}$$

$$N^T = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}.$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

$$A = N D N^T$$

$$= \begin{pmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2/\sqrt{5} & 0 & 6/\sqrt{5} \\ -1/\sqrt{5} & 0 & 12/\sqrt{5} \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{10}{5} & \frac{+10}{5} & 0 \\ 10/5 & \frac{25}{5} & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Eg:3 Given that α, β are the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, form the matrix whose eigenvalues are α^2, β^2.

Soln:-

Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

To find the eigen values of A.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$

$\Rightarrow (3 - 4\lambda + \lambda^2 - 8) = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda = 5, -1$

-5

$-5 \mid 1$

-4

Given that α, β are the eigen values of $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Then α^2, β^2 are the eigen values of $A^2 = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix}$$

<u>Homework:</u>

1) Reduce the quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ into a canonical form using an orthogonal transformation.

2) Show that A satisfies its own characteristic equation and hence find A^8 if $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

APPLICATIONS OF MATRICES.

Some of the applications of matrices include control theory, analysis of vibration, electric circuits, advanced dynamics and quantum mechanics.

Eigen values are used to determine the theoretical limit by calculating the eigen values and eigen vectors of the communication channel which are expressed in a matrix form, and then water filling on the eigen values.

Eigen values and eigenvectors are applied in designing bridges and car stereo systems.

Also eigen values and eigenvectors are used in decoupling three-phase systems by applying symmetrical component transformation in electrical engineering.

In mechanical engineering the eigenvalues and eigen vectors are applied fore reducing deformation effect. Eigen values are ~~mostly are~~ mostly used in machine learning for dimensionality reduction.

Example:-1 Stretching of an elastic membrane.

An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2+x_2^2=1$ is stretched so that a point $P:(x_1,x_2)$ goes over into the point $Q:(y_1,y_2)$ given by

$$Y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = AX = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In components $y_1 = 5x_1 + 3x_2$ and
$y_2 = 3x_1 + 5x_2$.

Find the principal directions. That means, the directions of the position vector X of P for which the direction of the position vector Y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Soln: Let us consider for vector X such that $Y=\lambda X$.

Since we know that $Y=AX$,

then $AX=\lambda X$.

To find the eigenvalue and eigenvectors. & $|A-\lambda I|=0$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$\Rightarrow (5-\lambda)(5-\lambda) - 9 = 0$

$\Rightarrow 25 - 5\lambda - 5\lambda + \lambda^2 - 9 = 0.$

$\therefore \lambda^2 - 10\lambda + 16 = 0$

$\Rightarrow (\lambda-8)(\lambda-2) = 0$

$\therefore$ The eigen values are 8, 2.

Case (i) when $\lambda = 8$

Case (ii) when $\lambda = 2$.

To find the eigen vectors.

Solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$\Rightarrow \left.\begin{array}{l} (5-\lambda)x_1 + 3x_2 = 0 \\ 3x_1 + (5-\lambda)x_2 = 0 \end{array}\right]$ (I)

Case (i) when $\lambda = 8$, (I) $\Rightarrow$

$-3x_1 + 3x_2 = 0.$ — (1)

$3x_1 - 3x_2 = 0.$ — (2)

Take (1) $-3x_1 = -3x_2$

$$\frac{x_1}{1} = \frac{x_2}{1}$$

$\therefore X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Case (ii) when $\lambda = 2$,

(I) $\Rightarrow$ $3x_1 + 3x_2 = 0.$

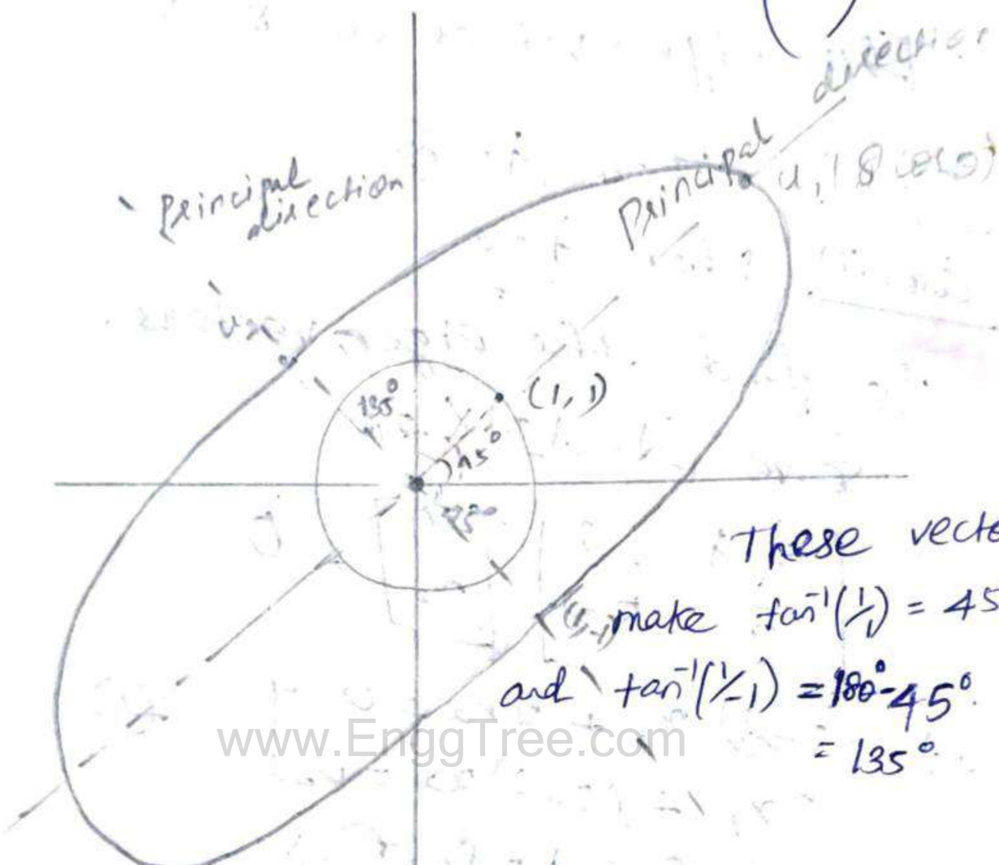
$3x_1 + 3x_2 = 0.$

$x_1 = -x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1}$

$\therefore X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$\therefore$ When $\lambda = 8$, eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $\lambda = 2$ eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



These vectors make $\tan^{-1}(1/1) = 45°$

and $\tan^{-1}(1/-1) = 180° - 45°$
$= 135°$

The boundary circle takes ellipse shape under this deformation.

Home work: Given $A = \begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$ in a $Y = AX$, find the principal directions and corresponding factors of extension or contraction.

Both were same

Take ①, $6x_1 + \sqrt{6}x_2 = 0$

$$6x_1 = -\sqrt{6}x_2$$

$$\frac{6x_1}{-\sqrt{6}} = \frac{x_2}{1}$$

$$\frac{x_1}{-1/\sqrt{6}} = \frac{x_2}{1} \qquad \therefore X_1 = \begin{pmatrix} -1/\sqrt{6} \\ 1 \end{pmatrix}$$

This vector make angle with positive x_1 direction is

$$\tan^{-1}\left(\frac{x_2}{x_1}\right) = \tan^{-1}\left(\frac{1}{-1/\sqrt{6}}\right)$$

$$= \tan^{-1}(-\sqrt{6})$$

$$= 180^\circ - \tan^{-1}\sqrt{6}$$

$$= 112 \cdot 2^\circ$$

This vector makes $112 \cdot 2^\circ$ angles with positive x_1 direction.

Case ii) Put $\lambda_2 = 8$,

$$-x_1 + \sqrt{6}x_2 = 0 \quad \text{— ③}$$

$$\sqrt{6}x_1 - 6x_2 = 0 \quad \text{— ④}$$

Both were same.

③ ⇒ $-x_1 = -\sqrt{6}x_2$

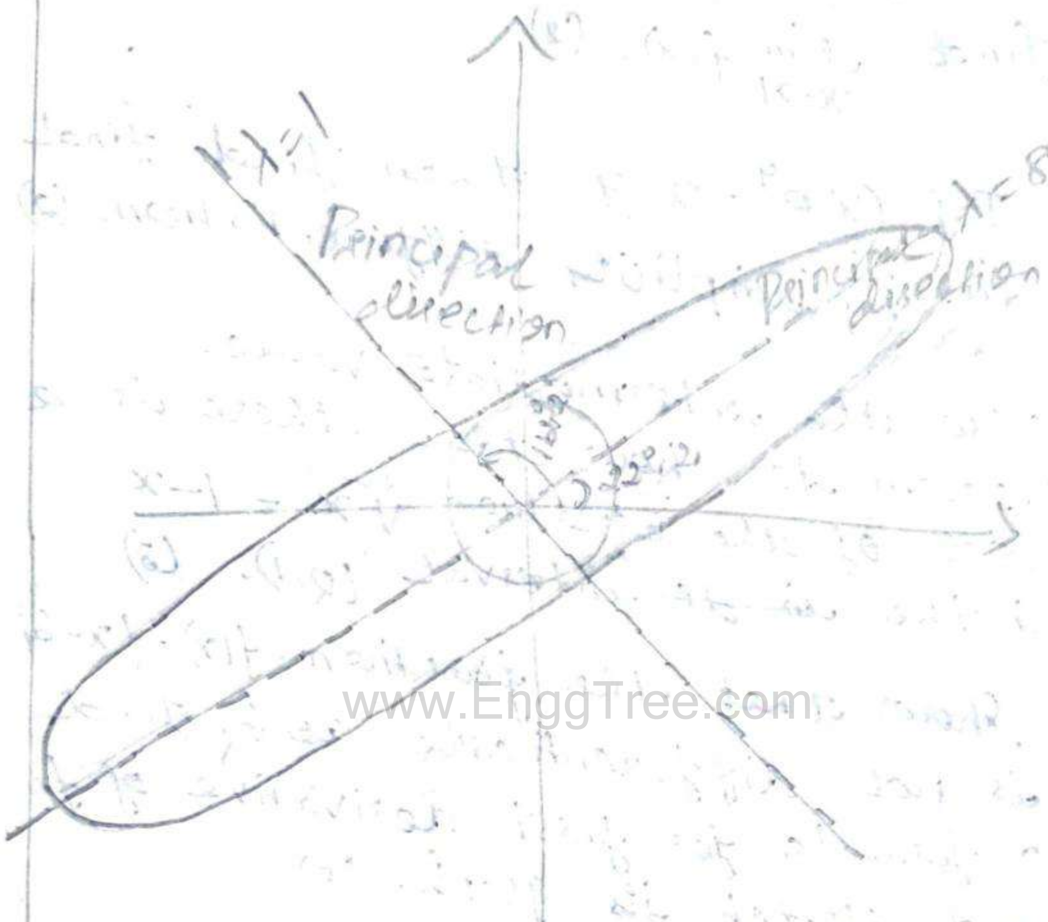
$$\frac{x_1}{\sqrt{6}} = \frac{x_2}{1} \qquad X_2 = \begin{pmatrix} \sqrt{6} \\ 1 \end{pmatrix}$$

This vector makes angle with positive x_1 direction is

$$\tan^{-1}\left(\frac{x_2}{x_1}\right) = \tan^{-1}\left(1/\sqrt{6}\right)$$

$$= 22.2^\circ.$$

This vector make 22.2° angles with positive x_1 direction.



Eg:- If the canonical form in three u, v, w is given by $3v^2 + 15w^2$ corresponding to a quadratic form, then state the nature, index, signature and rank of the quadratic form

Unit - II

Differential Calculus

Representation of a function

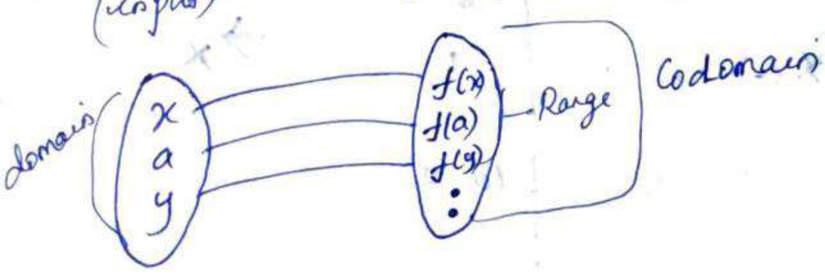
Definition: A function f is a rule that assigns to each element x in a set D exactly one element $f(x)$ in a ~~in~~ set E.

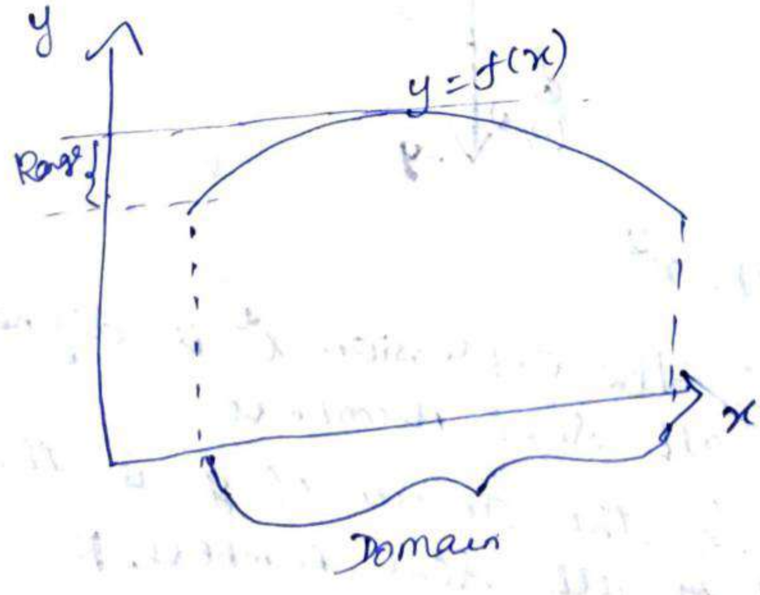
The set D is called the domain of the function.

The number $f(x)$ is the value of f at x.

The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

$$\underset{\text{(input)}}{x} \xrightarrow{f} \underset{\text{(output)}}{f(x)}$$





Ex:-1 Sketch the graph and find the domain and range of the following functions

a) $f(x) = 3x-2$.

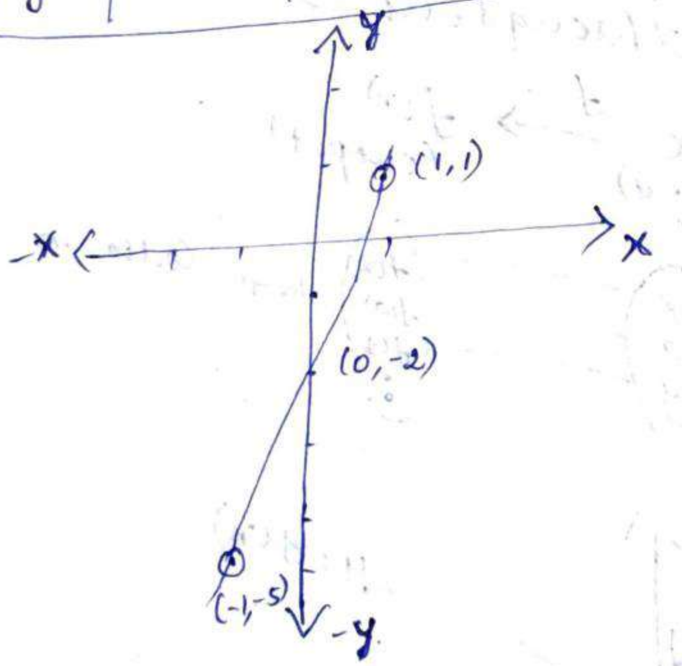
Soln: The expression $3x-2$ is defined for all real numbers.

So the domain of f is the set of all real numbers denoted by $\mathbb{R}$.

Range of f is $f(x) = 3x-2$. $\forall x \in R$.

$f(x) = y = 3x-2$.

x	-1	0	1
y	-5	-2	1



b) $g(x) = x^2$

Soln: The expression x^2 is defined for all real numbers.

So the domain of g is the set of all real numbers, R.

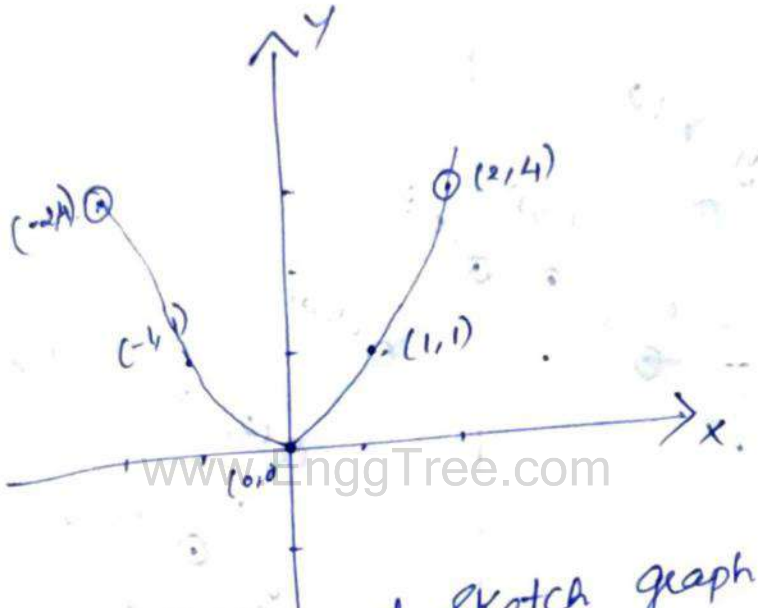
Range of g is all $g(x) = x^2, \forall x \in R$

i.e) All numbers of the form x^2.

$\therefore$ The range of g is $[0, \infty)$.

let $y = g(x) = x^2$.

$x =$	-1	0	1
y	1	0	1



c) Find the domain and sketch graph of the function $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0. \end{cases}$

Soln: Given $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0. \end{cases}$

i.e) $y = x+2, \quad x < 0$

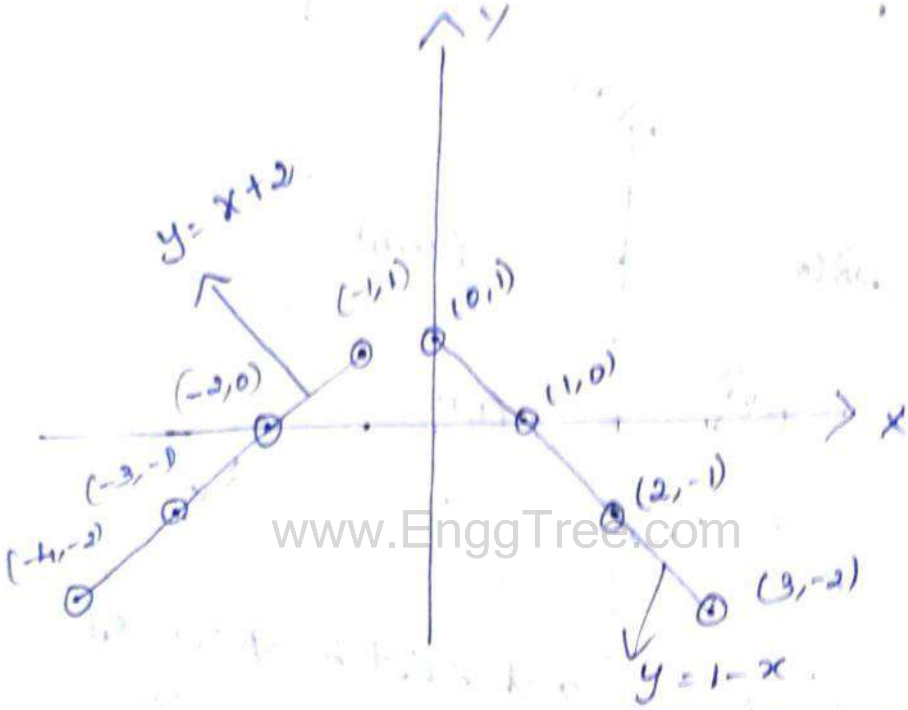
$\therefore y = 1-x, \quad x \geq 0$

The domain of f is $(-\infty, \infty)$

i.e) The set of all real numbers.

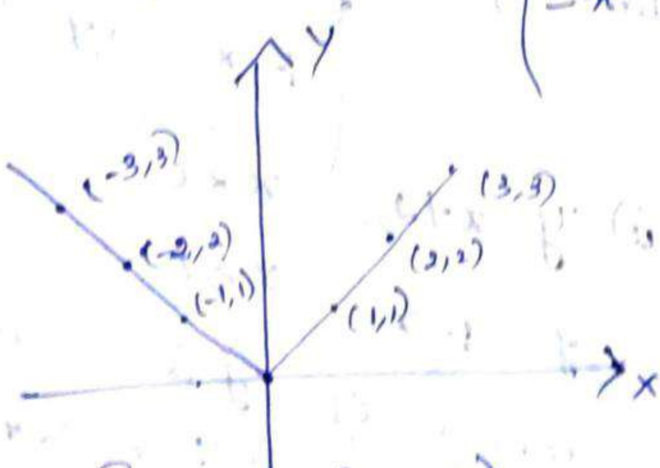
$x<0$	-1	-2	-3	-4
$y=x+2$	1	0	-1	-2

$x\geq 0$	0	1	2	3
$y=1-x$	1	0	-1	-2



d) Sketch the graph of the absolute value function $f(x) = |x|$.

Soln: Let $y = f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$



Range: $[0, \infty)$

domain: $\mathbb{R}$.

Ex:-2 Find the domain of the function $f(x) = \sqrt{x+2}$.

Soln: Since the square root of a negative numbers is not defined as a real numbers.

The square root of a number must be positive.

$\therefore x+2 \geq 0$

$\Rightarrow x \geq -2$.

$\therefore$ The domain of f is $\{x / x \geq -2\}$

$= [-2, \infty)$

Ex:-3 Find the domain of the function $\sqrt{3-x} - \sqrt{2+x}$.

Soln: For the negative numbers we cannot defined the square root.

$3-x \geq 0$, $\quad 2+x \geq 0$

$3 \geq x$, $\quad 2 \geq -x$.

ie) $x \leq 3$, $\quad -2 \leq x$.

$\Rightarrow -2 \leq x \leq 3$

$\therefore$ The domain is $[-2, 3]$

Ex:-4 Find the domain of the function $f(x) = \frac{1}{x^2-x}$

Soln: $f(x) = \frac{1}{x^2-x} = \frac{1}{x(x-1)}$

The function is not defined at $x=0$ and $x=1$.

$\therefore$ The domain is $\{x / x \neq 0, x \neq 1\}$.

ie) The domain is $(-\infty, 0) \cup (0,1) \cup (1,\infty)$.

Ex:-5 Find the domain $\frac{2x^2-5}{x^2+x-6}$.

Soln: $f(x) = \frac{2x^2-5}{x^2+x-6} = \frac{2x^2-5}{(x+3)(x-2)}$.

The function is not defined for $x=-3$ and $x=2$.

$\therefore$ The domain is $(-\infty,-3) \cup (-3,2) \cup (2,\infty)$

Ex:-6 Find the domain $f(x) = \frac{1}{\sqrt{x^2-5x}}$.

Soln: $x^2-5x \geq 0$

$x^2 \geq 5x$

$x \geq 5$

The domain is $[5,\infty)$

Homework:

1) Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$.

2). Find the domain of the function $f(x) = \sqrt{x+2}$.

Definition: A function $y=f(x)$ is an even function of x if $f(-x)=f(x)$ and Odd function of x if $f(-x) = -f(x)$ for every number x in its domain.

Note: $\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$

Eg:-1 Determine whether each of the following functions is even or odd.

a) $f(x) = x^3 + x$

Soln: Given $f(x) = x^3 + x$.

$f(-x) = (-x)^3 + (-x)$

$= -x^3 - x = -(x^3 + x) = -f(x)$

$\therefore f(x)$ is an odd function.

b) $f(x) = 1 - \cos x$.

Soln: Given $f(x) = 1 - \cos x$.

$f(-x) = 1 - \cos(-x)$

$= 1 - \cos x = f(x)$

$\therefore f(x)$ is an even function.

Homework: Determine whether the following is even or odd.

a) $f(x) = x\cos x$ (odd) b) $f(x) = x^2 + 1$ (even)

Eg:-2 Evaluate the difference quotient for the function

$f(x) = 4 + 3x - x^2$, $\dfrac{f(3+h) - f(3)}{h}$.

Soln: Let $f(x) = 4 + 3x - x^2$

$f(3+h) = 4 + 3(3+h) - (3+h)^2$

$= 4 + 9 + 3h - (9 + h^2 + 6h)$

$= 4 + 9 + 3h - 9 - h^2 - 6h$

$= 4 - 3h - h^2$

$f(3) = 4 + 3(3) - 3^2$

$= 4 + 9 - 9 = 4$.

$\dfrac{f(3+h) - f(3)}{h} = \dfrac{(\cancel{4} - 3h - h^2) - \cancel{4}}{h} = \dfrac{-\cancel{h}(3+h)}{\cancel{h}} = -(3+$

Homework:

1) Find the ~~evalu~~ difference quotient for the function $f(x) = \frac{x+3}{x+1}$, $\frac{f(x)-f(1)}{x-1}$

(Ans: $\frac{-1}{(x+1)}$)

Increasing and decreasing function.

A function f is called increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

A function f is called decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

Eg: (i) $f(x) = x^2$ is decreasing in $(-\infty, 0]$ and increasing in $[0, \infty)$

(ii) $f(x) = -x^3$ is decreasing in $(-\infty, \infty)$

Limit of a function

Suppose $f(x)$ is defined when x is near the number a. [(ie) f is defined on some open interval that contains a, except possibly at a] then the limit of that function is

$$\lim_{x \to a} f(x) = L.$$

It is defined as the limit of $f(x)$ as x approaches a, equal to L.

Eg: 1 Find the value of $\lim_{x\to 1} \frac{x-1}{x^2-1}$

Soln:

Let $f(x) = \frac{x-1}{x^2-1} = \frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}}$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{1+1} = \frac{1}{2} = 0.5$

Eg: 2 Find the value of

$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Soln:

Let $f(t) = \frac{\sqrt{t^2+9}-3}{t^2}$

$= \frac{\sqrt{t^2+9}-3}{t^2} \times \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

$= \frac{(\sqrt{t^2+9})^2-(3)^2}{t^2(\sqrt{t^2+9}+3)}$

$= \frac{\cancel{t^2}+\cancel{9}-\cancel{9}}{\cancel{t^2}(\sqrt{t^2+9}+3)}$

$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{9}+3}$

$= \frac{1}{3+3} = \frac{1}{6}$

Left limit: The left limit as x approaches "a" of $f(x)$ is L if the values of $f(x)$ gets as close to L as when x is very close to and left a, $x < a$.

i.e) $\lim_{x\to a^-} f(x) = L$.

Right limit: The right limit as x approaches "a" of $f(x)$ is L if the values of $f(x)$ gets as close to L as when x is very close to and right of a, $x > a$.

i.e) $\lim_{x \to a^+} f(x) = L$.

Defn: The limit of a function $f(x)$ can be defined using the right and left definitions of right and left limits of $f(x)$ as $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and

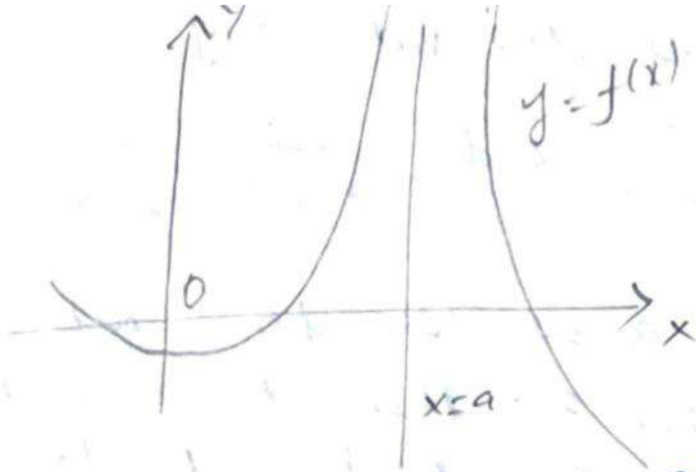
$\lim_{x \to a^+} f(x) = L$.

Infinite limits:

* Let f be a function defined on both sides of "a" except possibly at "a" itself.

Then $\boxed{\lim_{x \to a} f(x) = \infty}$

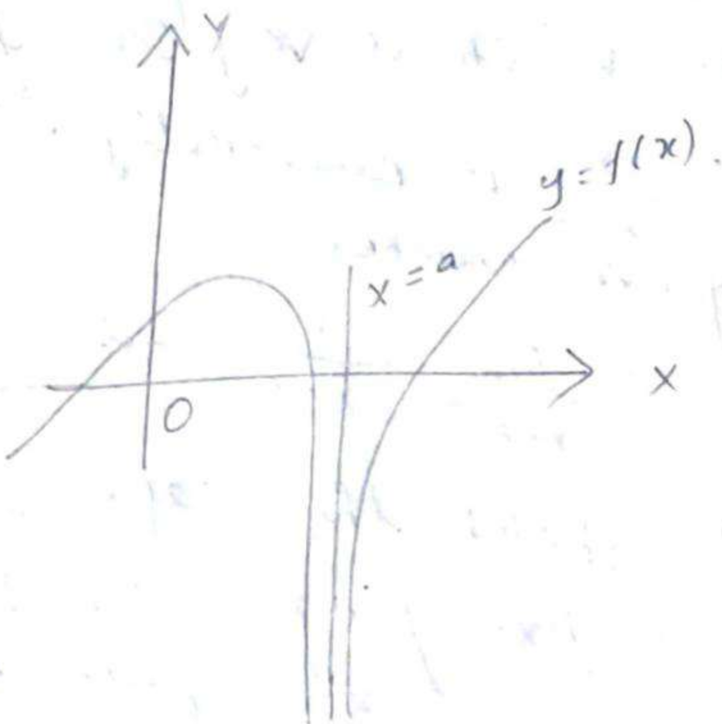
i.e) The values of $f(x)$ can be arbitrarily large, by taking x sufficiently close to "a" but not equal to "a"



* Let f be defined on both sides of "a" except at a, then

$$\boxed{\lim_{x \to a} f(x) = -\infty}$$

i.e) The values of $f(x)$ can be arbitrarily large negative by taking x sufficiently close to a but not equal to a.



Ex:-1 Find the values of

$\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$

Soln:

If x is close to 3 but larger than 3, then the denominator $x-3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/_{x-3}$ is a large positive number.

$$\therefore \lim_{x\to 3^+} \frac{2x}{x-3} = \infty.$$

If x is close to 3 but smaller than 3, then $x-3$ is a small negative number but $2x$ is a positive number which is very close to 6.

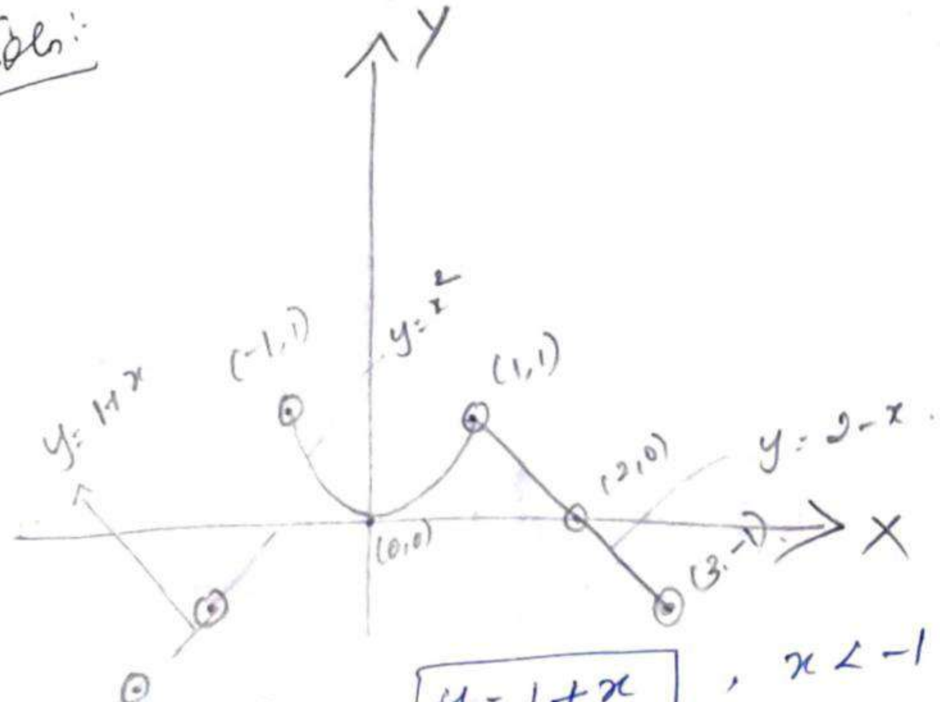
So $\frac{2x}{x-3}$ is numerically large negative number.

$$\therefore \lim_{x\to 3^-} \frac{2x}{x-3} = -\infty.$$

Ex:-2 Sketch the graph of the function $f(x) = \begin{cases} 1+x & , x < -1 \\ x^2 & , -1 \le x \le 1 \\ 2-x & , x \geq 1 \end{cases}$

and use it to determine the values of "a" for which $\lim_{x\to a} f(x)$ exists.

Soln:



let $y = 1+x$.

~~x~~	~~-1~~	~~0~~	~~1~~
~~y~~	~~0~~	~~1~~	~~2~~

$\boxed{y = 1+x}$, $x < -1$

x	-2	-3	-4
y	-1	-2	-3

$\boxed{y = x^2,}$ $-1 \le x \le 1$ $\boxed{y = 2-x}$, $x \geq 1$

x	-1	0	1
y	1	0	1

x	1	2	3
y	1	0	-1

Since the right and left limits are different at $a = -1$, it is observed that $\lim_{x \to a} f(x)$ exists for all "a" except when $a = -1$.

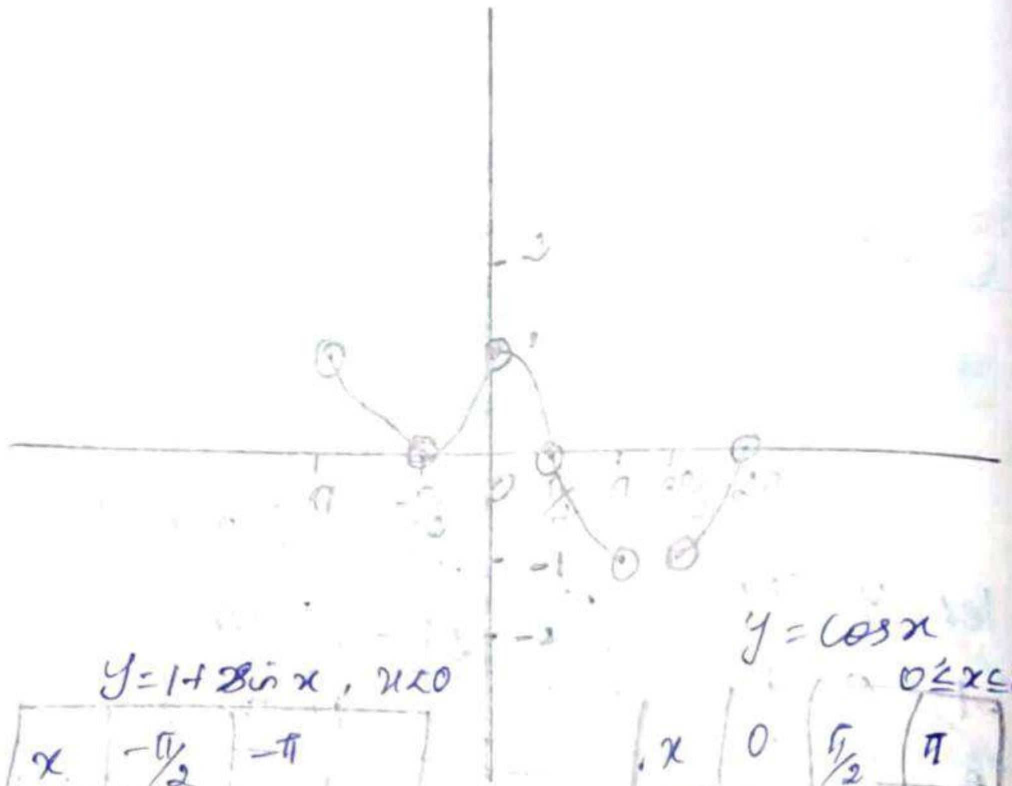
Homework:

1) Sketch the graph of the function

$$f(x) = \begin{cases} 1 + \sin x, & x < 0 \\ \cos x, & 0 \le x \le \pi \\ \sin x, & x > \pi \end{cases}$$

and use it to determine the values of "a" for which $\lim_{x \to a} f(x)$ exists.

Soln:



$y = 1 + 2\sin x,\ x < 0$

x	$-\pi/2$	$-\pi$
y	0	0

$y = \cos x$, $0 \le x \le \pi$

x	0	$\pi/2$	π
y	1	0	-1

$y = \sin x,\ x > \pi$

x	$3\pi/2$	2π
y	-1	0

Since the left and right limits are different at the point $a = \pi$, from the above graph, it is noticed that $\lim_{x \to a} f(x)$ exists for all "a" except when $a = \pi$.

Calculating limits using the limit laws.

1. $\lim_{x\to a} (f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

2. $\lim_{x\to a} (f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.

3. $\lim_{x\to a} (c f(x)) = c \lim_{x\to a} f(x)$

4. $\lim_{x\to a} (f(x)\, g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

5. $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.

6. $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$, where n is a positive integer.

7. $\lim_{x\to a} c = c$, where c is a constant.

8. $\lim_{x\to a} x = a$.

9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer.

10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer.
(If n is even, we assume that $a>0$)

11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer.
(If n is even, we assume that $\lim_{x\to a} f(x) > 0$)

Eg:-1 Find $\lim_{x \to 5} (2x^2 - 3x + 4)$

Soln: $\lim_{x \to 5} (2x^2 - 3x + 4)$

$= 2(5)^2 - 3(5) + 4$

$= 50 - 15 + 4$

$= 39$

Eg:-2 $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Soln:

$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)}$

$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$

$= \frac{-8 + 8 - 1}{5 + 6} = -1/11$.

Ex:-3 $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

Soln: $= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$

$= \lim_{x \to 1} (x+1)$

$= 1 + 1 = 2$.

Ex:-4 Evaluate: $\lim_{x \to 0} \frac{(3+x)^2 - 9}{x}$

Soln: $\lim_{x \to 0} \frac{(3+x)^2 - 9}{x}$

$= \lim_{x \to 0} \frac{((3+x)+3)((3+x)-3)}{x}$

$= \lim_{x \to 0} \left(\frac{(6+x)\cancel{x}}{\cancel{x}} \right) = 6 + 0$

$= 6$.

Ex:-5 Evaluate $\lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1}$

Soln:

$$\lim_{x \to 1} \frac{x^4-1}{x^3-1} = \lim_{x \to 1} \frac{(x^2)^2 - (1)^2}{x^3-1}$$

$$= \lim_{x \to 1} \frac{(x^2-1)(x^2+1)}{(x^3-1)}$$

$$= \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}(x^2+1)}{\cancel{(x-1)}(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{(x+1)(x^2+1)}{(x^2+x+1)}$$

$$= \frac{(1+1)(1+1)}{(1+1+1)} = 4/3.$$

Ex:6 Show that $\lim_{x \to 0} |x| = 0$.

Soln: We know that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Since $|x| = -x$, for $x < 0$

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0 \quad \text{--- (1)}$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{--- (2)}$$

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^+} |x| = 0$$

Hence $\lim_{x \to 0} |x| = 0$.

Ex:-7 If $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 0 \\ 8-2x & \text{if } x < 4 \end{cases}$

determine whether $\lim_{x \to 4}$ exists.

Soln:-

$f(x) = \sqrt{x-4}$ for $x > 4$.

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4}$

$= \sqrt{4-4} = 0$ —— (1)

$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8-2x)$

$= 8-2(4) = 0$ —— (2)

from (1) & (2) we get,

$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^-} f(x) = 0$

So $\lim_{x \to 4} f(x)$ exists.

$\lim_{x \to 4} f(x) = 0.$

Ex:8 Check whether

$\lim_{x \to -3} \frac{3x+9}{|x+3|}$ exist.

Soln:- Since $|x+3| = -(x+3)$

if $x+3 < 0$

$\Rightarrow x < -3$

$\therefore |x+3| = -(x+3)$ if $x < -3$.

Since $|x+3| = x+3$ if $x+3 > 0$

$\Rightarrow x > -3$

$\therefore |x+3| = x+3$ if $x > -3$.

$$\lim_{x\to -3^-} \frac{3x+9}{|x+3|} = \lim_{x\to -3^-} \frac{3x+9}{-(x+3)}$$

$$= \lim_{x\to -3^-} \frac{3\cancel{(x+3)}}{-\cancel{(x+3)}}$$

$$= -3 \quad \text{—} (1)$$

$$\lim_{x\to -3^+} \frac{3x+9}{|x+3|} = \lim_{x\to -3^+} \frac{3x+9}{x+3}$$

$$= \lim_{x\to -3^+} \frac{3\cancel{(x+3)}}{\cancel{(x+3)}}$$

$$= 3 \quad \text{—} (2)$$

from (1) & (2).

$$\lim_{x\to -3^-} \frac{3x+9}{|x+3|} \neq \lim_{x\to -3^+} \frac{3x+9}{|x+3|}.$$

$\therefore \lim_{x\to -3} \frac{3x+9}{|x+3|}$ does not exist.

Ex:9 Guess the value of the limit (if exists) for the function $\lim_{x\to 0} \frac{e^{5x}-1}{x}$ by evaluating the function at the given numbers $x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.001$ (correct to six decimal places)

~~Soln:~~ Ex:-10 If $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$, then find the value of $\lim_{x\to 1} f(x)$.

Soln:

$$\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$$

$$\Rightarrow \frac{\lim_{x\to 1} (f(x)-8)}{\lim_{x\to 1} (x-1)} = 10.$$

$$\Rightarrow \lim_{x\to 1} f(x) - \lim_{x\to 1} 8 = 10\left(\lim_{x\to 1} (x-1)\right)$$

$$\Rightarrow \lim_{x\to 1} f(x) - 8 = 10(0)$$

$$\Rightarrow \lim_{x\to 1} f(x) = 8.$$

Home work:

1) If the function $f(x)$ satisfies $\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim_{x\to 1} f(x)$. (Ans:- 2)

The Squeeze theorem:

If $f(x) \le g(x) \le h(x)$ where x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$,

then $\lim_{x\to a} g(x) = L$.

[The squeeze theorem is otherwise called as sandwich theorem or the pinching theorem. It states that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a]

Ex:-1 Show that $\lim_{x\to 0} x^2 \sin 1/x = 0$ by using ~~us~~ squeeze theorem.

Soln: Here we cannot use

$\lim_{x\to 0} x^2 \cdot \sin(1/x) = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin(1/x)$

Because $\lim_{x\to 0} \sin(1/x)$ does not exist.

Therefore, to apply squeeze theorem, we have to find a function f smaller than $g(x) = x^2 \cdot \sin(1/x)$.

and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0 as $x \to 0$.

we have,

$$-1 \le \sin \frac{1}{x} \le 1$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$(\because x^2 \ge 0)$ for all x.

Here

$$\lim_{x \to 0} (-x^2) = 0$$

$$\lim_{x \to 0} x^2 = 0$$

Taking $f(x) = -x^2$ and

$g(x) = x^2 \sin\left(\frac{1}{x}\right)$

and $h(x) = x^2$ in the Squeeze theorem.

we have

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Continuity

Definition: A function f is continuous at a number a if $\lim_{x \to a} f(x) = f(a)$.

Note:- If f is continuous at a, then

1. $f(a)$ should exist
2. $\lim_{x \to a} f(x)$ exists both on the left and right
   i.e) $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exists.
3) $\lim_{x \to a} f(x) = f(a)$.

Definition: A function f is continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$.

and f is continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$.

Thm: If f and g are continuous at a and c is a constant then the following functions are also continuous at a

a) $f+g$ b) $f-g$ c) cf d) fg

e) f/g if $g(a) \neq 0$.

Thm: Any polynomial is continuous everywhere (i.e) it is continuous on $R = (-\infty, \infty)$.

Any rational function is continuous whenever it is defined (that is) it is continuous on its domain.

Example: Where are each of the following function discontinuous.

a) $f(x) = \frac{x^2 - x - 2}{x - 2}$

Soln: Here $f(2)$ is not defined
Hence f is discontinuous at $x = 2$.

b) $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Soln: Here $f(0) = 1$.
but $\lim_{x \to 0} f(x) = \lim_{x \to 0} 1/x^2$ does not exist

$\therefore f$ is discontinuous at 0.

$$f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Soln:

Here $f(2) = 1$.

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-x-2}{x-2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}} = (1+2) = 3.$$

$\lim_{x \to 2} f(x) = 3$ It exists

So f is not continuous at $x = 2$

Ex: 2 Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous in the interval $[-1, 1]$.

Solution:

$$\lim_{x \to -1^+} f(x) = 1 = f(-1)$$

$$\lim_{x \to 1^-} f(x) = 1 = f(1).$$

So, f is continuous from the right at -1 and f is continuous from the left at 1

$\therefore f$ is continuous on $[-1, 1]$

Ex 1.3 Show that the function is Continuous at the number "a" by using definition of continuity and properties

$f(x) = 3x^4 - 5x + \sqrt[3]{x^2+4}$

Soln:
$$\lim_{x\to 2} f(x) = \lim_{x\to 2} [3x^4 - 5x + \sqrt[3]{x^2+4}]$$
$$= 3\lim_{x\to 2}(x^4) - 5\lim_{x\to 2}(x) + \lim_{x\to 2}\sqrt[3]{x^2+4}$$
$$= 3(2)^4 - 5(2) + \sqrt[3]{2^2+4}$$
$$= 3(16) - 10 + \sqrt[3]{8}$$
$$= 48 - 10 + 2$$

$$= 50 - 10 = 40.$$

$$f(2) = 3(2)^4 - 5(2) + \sqrt[3]{2^2+4}$$
$$= 48 - 10 + 2 = 40.$$
$$\therefore \lim_{x\to 2} f(x) = f(2).$$

f is Continuous at $a=2$.

Ex 1.4 Show that the function $f(x) = x + \sqrt{x-4}$ is Continuous on the interval $[4,\infty)$

Soln: for $a \geq 4$
$$\lim_{x\to a} f(x) = \lim_{x\to a} (x + \sqrt{x-4})$$
$$= \lim_{x\to a}(x) + \lim_{x\to a}\sqrt{x-4}$$

$= a + \sqrt{a-4}$

$= f(a)$

$\therefore f$ is Continuous on $[4, \infty)$.

~~The~~

Ex:-5 For what value of the Constant C is the function f is Continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} Cx^2 + 2x, & x < 2 \\ x^3 - Cx, & x \geq 2 \end{cases}$$

Soln: The given function $f(x)$ is Continuous on $(-\infty, \infty)$

Then the function $f(x)$ is Continuous at $x = 2$.

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) \quad \text{—①}$$

Now, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^3 - Cx$

$= 8 - 2C$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} Cx^2 + 2x$

$= 4C + 4$ —②

$\therefore$ ① $\Rightarrow$ $8 - 2C = 4C + 4$

$-4C - 2C = 4 - 8$

$-6C = -4$

$C = \frac{-4}{-6} = \frac{2}{3}$ $\boxed{C = \frac{2}{3}}$

Ex-6 Find the values of a and b that makes $f(x)$ is continuous everywhere

$$f(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{if } x<2 \\ ax^2-bx+3 & \text{if } 2 \le x < 3 \\ 2x-a+b & \text{if } x \geq 3 \end{cases}$$

Soln: The given function $f(x)$ is continuous on $(-\infty, \infty)$

$$\therefore \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

f is continuous at $x=2$,

then $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$ — (1)

Now $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax^2 - bx + 3)$

$$= 4a - 2b + 3$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(\frac{x^3-8}{x-2} \right)$$

$$= \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}$$

$$= 4+4+4 = 12.$$

Now, (1) $\Rightarrow$ $4a - 2b + 3 = 12$

$$\Rightarrow 4a - 2b = 12 - 3$$

$$\Rightarrow 4a - 2b = 9 \quad \text{— (I)}$$

Also The function $f(x)$ is continuous at $x=3$,

Then $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x)$ —— ②

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b$$

$$= 2(3) - a + b$$

$$= 6 - a + b$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2 - bx + 3)$$

$$= 9a - 3b + 3$$

② $\Rightarrow$ $6 - a + b = 9a - 3b + 3$

$$-a - 9a + b + 3b = 3 - 6$$

$$-10a + 4b = -3.$$

$$10a - 4b = 3 \text{ —— (II)}$$

(I) $\times 2 \Rightarrow$ $8a - 4b = 18$

(II) $\Rightarrow$ $\underline{\overset{(-)}{10a} - \overset{(+)}{4b} = \overset{(-)}{3}}$

(I) - (II) $-2a = 15$

$$\boxed{a = \frac{-15}{2}}$$

(I) $\Rightarrow$ $4\left(\frac{-15}{2}\right) - 2b = 9$

$\Rightarrow$ $-30 - 2b = 9$

$\Rightarrow$ $-2b = 9 + 30$

$\Rightarrow$ $\boxed{b = -39/2}$

The intermediate value theorem.

Suppose that f is Continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c)=N$.

Example: 1 Prove that the equation $x^3-15x+1=0$ has atmost one real root in the interval $[-2,2]$.

Soln: Let $f(x)=x^3-15x+1$

$f(-2)=(-2)^3-15(-2)+1$

$=-8+30+1=23$ (+)ve

$f(-1)=-1+15+1=15=(+)$ ve

$f(0)=1=(+)$ve

$f(1)=(1)^3-15(1)+1=-13=(-)$ve

$f(2)=(2)^3-30+1=-21=(-)$ve

Hence $f(0)>0>f(1)$

f Changes sign between 0 and 1.

$\therefore$ By intermediate value theorem there is a number C between 0 and 1 such that $f(C)=0$

Hence there is atmost one real root in the interval $[-2, 2]$.

Example:-2 Use the intermediate value theorem to show that there is a root of the equation $\sqrt[3]{x} = 1-x$ in the interval $(0,1)$.

Soln:- Let $f(x)$ be $\sqrt[3]{x} = 1-x$

Then $f(x) = \sqrt[3]{x} + x - 1$
$= 0$

$f(0) = -1$ (−)ve

$f(1) = \sqrt[3]{1} + 1 - 1$
$= 2 - 1 = 1$ (+)ve

Hence $f(0) > f(1)$

f changes sign between 0 and 1.

$\therefore$ By intermediate value theorem there is a number C between 0 and 1 such that $f(C) = 0$.

$\therefore$ There exists a root of the equation $\sqrt[3]{x} = 1-x$ in the interval $(0,1)$.

Homework: Using intermediate value theorem, show that there is a root of the given equation in the given interval $f(x) = x^4 + x - 3 = 0$ at $(1,2)$

Derivatives of polynomials, exponential, logarithmic and Trigonometric functions.

1. Derivative of a constant function.

$\frac{d}{dx}(c) = 0$, where c is a constant.

2 Power rule.

If n is any real number then $\frac{d}{dx}(x^n) = nx^{n-1}$.

3. The constant multiple rule.

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}(c\,f(x)) = c \cdot \frac{d}{dx}(f(x))$$

4. The Sum rule.

If f and g are both differentiable, then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

5) The difference rule:-

If f and g are both differentiable, then

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

6) Derivative of the natural exponential function.

$$\frac{d}{dx}(e^x) = e^x.$$

7) The product rule:

If f and g are both differentiable, then

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

8) The quotient rule:

If f and g are differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}.$$

Differentiation of some functions.

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(\sin x) = \cos x$.
5. $\frac{d}{dx}(\cos x) = -\sin x$.
6. $\frac{d}{dx}(\tan x) = \sec^2 x$.
7. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$
9. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
10. $\frac{d}{dx}(\log x) = 1/x$.

Eg:-1 (i) $f(x) = \sqrt{30}$ [find the derivative of the following function]

(ii) $x^{100} = f(x)$

(iii) $\sqrt{x} = f(x)$

(iv) $f(x) = x^8 - 12x^5 - 4x^4 + 10x^3 - 6x + 5$

Soln:

(i) $\frac{d(f(x))}{dx} = \frac{d}{dx}(\sqrt{30})$

$= 0.$ ($\because \sqrt{30}$ is Constant)

(ii) $\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^{100})$

$= 100x^{99}$

(iii) $f(x) = \sqrt{x}$

$= x^{1/2}$

$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^{1/2})$

$= \frac{1}{2}x^{1/2-1}$

$= \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{-1/2}$

$= \frac{1}{2}\frac{1}{x^{1/2}}$

$= \frac{1}{2\sqrt{x}}$

(iv) $f(x) = x^8 - 12x^5 - 4x^4 + 10x^3 - 6x + 5$

$\frac{d(f(x))}{dx} = 8x^7 - 12(5x^4) - 4(4x^3) + 10(3x^2) - 6(1) + 0$

$= 8x^7 - 60x^4 - 16x^3 + 30x^3 - 6$

Eg:-2 Differentiate the following.

(i) $y = a^x$, (ii) $y = e^x - x$

Soln:

(i) $y = a^x \Rightarrow y = e^{\log a^x} = e^{x \log a}$

$\frac{dy}{dx} = e^{x\log a} \times \log a$

$= e^{\log a^x} \times \log a$

$= a^x \log a$

(ii) $y = e^x - x$.

$\frac{dy}{dx} = e^x - 1$.

Eg:-3 Differentiate the following

(i) $f(x) = (x^3 + 2x) e^x$

(ii) $f(x) = \frac{x^2}{(1+2x)}$

(iii) $f(x) = x^3 \sin x$.

(iv) $f(x) = \operatorname{cosec} x + e^x \cot x$.

Soln:

(i) $\frac{d f(x)}{dx} = (x^3 + 2x)\frac{d(e^x)}{dx} + e^x \frac{d}{dx}(x^3 + 2x)$

$= (x^3 + 2x) e^x + e^x (3x^2 + 2)$

$= e^x (x^3 + 3x^2 + 2x + 2)$

(ii) $\frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\left(\frac{x^2}{1+2x}\right)$

$= \frac{(1+2x)\frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(1+2x)}{(1+2x)^2}$

$$= \frac{(1+2x)(2x) - x^2(0+2(1))}{(1+2x)^2}$$

$$= \frac{2x + 4x^2 - 2x^2}{1+4x+4x^2}$$

$$= \frac{2x+2x^2}{1+4x+4x^2}$$

(iii) $f(x) = x^3 \sin x$

$$\frac{d f(x)}{dx} = x^3 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^3)$$

$$= x^3 \cos x + \sin x (+3x^2)$$

$$= x^3 \cos x + 3x^2 \sin x.$$

(iv) $f(x) = \operatorname{Cosec} x + e^x \cot x.$

$$\frac{d(f(x))}{dx} = \frac{d}{dx}(\operatorname{cosec} x) + \frac{d}{dx}(e^x \cot x)$$

$$= -\operatorname{Cosec} x \cot x + e^x \frac{d}{dx}(\cot x) + \cot x \frac{d}{dx}(e^x)$$

$$= -\operatorname{cosec} x \cot x + e^x(-\operatorname{Cosec}^2 x) + \cot x \times e^x.$$

$$= -\operatorname{Cosec} x \cot x - e^x \operatorname{cosec}^2 x + e^x \cot x.$$

The Chain Rule

Ex) Differentiate the following

(i) $y = \sin(x^2)$

(ii) $y = (x^3-1)^{100}$

(iii) find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

(iv) $y = x^2 e^{2x}(x^2+1)^4$

Soln:

(i) $\frac{dy}{dx} = \cos(x^2) \times \frac{d}{dx}(x^2)$

$= 2x\cos(x^2)$

(ii) $\frac{dy}{dx} = \frac{d}{dx}(x^3-1)^{100}$

$= 100(x^3-1)^{99} \times \frac{d}{dx}(x^3-1)$

$= 100(x^3-1)^{99} \times (3x^2)$

$= 300x^2(x^3-1)^{99}$

(iii) $g'(t) = \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)^9$

$= 9\left(\frac{t-2}{2t+1}\right)^8 \times \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$

$= 9\left(\frac{t-2}{2t+1}\right)^8 \times \frac{(2t+1)\frac{d}{dt}(t-2)-(t-2)\frac{d}{dt}(2t+1)}{(2t+1)^2}$

$= 9\frac{(t-1)^8}{(2t+1)^8} \times \frac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}$

$$= \frac{9(t-1)^8(2t+1-2t+4)}{(2t+1)^{10}}$$

$$= \frac{9(t-1)^8(5)}{(2t+1)^{10}}$$

$$= \frac{45(t-1)^8}{(2t+1)^{10}}$$

(iv) $y = x^2 e^{2x}(x^2+1)^4$

$$\frac{dy}{dx} = x^2 e^{2x}\frac{d}{dx}(x^2+1)^4$$
$$+ x^2(x^2+1)^4\frac{d}{dx}(e^{2x})$$
$$+ (x^2+1)^4 e^{2x}\frac{d}{dx}(x^2)$$

$$= x^2 e^{2x}\left(4(x^2+1)^3 \times \frac{d}{dx}(x^2+1)\right)$$
$$+ x^2(x^2+1)^4 \cdot e^{2x} \cdot \frac{d}{dx}(2x)$$
$$+ (x^2+1)^4 e^{2x} \times 2x.$$

$$= x^2 e^{2x}\left(4(x^2+1)^3 \times 2x\right)$$
$$+ x^2(x^2+1)^4 \times e^{2x} \times 2.$$
$$+ (x^2+1)^4 e^{2x} \times 2x$$

$$= 2x e^{2x}\left(4x^2(x^2+1)^3 + x(x^2+1)^4 + (x^2+1)^4\right).$$

$$= 2x e^{2x}(x^2+1)^3\left(4x^2 + x(x^2+1) + (x^2+1)\right)$$

$$= 2x e^{2x}(x^2+1)^3 \left(4x^2 + x^3 + x + x^2 + 1\right)$$

$$\frac{dy}{dx} = 2x e^{2x}(x^2+1)^3 \left(x^3 + 5x^2 + x + 1\right)$$

Homework:

1. Differentiate $y = (2x+1)^5 (x^3 - x + 1)$!
2. Differentiate $y = e^{\sin x}$.

Implicit Differentiation

An implicit function is a function $y = f(x)$ which is defined by an equation of the form

$$F(x,y) = 0.$$

Ex:-1 If $xy = c^2$, then find $\frac{dy}{dx}$.

Soln:

$$xy = c^2$$

$$x \times \frac{dy}{dx} + y \frac{d(x)}{x} = 0.$$

$$x \frac{dy}{dx} + y(1) = 0.$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Ex:-2 If $xe^y = x - y$, then find $\frac{dy}{dx}$ by implicit differentiation.

Soln: $xe^y = x - y$

$$x\frac{d}{dx}(e^y) + e^y\frac{d(x)}{dx} = \frac{d(x)}{dx} - \frac{d}{dx}(y)$$

$$x \times e^y \times \frac{dy}{dx} + e^y \times (1) = 1 - \frac{dy}{dx}$$

$$xe^y\frac{dy}{dx} + e^y = 1 - \frac{dy}{dx}$$

$$xe^y\frac{dy}{dx} + \frac{dy}{dx} = 1 - e^y$$

$$\frac{dy}{dx}(1 + xe^y) = 1 - e^y$$

$$\frac{dy}{dx} = \frac{1 - e^y}{1 + xe^y}$$

Ex 1.3 Find the first two derivatives for $x^4 + y^4 = 16$.

Soln: $x^4 + y^4 = 16$

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(y^4) = \frac{d}{dx}(16)$$

$$4x^3 + 4y^3 \times \frac{dy}{dx} = 0$$

$$4y^3\frac{dy}{dx} = -4x^3$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3}$$

$$y' = \frac{dy}{dx} = -\frac{x^3}{y^3}$$

$$y'' = \frac{d^2y}{dx^2} = -\left[\frac{y^3(3x^2) - x^3 \times 3y^2 \times \left(\frac{dy}{dx}\right)}{(y^3)^2}\right]$$

$$y'' = -\left[\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}\right]$$

$$= -\left[\frac{3x^2y^3 - 3x^3\left(-\frac{x^3}{y}\right)}{y^6}\right]$$

$$= -\left[\frac{3x^2y^4 + 3x^6}{y^7}\right]$$

$$= \frac{-3x^2(y^4+x^4)}{y^7}$$

$$= \frac{-3x^2 \times (16)}{y^7}$$

$$y'' = -48\frac{x^2}{y^7}$$

Ex:4 Find y' for $\cos(xy) = 1 + \sin y$.

Soln: $\cos(xy) = 1 + \sin y$. —(1)

differentiate (1) with respect to 'x'

$$-\sin xy \frac{d}{dx}(xy) = 0 + \cos y \times \frac{dy}{dx}$$

$$-\sin(xy)\left(x\frac{dy}{dx} + y(1)\right) = \cos y \frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - y\sin(xy) = \cos y \times \frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - \cos y\frac{dy}{dx} = y\sin(xy)$$

$$\Rightarrow \frac{dy}{dx}\left(-x\sin(xy) - \cos y\right) = y(\sin(xy))$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin(xy)}{-x\sin(xy) - \cos y}$$

Derivative of inverse Trigonometric functions

(1) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(2) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(3). $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

4) $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

5) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{(x^2-1)}}$

6) $\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

The hyperbolic functions are defined as,

1. $\sinh x = \frac{e^x - e^{-x}}{2}$

2. $\cosh x = \frac{e^x + e^{-x}}{2}$

3. $\operatorname{sech} x = \frac{1}{\cosh x}$

4. $\tanh x = \frac{\sinh x}{\cosh x}$

5. $\coth x = \frac{\cosh x}{\sinh x}$

6. $\operatorname{cosech} x = \frac{1}{\sinh x}$

7. $\cosh^2 x - \sinh^2 x = 1$

8. $\cosh^2 x + \sinh^2 x = \cosh 2x$

9. $2 \sinh x \cosh x = \sinh 2x$

hyperbolic identities

1. $\sinh(-x) = -\sinh x$
2. $\cosh(-x) = \cosh x$
3. $\cosh^2 x - \sinh^2 x = 1$
4. $1 - \tanh^2 x = \operatorname{sech}^2 x$
5. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
6. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$

Derivatives of hyperbolic functions

1) $\frac{d}{dx}(\sinh x) = \cosh x$

2) $\frac{d}{dx}(\cosh x) = \sinh x$

3) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x.$

4) $\frac{d}{dx}(\cot hx) = -\operatorname{cosec}h^2 x$

5) $\frac{d}{dx}(\sec hx) = -\sec hx \tan hx$

6) $\frac{d}{dx}(\operatorname{cosec} hx) = -\operatorname{cosec}hx \cot hx$

Inverse hyperbolic functions

1) $\sin h^{-1}(x) = \log(x+\sqrt{x^2+1})$

2) $\cos h^{-1}(x) = \log(x+\sqrt{x^2-1})$

3) $\tan h^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$.

Differentiation of inverse hyperbolic functions

1) $\frac{d}{dx}(\sin h^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

2) $\frac{d}{dx}(\cos h^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

3) $\frac{d}{dx}(\tan h^{-1}x) = \frac{1}{1-x^2}$

4) $\frac{d}{dx}(\cot h^{-1}x) = \frac{-1}{(x^2-1)}$

5) $\frac{d}{dx}(\sec h^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}$

6) $\frac{d}{dx}(\operatorname{cosec} h^{-1}x) = \frac{-1}{x\sqrt{1+x^2}}$

Ex:-1 find the derivative of $\tanh^{-1}(\tan(x/2))$

Soln: Let $f(x) = \tanh^{-1}(\tan x/2)$

~~Let~~ Then $\frac{d(f(x))}{dx} = \frac{1}{1-(\tan x/2)^2} \times \frac{d}{dx}(\tan(x/2))$

$= \frac{1}{1-\tan^2 x/2} \sec^2 x/2 \cdot \frac{d}{dx}(x/2)$

$= \frac{1}{1-\tan^2 x/2} \times \sec^2(x/2) \times 1/2$

$= \frac{1 \cancel{\sec^2(x/2)}}{2 \cancel{\sec^2(x/2)}} = 1/2.$

Ex:-2. Find the derivative of

$f(x) = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right).$

Soln: Let $f(x) = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right).$

Let $u = \frac{b+a\cos x}{a+b\cos x}.$

Hence $f(x) = \cos^{-1}(u).$

Differentiating using chain rule, we have $f'(x) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}.$

$\frac{du}{dx} = \frac{(a+b\cos x)(-a\sin x) - (b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}.$

$$= \frac{-a^2\sin x - ab\cos x + b^2\sin x - ab\sin x\cos x}{(a+b\cos x)^2}$$

$$= \frac{-\sin x(a^2 + ab\cos x - b^2 - ab\cos x)}{(a+b\cos x)^2}$$

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{b+a\cos x}{a+b\cos x}\right)^2}} \left(\frac{-\sin x(a^2-b^2)}{(a+b\cos x)^2}\right)$$

$$= +\frac{1}{\sqrt{1-\left(\frac{b^2+a^2\cos^2 x+2ab\cos x}{a^2+b^2\cos^2 x+2ab\cos x}\right)}} \left(\frac{-\sin x(a^2-b^2)}{(a+b\cos x)^2}\right)$$

$$= \frac{1}{\sqrt{\frac{a^2+b^2\cos^2 x+2ab\cos x-b^2-a^2\cos^2 x-2ab\cos x}{(a+b\cos x)^2}}} \quad \frac{\sin x \times (a^2-b^2)}{(a+b\cos x)^2}$$

$$= \frac{\sin x \times (a+b\cos x)}{\sqrt{a^2(1-\cos^2 x) - b^2(1-\cos^2 x)}} \times \frac{(a^2-b^2)}{(a+b\cos x)^2}$$

$$= \frac{\sin x\,(a+\cancel{b\cos x})}{\sqrt{1-\cos^2 x}\sqrt{(a^2-b^2)}} \times \frac{(a^2-b^2)}{(a+b\cos x)^{2}}$$

$$= \frac{\cancel{\sin x} \times \sqrt{a^2-b^2}}{\cancel{\sin x}\ (a+b\cos x)}$$

$$f'(x) = \frac{\sqrt{a^2-b^2}}{(a+b\cos x)}$$

$(\because \sin^2 x+\cos^2 x = 1$
$\Rightarrow \sin^2 x = 1-\cos^2 x$
$\Rightarrow \sin x = \sqrt{1-\cos^2 x}$

Tangent line

A line that touches a curve (or) a closed surface at a single point is called tangent line.

Equation of the tangent line at a point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

where m is the slope of the curve $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

Eg:-1 Find the equation of the tangent line to the parabola $y = x^2$ at the point $(1,1)$.

Soln: The equation of the tangent line is $y - y_1 = m(x - x_1)$ — (1) where

$$m = \left(\frac{dy}{dx}\right)_{(1,1)}$$

Given that $y = x^2$

$$\frac{dy}{dx} = 2x.$$

$m = \left(\frac{dy}{dx}\right)_{(1,1)} = 2(1) = 2$

(1) $\Rightarrow$ $y - y_1 = 2(x-1)$

$y - 1 = 2x - 2$

$2x - y = -1 + 2$

$2x - y = 1$, which is a required tangent line.

__eg:2__ Find the equation of the tangent line to the curve $y = x^4 + 2x^2 - x$ at the point $(1,2)$.

__Soln:__ The equation of the tangent line is

$y - y_1 = m(x - x_1)$ —— (1)

$m = \left(\frac{dy}{dx}\right)_{(1,2)}$

Given curve, $y = x^4 + 2x^2 - x$.

$\frac{dy}{dx} = 4x^3 + 4x - 1$

$m = \left(\frac{dy}{dx}\right)_{(1,2)} = 4(1)^3 + 4(1) - 1$

$= 4 + 4 - 1$

$\boxed{m = 7}$

(1) $\Rightarrow$ $(y - 2) = 7(x - 1)$

$y - 2 = 7x - 7$

$7x - y = -2 + 7$

$7x - y = 5$, which is a required tangent line.

Eg:-3 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where?

Soln:-

The horizontal tangents occurs where the slope $\frac{dy}{dx}$ is zero.

Given $y = x^4 - 2x^2 + 2$

Now, $\frac{dy}{dx} = 4x^3 - 4x$
$= 4x(x^2 - 1)$

To get the horizontal tangents occuring points,

Take $\frac{dy}{dx} = 0$

$\Rightarrow 4x(x^2 - 1) = 0$

$\Rightarrow x = 0$ & $x = \pm 1$.

$\therefore$ The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$ and -1.

The corresponding points on the curve are $(0, 2), (1, 1), (-1, 1)$.

Note: The tangent line is horizontal then the slope m is 0

i.e) $\frac{dy}{dx} = 0$.

Normal line

Eg: 3 Find the tangent line to the equation $x^3 + y^3 = 6xy$ at the point $(3,3)$ and at what point the tangent line horizontal in the first quadrant.

Soln: The tangent line equation of the given curve is $y - y_1 = m(x - x_1)$ — ①

$$m = \left(\frac{dy}{dx}\right)_{(3,3)}$$

Given Curve is

$$x^3 + y^3 = 6xy$$

Now,
$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6\left(x\frac{dy}{dx} + y\right)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

$$3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$m = \frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$m = \left(\frac{dy}{dx}\right)_{(3,3)} = \frac{18 - 3(9)}{3(9) - 18}$$

$$= \frac{18 - 27}{27 - 18} = -\frac{9}{9} = -1$$

$\therefore \boxed{m = -1}$

$\therefore$ The equation of the tangent line is

$$y - 3 = (-1)(x - 3)$$

$$y - 3 = -x + 3$$

$$y + x = 3 + 3$$

$\boxed{x + y = 6}$ which is the required tangent line.

Take $\frac{dy}{dx} = 0$

$$\frac{6y - 3x^2}{3y^2 - 6x} = 0$$

$$6y - 3x^2 = 0$$

$$6y = 3x^2$$

$$x^2 = 2y$$

$$y = \frac{x^2}{2}$$

Substituting $y = \frac{x^2}{2}$ in the equation of the curve we get

$$x^3 + y^3 = 6xy$$

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\left(\frac{x^2}{2}\right)$$

$$x^3 + \frac{x^6}{8} = \cancel{6}^{3}\frac{x^3}{\cancel{2}}$$

$$\cancel{8}x^3 + \frac{x^6}{8} = 3x^3$$

$$\frac{x^6}{8} = 3x^3 - x^3$$

$$\frac{x^6}{8} = 2x^3$$

$$x^6 = 16x^3$$

$$x^3 = 16 \quad (\because x \neq 0 \text{ in the } 1^{st} \text{ quadrant})$$

$$x = (16)^{1/3}.$$

$$\Rightarrow y = \frac{((16)^{1/3})^2}{2} = \frac{(16)^{2/3}}{2}$$

$$y = \frac{(2^{4})^{2/3}}{2}$$

$$= \frac{2^{8/3}}{2} = 2^{8/3 - 1}$$

$$= 2^{\frac{8-3}{3}} = 2^{5/3}.$$

Thus the tangent is horizontal at $(2^{4/3}, 2^{5/3})$ which is approximately $(2.5198, 3.1748)$.

Normal line

The normal line to a curve C at a point P is the line passing through P that is perpendicular to the tangent line.

$\therefore$ The equation of the normal line is

$$y-y_1 = -\frac{1}{m}(x-x_1).$$

Eg:-1 Find the equation of the normal line to the curve $y = x\sqrt{x}$ at the point (1,1).

Soln:-

The equation of the normal line is given by,

$$y-y_1 = -\frac{1}{m}(x-x_1) \quad \text{--- (1)}$$

Given curve

$$y = x\sqrt{x}.$$

$$y = x\, x^{1/2}$$

$$= x^{1+1/2} = x^{3/2}$$

$$y = x^{3/2}.$$

Now, $\frac{dy}{dx} = \frac{3}{2} x^{3/2 - 1}$

$$= \frac{3}{2} x^{1/2}$$

$$m = \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{3}{2}(1)^{1/2} = \frac{3}{2}$$

$\boxed{m = 3/2}$

$$(1) \Rightarrow y - 1 = \frac{-1}{3/2}(x-1)$$

$$\Rightarrow y - 1 = \frac{-2}{3}(x-1)$$

$$\Rightarrow 3(y-1) = -2(x-1)$$

$$\Rightarrow 2x + 3y = 2 + 3$$

$\Rightarrow 2x + 3y = 5$ which is the required equation of normal line.

Maximum and Minimum Values

Defn:- Let c be a number in the domain D of a function f. Then $f(c)$ is the

* **Absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D

* **Absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D.

Defn:- The number $f(c)$ is a

* **Local maximum** value of f if $f(c) \geq f(x)$ when x is near c.

* Local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Defn: A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ does not exist

Eg:-1 Find the critical numbers of $f(x) = x^{3/5}(4-x)$.

Soln:-

$$f(x) = x^{3/5}(4-x)$$

$$f(x) = 4x^{3/5} - x^{3/5+1}$$

$$= 4x^{3/5} - x^{8/5}$$

$$f'(x) = 4 \times 3/5 \, x^{3/5-1} - 8/5 \, x^{8/5-1}$$

$$= \frac{12}{5} x^{-2/5} - 8/5 \, x^{3/5}$$

$$f'(x) = \frac{12}{5} \frac{1}{x^{2/5}} - 8/5 \, x^{3/5}$$

To get the critical numbers, Take $f'(x) = 0$.

$$\frac{12}{5}\left(1/x^{2/5}\right) - 8/5 \, x^{3/5} = 0$$

$(\times)$ by $5x^{2/5}$, $12 - 8x = 0$

$$12 = 8x$$

$$8x = 12 \Rightarrow x = 12/8$$

$x = 3/2$

and $f'(x)$ does not exist when

$x = 0$.

Thus the critical numbers are $3/2$ and 0.

Eg:2 Find the critical points of $y = 5x^3 - 6x$.

Soln:

$$y = 5x^3 - 6x$$

$$y' = 5(3)x^2 - 6$$

To get the critical points

Take $y' = 0$

$$15x^2 = 6.$$

$$x^2 = 6/15$$

$$x = \pm\sqrt{6/15}$$

$$x = \pm\sqrt{2/5}.$$

Thus the critical numbers are $+\sqrt{2/5}$ and $-\sqrt{2/5}$

Defn: If f has a local maximum or minimum at C, then C is a critical number of f

The Closed interval method:

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$.

(i) Find the values of f at the critical numbers of f in (a,b).

(ii) Find the values of f at the end points of the interval

(iii) The largest of the values from steps (i) and (ii) is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex 1.1 Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \leq x \leq 4$

Soln:

Since f is continuous on $[-\frac{1}{2}, 4]$, we can use the closed interval method. To find the critical numbers

Given that $f(x) = x^3 - 3x^2 + 1$

$$f'(x) = 3x^2 - 6x$$

$$= 3x(x-2)$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x)=0$

i.e) $3x(x-2)=0$

$\Rightarrow x=0$ or $x=2$

The values of f at these critical numbers are

$f(0) = x^3-3x^2+1 = 1$

$f(2) = (2)^3-3(2)^2+1$

$= 8-12+1$

$= -3$

The values of f at the end points of the interval are $f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2+1$

$= -\frac{1}{8} - \frac{3}{4}+1$

$= \frac{-1-6+8}{8} = \frac{1}{8}.$

$f(4) = (4)^3 - 3(4)^2+1$

$= 64-48+1$

$= 17.$

Comparing these four ~~va~~ numbers, we see that the absolute maximum value is

$$f(1) = 17.$$

absolute minimum value is

$$f(2) = -3.$$

Rolle's theorem

Let $f(x)$ be a real function defined on the interval $[a,b]$, such that

(i) $f(a) = f(b)$

(ii) f is continuous in the interval $[a,b]$.

(iii) $f(x)$ differentiable in the open interval (a,b).

Then there is some point c in the open interval (a,b) such that $f'(c) = 0$.

Algebraic interpretation of Rolle's theorem

Let $f(x)$ be a polynomial in x and let the roots of $f(x) = 0$ be $x = a$ and $x = b$. Then according to Rolle's theorem, atleast one root of $f'(x) = 0$ lies between a and b.

Ex:-1 Verify Rolle's theorem for $f(x) = 3x^4 - 4x^2 + 5$ in $[-1, 1]$

Soln:- $f(x) = 3x^4 - 4x^2 + 5$

Clearly $f(x)$ is Continuous and derivable in $[-1, 1]$

Now, $f(-1) = 3 - 4 + 5 = 4$

$f(1) = 3 - 4 + 5 = 4$

$f(-1) = f(1)$

Conditions for Rolle's theorem holds good.

$f'(x) = 12x^3 - 8x$

$f'(x) = 0$

$12x^3 - 8x = 0$

$4x(3x^2 - 2) = 0$

$x = 0, \quad x^2 = 2/3$

$\Rightarrow x = \pm\sqrt{2/3}$.

Then $f'(0) = 0$, $f'\left(\sqrt{2/3}\right) = 0$, $f'\left(-\sqrt{2/3}\right) = 0$

Hence $-1 < 0 < 1$, $-1 < \sqrt{2/3} < 1$,

$-1 < -\sqrt{2/3} < 1$

$\therefore$ Verified.

Ex:-2 Prove that the given equation $x^3 + x - 1 = 0$ has exactly one real root

Soln:-

First we use the intermediate Value theorem to prove that a root exists.

Let $f(x) = x^3 + x - 1$,

Then $f(0) = -1 < 0$

and $f(1) = 1 > 0$

Since f is a polynomial, it is continuous.

f changes sign between 0 and 1 such that $f(c) = 0$.

$\therefore$ the equation has a root.

To show that the equation has no other real root, we use Rolle's theorem and argue by contradiction.

Suppose that it has two real roots a and b.

Then $f(a) = 0 = f(b)$.

Since f is polynomial, it is differentiable on (a, b) and continuous on $[a, b]$.

Then the conditions for Rolle's theorem holds good.

Thus by Rolle's theorem there exists c between a and b such that $f'(c) = 0$.

But $f'(x) = 3x^2+1 \geq 1$

for all x (Since $x^2 \geq 0$).

So $f'(x)$ Can never be 0.

This gives a Contradiction

Therefore, the equation Cannot have two real roots.

Theorem:

(The mean Value theorem).

Let f be a function that satisfies the following 2 assumptions.

(i) f is Continuous on the Closed interval $[a,b]$

(ii) f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Ex:-1 Verify the Lagrange's mean Value theorem for the function

$$f(x) = x^3+x^2 \text{ in } [-1,2]$$

Soln:- $f(x) = x^3+x^2$ in $[-1,2]$.

Clearly $f(x)$ is Continuous and derivable in $[-1,2]$.

$f'(x) = 3x^2 + 2x$.

$\therefore$ Conditions of mean value theorem holds good.

$f(-1) = -1+1 = 0$

$f(2) = 8+4 = 12$.

Now, $f'(c) = \frac{f(b)-f(a)}{b-a}$ holds where $a=-1, b=2$.

$$3c^2 + 2c = \frac{f(2)-f(-1)}{2-(-1)}$$

$$3c^2 + 2c = \frac{(12)-(0)}{2+1} = 12/3 = 4.$$

$$3c^2 + 2c - 4 = 0$$

$$c = \frac{-2 \pm \sqrt{4+48}}{2(3)} = \frac{-2 \pm \sqrt{52}}{6}$$

$$= \frac{\cancel{2}\left(-1 \pm \sqrt{13}\right)}{\cancel{2} \cdot 3}$$

$$\therefore c = \frac{-1+\sqrt{13}}{3}, \quad c = \frac{-1-\sqrt{13}}{3}$$

$c = \frac{-1+\sqrt{13}}{3} = 0.868$ lies in $[-1,2]$

$c = \frac{-1-\sqrt{13}}{3} = -1.535$ not lies in $[-1,2]$.

we have $c = \frac{-1+\sqrt{13}}{3}$ which lies $[-1,2]$

Hence verified.

Increasing / Decreasing Test:

(i) If $f'(x) > 0$ on an interval, then f is increasing on that interval

(ii) If $f'(x) < 0$, on an interval, then f is decreasing on that interval.

First derivative Test

Suppose that c is a critical number of a continuous function f.

(i) If f' changes from positive to negative at c, then f has a local maximum at c.

(ii) If f' changes from negative to positive at c, then f has a local minimum at c.

(iii) If f' does not change sign at c, then f has no local maximum or minimum at c.

The first derivative test is a consequence of the increasing/decreasing test.

Definition :- If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I.

* If the graph of f lies below all of its tangents on an interval I, it is called concave downward on I.

Concavity test:

* If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

* If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition: A point P on a curve $y = f(x)$ is called an inflection point of f is continuous and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The second derivative test:

Suppose f'' is continuous near c.

(i) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c

(ii) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

1) Eg: 1 For the function $f(x) = 2x^3 + 3x^2 - 36x$,

(i) Find the intervals on which it is increasing or decreasing

(ii) Find the local maximum and local minimum values of f

(iii) Find the intervals of concavity and the inflection points

Soln:

(i) $f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

$= 6(x^2 + x - 6)$

$= 6(x+3)(x-2)$ Then the critical points are $-3, 2$

Interval	$(x+3)$	$(x-2)$	$f'(x)$	f
$x < -3$	$-$	$-$	$+$	Increasing on $(-\infty, -3)$
$-3 < x < 2$	$+$	$-$	$-$	decreasing on $(-3, 2)$
$x > 2$	$+$	$+$	$+$	Increasing on $(2, \infty)$

(ii) f changes from increasing to decreasing at $x = -3$ and from decreasing to increasing at $x = 2$.

So $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3)$
$= -54 + 27 + 108$
$= -27 + 108$
$= 81$ is a local maximum value.

and $f(2) = 2(2)^3 + 3(2)^2 - 36(2)$
$= 2(8) + 3(4) - 72$
$= 16 + 12 - 72$
$= 28 - 72$
$= -44$ is local minimum Value

$$\begin{array}{r} 72 \\ 28 \\ \hline 44 \end{array}$$

(iii) $f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6$
$= 6(2x+1)$

~~Then~~ let $f''(x) = 0$
Then $(2x+1) = 0$
$\Rightarrow x = -\frac{1}{2}$

Interval	~~$f''(x)$~~ $(2x+1)$	$f''(x)$	Concavity
$x < -\frac{1}{2}$	$-$	$-$	Concave downward.
$x > -\frac{1}{2}$	$+$	$+$	Concave upward

The curve changes concave downward to upward at $x = -\frac{1}{2}$
The Point of inflection is $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$

$f(-1/2) = 2(-1/2)^3 + 3(-1/2)^2 - 36(-1/2)$

$= 2(-1/8) + 3(1/4) + 36/2$

$= -2/8 + 3/4 + 18$

$= -1/4 + 3/4 + 18$

$= +2/4 + 18$

$= 1/2 + 18 = \frac{1+36}{2} = 37/2.$

The inflection point at $(-1/2, f(-1/2))$

$= (-1/2, 37/2)$

Eg:-2 Find the local maximum and local minimum values of $f(x) = \sqrt{x} - \sqrt[4]{x}$ using both the first and second derivative tests.

Soln:-

$f(x) = \sqrt{x} - \sqrt[4]{x}$

$f'(x) = \frac{1}{2} x^{-1/2} - \frac{1}{4} x^{-3/4}.$

$= \frac{x^{-3/4}}{4}\left(2 x^{-1/2} x^{3/4} - 1\right)$

$= \frac{2x^{1/4} - 1}{4\, x^{3/4}} = \frac{2\sqrt[4]{x} - 1}{4\sqrt[4]{x^3}}.$

$f'(x) = \dfrac{2\sqrt[4]{x} - 1}{4\sqrt[4]{x^3}}$, Here $f'(0)$ does not exist.

The Critical points are Obtained by assuming that $f'(x) = 0$.

(ie) $\dfrac{2\sqrt[4]{x} - 1}{4\sqrt[4]{x^3}} = 0$

$\Rightarrow 2\sqrt[4]{x} - 1 = 0$

$\Rightarrow 2\sqrt[4]{x} = 1$

$\Rightarrow x^{1/4} = 1/2$

$\Rightarrow x = (1/2)^4 \Rightarrow x = 1/16$

Critical Points are $0, 1/16$

Interval	$2(\sqrt[4]{x}) - 1$	$f'(x)$
$0 < x < 1/16$.	$-$	$-$
$x > 1/16$.	$+$	$+$

Since.
$f'(x)$ changes negative to positive at $x = 1/16$, the local minimum value is

$$f(1/16) = \sqrt{(1/16)} - \sqrt[4]{1/16}$$
$$= \sqrt{(1/4)^2} - \sqrt[4]{(1/2)^4}$$
$$= 1/4 - 1/2 = -1/4$$

Second derivative test:

$$\cancel{f''(x) = \frac{-1}{4} x^{-3/2}} \qquad f'(x) = \frac{1}{2} x^{-1/2} - \frac{1}{4} x^{-3/4}$$

$$f''(x) = \frac{1}{2}\left(\frac{-1}{2}\right) x^{-1/2 - 1} - \frac{1}{4}\left(\frac{-3}{4}\right) x^{-3/4 - 1}$$
$$= \frac{-1}{4} x^{-3/2} + \frac{3}{16} x^{-7/4}$$
$$= \frac{-1}{4\sqrt{x^3}} + \frac{3}{16\sqrt[4]{x^7}}$$

At $x = 1/16$,

$$f'(1/16) = \frac{2\sqrt[4]{1/16} - 1}{4\sqrt[4]{(1/16)^3}}$$
$$= \frac{2(1/2) - 1}{4\left(\sqrt[4]{(1/16)^3}\right)} = 0$$

$$f''(1/16) = \frac{-1}{4\sqrt{(1/16)^3}} + \frac{3}{16\sqrt[4]{(1/16)^7}}$$

$$= \frac{-1}{4\left(\frac{1}{(2^4)^3}\right)^{1/2}} + \frac{3}{16\left(\frac{1}{(2^4)^7}\right)^{1/4}}$$

$$= \frac{-1}{4\frac{1}{(2^{12})^{1/2}}} + \frac{3}{16\left(\frac{1}{(2^{28})^{1/4}}\right)}$$

$$= \frac{-1}{4\left(\frac{1}{2^6}\right)} + \frac{3}{16\left(\frac{1}{2^7}\right)}$$

$$= \frac{-1(2^6)}{2^2} + \frac{3(2^7)}{2^4}$$

$$= -(2^{6-2}) + 3(2^{7-4})$$

$$= -2^4 + 3(2^3)$$

$$= -16 + 3(8)$$

$$= -16 + 24$$

$$= 8 > 0.$$

$\therefore$ f has a local ~~maximum~~ minimum at $x = 1/16$.

The local minimum value of f is $f(1/16) = -1/4$.

Logarithmic differentiation

Eg:-1 Use logarithmic differentiation to find the first derivative of

$f(x) = (5-3x^2)^7 \sqrt{6x^2+8x-12}$.

Soln:

Take log on both sides.

$\log f(x) = \log\left[(5-3x^2)^7 \sqrt{6x^2+8x-12}\right]$

$= \log(5-3x^2)^7 + \log\left(\sqrt{6x^2+8x-12}\right)$

$= 7\log(5-3x^2) + \frac{1}{2}\log(6x^2+8x-12)$

differentiate on both sides w.r.to 'x'

$\frac{f'(x)}{f(x)} = \frac{7(-6x)}{(5-3x^2)} + \frac{1}{2}\frac{(12x+8)}{(6x^2+8x-12)}$

$f'(x) = f(x)\left[\frac{-42x}{(5-3x^2)} + \frac{(6x+4)}{6x^2+8x-12}\right]$

$$f'(x) = (5-3x^2)^7 \left(\sqrt{6x^2+8x-12}\right) \times \left(\frac{-42x}{(5-3x^2)} + \frac{(6x+4)}{6x^2+8x-12}\right).$$

Eg:- 2

$$y = \frac{\sin(3x+x^2)}{(6-x^4)^3} \text{ find } y'$$

by using logarithmic differentiation.

Soln:

Take log on both sides,

$$\log y = \log\left(\frac{\sin(3x+x^2)}{(6-x^4)^3}\right)$$

$$\frac{y'}{y} = \log(\sin(3x+x^2)) - \log(6-x^4)^3$$

$$= \frac{\cos(3x+x^2)\times(3+2x)}{\sin(3x+x^2)} - 3\left(\frac{-4x^3}{(6-x^4)}\right)$$

$$y' = y\left(\frac{(3+2x)(\cos(3x+x^2))}{\sin(3x+x^2)} + \frac{12x^3}{(6-x^4)}\right)$$

$$= \frac{\sin(3x+x^2)}{(6-x^4)^3}\left(3+2x\cot(3x+x^2) + \frac{12x^3}{6-x^4}\right)$$

Eg: 7 Use logarithmic differentiation to find the first derivative of the given function.

$$f(x) = (2x - e^{8x})^{\sin(2x)}$$

Soln:

Take log on both sides.

$$\log f(x) = \log (2x - e^{8x})^{\sin(2x)}$$

$$\cancel{\log f(x)} = \sin 2x \log (2x - e^{8x})$$

diff both sides w.r.to 'x'.

$$\frac{f'(x)}{f(x)} = 2\cos 2x \log (2x - e^{8x}) + \sin 2x \frac{(2 - 8e^{8x})}{(2x - e^{8x})}$$

$$f'(x) = f(x) \left(2\cos 2x \log (2x - e^{8x}) + \sin 2x \frac{(2 - 8e^{8x})}{(2x - e^{8x})} \right)$$

$$= (2x - e^{8x})^{\sin(2x)} \left(2\cos 2x \log (2x - e^{8x}) + \sin 2x \frac{(2 - 8e^{8x})}{2x - e^{8x}} \right)$$

Home work:

1. $f(x) = \dfrac{\sqrt{5x+8}\ \sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}$.

(Ans:- $f(x)\left(\frac{5}{2(5x+8)} + \frac{12\sin(4x)}{1-9\cos 4x} - \frac{\frac{1}{2}x+5/2}{x^2+10x}\right)$.

2. $f(x) = (3x-7)^{4x}$; (Ans:- $f(x)\left(4\log(3x-7) + \frac{12x}{3x-7}\right)$.

Jan (2022).

4). If $f(x) = xe^x$, then find $f'(x)$. Also find the n^{th} derivative $f^{(n)}(x)$.

Soln:

$f(x) = xe^x$

$f'(x) = 1(e^x) + xe^x$

$f''(x) = e^x + 1e^x + xe^x$.

$f'''(x) = e^x + e^x + e^x + xe^x$

$\vdots$

$f^{(n)}(x) = \underbrace{e^x + e^x + \cdots + e^x}_{(n \text{ times})} + xe^x$.

5). Differentiate the function $f(x) = \dfrac{\sec x}{1+\tan x}$ for what value of x, the graph of $f(x)$ has a horizontal tangents?

Soln:

$$\cancel{f'(x) = \frac{\quad}{(1+\tan x)^2}}$$

$$f(x) = \frac{\frac{1}{\cos x}}{1+\frac{\sin x}{\cos x}}$$

$$= \frac{1}{\cancel{\cos x}\left(\frac{\cos x+\sin x}{\cancel{\cos x}}\right)}$$

$$= \frac{1}{(\cos x+\sin x)}$$

$$f(x) = (\cos x+\sin x)^{-1}$$

$$f'(x) = -1\,(\cos x+\sin x)^{-2} \times \frac{d}{dx}(\cos x+\sin x)$$

$$= \frac{-1(\cos x-\sin x)}{(\cos x+\sin x)^2}$$

$$= \frac{\sin x-\cos x}{(\cos x+\sin x)^2}$$

If $\frac{d}{dx}(f(x)) = 0$, i.e) $f'(x) = 0$, then $f(x)$ have the horizontal tangents

$\therefore \quad f'(x) = 0$

$$\frac{\sin x - \cos x}{(\cos x + \sin x)^2} = 0$$

$\Rightarrow \sin x - \cos x = 0$

$\Rightarrow \sin x = \cos x.$

$\Rightarrow \frac{\sin x}{\cos x} = 1$

$\Rightarrow \tan x = 1$

$\Rightarrow x = \pi/4.$

$\therefore x$ takes the values

$n\pi + \pi/4, \ n \in Z$

Unit - III

Functions of Several Variables:

Partial differentiation:

Let $z = f(x,y)$ be a function of two independent variables.

Differentiating with respect to x partially is denoted by $\frac{\partial z}{\partial x}$ by keeping y as a constant.

Differentiating with respect to y partially is denoted by $\frac{\partial z}{\partial y}$ by keeping y as a constant.

It is represented by,

$$\frac{\partial z}{\partial x} = \underset{\Delta x \to 0}{Lt} \frac{\partial(x+\Delta x, y) - \partial(x,y)}{\Delta x};$$

$$\frac{\partial z}{\partial y} = \underset{\Delta y \to 0}{Lt} \frac{\partial(x, y+\Delta y) - \partial(x,y)}{\Delta y}.$$

Note!

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

If $z = f(x,y)$ and its partial

derivatives

Eg:1 If $u = \frac{y}{z} + \frac{z}{x}$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Soln: Given $u = \frac{y}{z} + \frac{z}{x}$

$\cancel{\frac{\partial u}{\partial x} = -\frac{y}{x^2} +}$ $\frac{\partial u}{\partial x} = 0 - \frac{z}{x^2}$

$$\frac{\partial u}{\partial x} = -\frac{z}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= x\left(-\frac{z}{x^2}\right) + y\left(\frac{1}{z}\right) + z\left(-\frac{y}{z^2} + \frac{1}{x}\right)$$

$$= -\frac{z}{x} + \frac{y}{z} - \frac{y}{z} + \frac{z}{x}$$

$$= 0.$$

Eg:2 If $u = (x^2+y^2+z^2)^{-1/2}$,

prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Soln: Given $u = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2} \cdot 2x + (x^2+y^2+z^2)^{-3/2} \cdot 1\right]$$

$$= -(x^2+y^2+z^2)^{-5/2}\left[-3x^2 + x^2+y^2+z^2\right]$$

$$= -(x^2+y^2+z^2)^{-5/2}\left[-2x^2+y^2+z^2\right]$$

$$\frac{\partial^2 u}{\partial x^2} = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2) \quad \text{--- (1)}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(2y^2-x^2-z^2) \quad \text{--- (2)}$$

$$\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(2z^2-x^2-y^2). \quad \text{--- (3)}$$

Adding (1), (2) and (3), we get,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$= (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2)$$

$$= (x^2+y^2+z^2)^{-5/2}(0)$$

$$= 0.$$

Note: The equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$ is known as Laplace equation in three dimensions.

Eg. 3 Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when $u(x,y) = x^y + y^x$

$y = a^x$

$\log y = \log a^x = x\log a$

diff, $\frac{1}{y}\frac{dy}{dx} = 1 \log a$

$\frac{dy}{dx} = y\log a$

$\frac{dy}{dx} = a^x \log a$

Soln Given $u(x,y) = x^y + y^x$.

$$\frac{\partial u}{\partial x} = y x^{y-1} + y^x \log y$$

$$\frac{\partial u}{\partial y} = x^y \log x + x y^{x-1}.$$

Eg:-4 If $x = r\cos\theta$, $y = r\sin\theta$ then find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$.

Soln: Given that $x = r\cos\theta$

$\Rightarrow x^2 = r^2\cos^2\theta$.

$y = r\sin\theta$

$\Rightarrow y^2 = r^2\sin^2\theta$.

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2+y^2 = r^2$

$\Rightarrow r^2 = x^2+y^2$

$\Rightarrow r = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{1/2-1} \times 2x$

$= \frac{2x\,(x^2+y^2)^{-1/2}}{2} = \frac{x}{\sqrt{x^2+y^2}} = x/r$.

$\frac{\partial r}{\partial x} = x/r$.

Similarly, $\frac{\partial r}{\partial y} = y/r$.

Eg:5 If $u = f(x-y, y-z, z-x)$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln:

Let $x-y = x_1$, $y-z = x_2$, $z-x = x_3$.

$u = f(x-y, y-z, z-x) = f(x_1, x_2, x_3)$.

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial x}$ — (1)

$x_1 = x-y$, $\quad x_2 = y-z$, $\quad x_3 = z-x$

$\frac{\partial x_1}{\partial x} = 1$, $\quad \frac{\partial x_2}{\partial x} = 0$ $\quad \frac{\partial x_3}{\partial x} = -1$

(1) $\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_3} \quad \text{--- (I)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial y} \quad \text{--- (2)}$$

$$\frac{\partial x_1}{\partial y} = -1, \quad \frac{\partial x_2}{\partial y} = 1, \quad \frac{\partial x_3}{\partial y} = 0$$

(2) $\Rightarrow$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \quad \text{--- (II)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial z} \quad \text{--- (3)}$$

$$\frac{\partial x_1}{\partial z} = 0, \quad \frac{\partial x_2}{\partial z} = -1, \quad \frac{\partial x_3}{\partial z} = 1$$

(3) $\Rightarrow$ $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_3} - \frac{\partial u}{\partial x_2} \quad \text{--- (III)}$$

Adding (I), (II) & (III) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Eg. 6 If $u = f(2x-3y, 3y-4z, 4z-2x)$ then find $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$.

Soln: Let $x_1 = 2x-3y$, $x_2 = 3y-4z$,

$x_3 = 4z-2x$.

$u = f(2x-3y, 3y-4z, 4z-2x)$

$= f(x_1, x_2, x_3)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial x} \quad \text{--- (1)}$$

$$\frac{\partial x_1}{\partial x} = 2, \quad \frac{\partial x_2}{\partial x} = 0, \quad \frac{\partial x_3}{\partial x} = -2$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial y} \quad \text{—— (2)}$$

$$\frac{\partial x_1}{\partial y} = -3, \quad \frac{\partial x_2}{\partial y} = 3, \quad \frac{\partial x_3}{\partial z} = 0$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial z} \quad \text{—— (3)}$$

$$\frac{\partial x_1}{\partial z} = 0, \quad \frac{\partial x_2}{\partial z} = -4, \quad \frac{\partial x_3}{\partial z} = 4.$$

$$(1) \Rightarrow \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial x_1} - 2\frac{\partial u}{\partial x_3}.$$

$$(2) \Rightarrow \frac{\partial u}{\partial y} = -3\frac{\partial u}{\partial x_1} + 3\frac{\partial u}{\partial x_2}.$$

$$(3) \Rightarrow \frac{\partial u}{\partial z} = -4\frac{\partial u}{\partial x_2} + 4\frac{\partial u}{\partial x_3}$$

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}.$$

$$= \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_3} - \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}$$

$$= 0.$$

Eg: 7 If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$.

find $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$.

Soln:

Let $x_1 = \frac{y-x}{xy}$, $x_2 = \frac{z-x}{xz}$.

$$u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right) = f(x_1, x_2)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial x} \quad \text{—— (1)}$$

$$x_1 = \frac{y-x}{xy}$$

$$\frac{\partial x_1}{\partial x} = \frac{xy(-1)-(y-x)(y)}{x^2y^2}$$

$$= \frac{-xy - y^2 + xy}{x^2y^2} = \frac{-y^2}{x^2y^2} = -\frac{1}{x^2}$$

$$x_2 = \frac{z-x}{xz}$$

$$\frac{\partial x_2}{\partial x} = \frac{(xz)(-1)-(z-x)(z)}{x^2z^2}$$

$$= \frac{-xz - z^2 + xz}{x^2z^2} = \frac{-z^2}{x^2z^2} = -\frac{1}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y}$$

$$\frac{\partial x_1}{\partial y} = \frac{xy(1)-(y-x)(x)}{x^2y^2}$$

$$= \frac{xy - xy + x^2}{x^2y^2} = \frac{1}{y^2}$$

$$\frac{\partial x_2}{\partial y} = 0.$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial z}.$$

$$\frac{\partial x_1}{\partial z} = 0$$

$$\frac{\partial x_2}{\partial z} = \frac{(xz)(1)-(z-x)(x)}{x^2z^2}$$

$$= \frac{xz - xz + x^2}{x^2z^2} = \frac{x^2}{x^2z^2} = \frac{1}{z^2}$$

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}$$

$$= 0.$$

Eg: 8 If $z = f(x,y)$, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, then show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

Soln:

$\cancel{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}}$ $\quad x = e^u + e^{-v}, \quad y = e^{-u} - e^v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial x}{\partial u} = e^u, \quad \frac{\partial y}{\partial u} = -e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial x}{\partial v} = -e^{-v} \quad \frac{\partial y}{\partial v} = -e^v$$

$$\frac{\partial z}{\partial u} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}.$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^v\frac{\partial z}{\partial y}$$

$$= (e^u + e^{-v})\frac{\partial z}{\partial x} + (-e^{-u} + e^v)\frac{\partial z}{\partial y}.$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Homogeneous function:

A function $f(x,y)$ is homogeneous if $f(tx, ty) = t^k f(x,y)$ and k is any real number and k is the degree of the homogeneous function

Euler's theorem for homogeneous function.

If u is a homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Proof: It is given that u is a homogeneous function of degree n in terms of x and y.

Let us consider $u = x^n f(y/x)$. — (1)

Differentiate (1) partially with respect to x, we get.

$$\frac{\partial u}{\partial x} = nx^{n-1} f(y/x) + x^n f'(y/x)(-y/x^2).$$

$$= nx^{n-1} f(y/x) - yx^{n-2} f'(y/x). \quad \text{— (2)}$$

Now differentiate (1) partially with respect to y, then we get.

$$\frac{\partial u}{\partial y} = x^n f'(y/x) \cdot 1/x$$

$$= x^{n-1} f'(y/x). \quad \text{— (3)}$$

from ②, $x\frac{\partial u}{\partial x} = nx^n f(y/x) - yx^{n-1}f'(y/x)$ — ④

from ③, $y\frac{\partial u}{\partial y} = yx^{n-1}f'(y/x)$ — ⑤

④ + ⑤

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f(y/x) - yx^{n-1}f'(y/x) + yx^{n-1}f'(y/x)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f(y/x)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Hence Euler's theorem is proved.

Eg 4 If $u = \sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$ by using Euler's theorem.

Soln: Given that $u = \sin^{-1}\left(\frac{x^3-y^3}{x+y}\right)$ degree 2.

$$\sin u = \frac{x^3-y^3}{x+y}$$

$f(tx,ty) = t^{②}f(x,y)$

$$z = \frac{(tx)^3-(tx)^3}{(tx+ty)} = \frac{t^3(x^3+y^3)}{t(x+y)} = \frac{t^2(x^3+y^3)}{x+y}$$

Let $z = \sin u = \frac{x^3-y^3}{x+y}$

then $\frac{\partial z}{\partial u} = \cos u.$

z is a homogeneous function of degree 2 in x & y.

By Euler's theorem on homogeneous function $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 2z$

$x \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + y \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = 2z$

$x \cos u \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \cos u = 2 \sin u$

$\div \cos u$,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u.$$

Eg:-2 Verify Euler's theorem for the function

$$u = x^2+y^2+2xy.$$

Soln: Given $u = x^2+y^2+2xy$

It is clear that u is a homogeneous function of degree 2 in x and y

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 2u$$

$\frac{\partial u}{\partial x} = 2x+2y$

$= 2(x+y)$.

$\frac{\partial u}{\partial y} = 2y+2x = 2(x+y)$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2(x+y)x + 2y(x+y)$

$= 2x^2+2xy+2yx+2y^2$

$= 2x^2+2y^2+4xy$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2(x^2+y^2+2xy)$$
$$= 2u.$$

Hence verified

Eg:- 3 If $z = xf(y/x)$ then find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ using Euler's theorem.

Soln:- Given $z = xf(y/x)$

$$z(x,y) = xf(y/x)$$
$$z(tx, ty) = tx f\left(\frac{ty}{tx}\right)$$
$$= t\,x f(y/x)$$

z is a homogeneous function with degree 1.

By Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z.$$

Total differential Coefficient:

If $u = f(x,y)$ is a function of x & y, where $x = f(t)$; $y = g(t)$ then we can express u as a function of t alone by substituting the values of x and y in $f(x,y)$

Then we can find the ordinary derivative $\frac{du}{dt}$ which is called the total derivative of u.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Ex:1 find $\frac{du}{dt}$ if $u = x^3 y^4$, where $x = t^3$, $y = t^2$

Soln: Given $u = x^3 y^4$,

$x = t^3$, $y = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \quad \text{—(1)}$$

$\frac{\partial u}{\partial x} = 3x^2 y^4$; $\frac{dx}{dt} = 3t^2$

$\frac{\partial u}{\partial y} = 4x^3 y^3$; $\frac{dy}{dt} = 2t$.

$$(1) \Rightarrow \frac{du}{dt} = (3x^2y^4)(3t^2) + (4x^3y^3)(2t)$$

$$= 9(t^3)^2(t^2)^4 t^2 + 4(t^3)^3(t^2)^3 2t$$

$$= 9t^{16} + 4(2)t^{16}$$

$$= 9t^{16} + 8t^{16}$$

$$\frac{du}{dt} = t^{16}(9+8) = 17t^{16}.$$

Ex:2 Find $\frac{du}{dt}$ in terms of t if $u = x^3 + y^3$ where $x = at^2$, $y = 2at$.

Soln: $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ —— (1)

$x = at^2$, $y = 2at$

$\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$.

$u = x^3 + y^3$

$\frac{\partial u}{\partial x} = 3x^2$ $\frac{\partial u}{\partial y} = 3y^2$

$$(1) \Rightarrow \frac{du}{dt} = 3x^2 \cdot (2at) + 3y^2(2a)$$

$$= 3(at^2)^2\, 2at + 3(2at)^2\, 2a$$

$$= 6a^3t^5 + 24a^3t^2$$

$$= 6a^3t^2(t^3 + 3).$$

Differentiation from implicit function

Let $u = f(x, y) = c$ be a given implicit function of x and y.

We know that

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

Since $u = c$, is a constant

then $du = 0$.

$$\therefore \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0.$$

$$\frac{\partial f}{\partial y} dy = -\frac{\partial f}{\partial x} dx$$

$$\frac{dy}{dx} = -\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right).$$

Ex:1 find $\frac{dy}{dx}$ if $x^3+y^3 = 3axy$

Soln:- $f(x,y) = x^3+y^3 - 3axy$

$$\frac{dy}{dx} = -\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right)$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{dy}{dx} = -\left(\frac{3x^2-3ay}{3y^2-3ax}\right)$$

$$\frac{dy}{dx} = -\frac{3}{3}\left(\frac{x^2-ay}{y^2-ax}\right)$$

Ex:2 find $\frac{dy}{dx}$ if $x^y + y^x = c$

Where c is a constant.

Soln:- $\frac{dy}{dx} = -\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right)$

$$f(x,y) = x^y + y^x - c.$$

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$$

$$\frac{\partial f}{\partial y} = y^x \log y + xy^{x-1}$$

$$\frac{dy}{dx} = -\left(\frac{yx^{y-1} + y^x \log y}{y^x \log y + xy^{x-1}}\right)$$

JACOBIANS

Defn: If $u_1, u_2, \ldots u_n$ are functions of n variables $x_1, x_2, \ldots x_n$ then the Jacobian of the transformation from $x_1, x_2, \ldots x_n$ to $u_1, u_2, \ldots u_n$ is defined by

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

and is denoted by the symbol

$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)}$ or $J_{(u_1, u_2, \ldots u_n)}$.

In Particular
$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

Properties of Jacobians:

Property:-1 If u and v are the functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1.$$ (Inverse Property of Jacobians)

If J_1 is the Jacobian of $u(x,y)$ and $v(x,y)$ and J_2 is the Jacobian of $x(u,v)$ and $y(u,v)$ then $J_1 J_2 = 1$.

Property:-2 If u, v are functions of x, y and x, y are themselves functions of r, s, then

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(r,s)}$$

Property:-3 If u, v, w are functionally dependent functions of three variables x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

Ex:-1 If $x = r\cos\theta$, $y = r\sin\theta$. Then find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

Soln:- Given $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

(i) $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta)$$
$$= r.$$

(ii) By Property 1 we have

$$\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = 1 \Big/ \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}.$$

Ex:2 If $x = u^2 - v^2$, $y = 2uv$ find the Jacobian of x, y with respect to u and v.

Soln:
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = 2u$, $\frac{\partial x}{\partial v} = -2v$

$\frac{\partial y}{\partial u} = 2v$, $\frac{\partial y}{\partial v} = 2u$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4u^2 - (-4v^2)$$

$$= 4u^2 + 4v^2.$$

Ex:3 If $x = r\cos\theta$ and $y = r\sin\theta$ then find $\frac{\partial r}{\partial x}$.

Soln:
$$\cancel{\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

Ex:3 Let $u = 3x + 2y - z$
$v = x - 2y + z$
$w = x(x + 2y - z)$

Are u, v and w are functionally related? If so find this relationship.

Soln:
first let us find

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$\frac{\partial u}{\partial x} = 3$, $\frac{\partial v}{\partial x} = 1$ $\frac{\partial w}{\partial x} = x(1) + (1)x$
$= 2x.$

$\frac{\partial u}{\partial y} = +2$ $\frac{\partial v}{\partial y} = -2$ $\frac{\partial w}{\partial y} = 2x$

$\frac{\partial u}{\partial z} = -1$ $\frac{\partial v}{\partial z} = 1$ $\frac{\partial w}{\partial z} = -x.$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3 & +2 & -1 \\ 1 & -2 & 1 \\ 2x & 2x & -x \end{vmatrix}$$

$$= 3(2x - 2x) - 2(-x - 2x)$$
$$-1(2x + 4x)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 3(0) - 2(-3x) - 1(6x)$$
$$= 0 + 6x - 6x$$
$$= 0.$$

Hence u, v, w are functionally related.

They are functionally dependent.

Eg: If $x = uv$, $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$

Soln: Given that $x = uv$, $y = u/v$

$\frac{\partial x}{\partial u} = v$ $\quad \frac{\partial x}{\partial v} = u$, $\quad \frac{\partial y}{\partial u} = 1/v$, $\quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = -v\left(u/v^2\right) - u/v$$
$$= -u/v - u/v$$
$$= -\frac{2u}{v}$$

Taylor's theorem for functions of two variables.

Let $f(x,y)$ be a function of two variables x and y. Then $f(x+h, y+k)$ can be expanded in series of powers of h and k.

Considering $f(x+h, y+k)$ as a function of two variables x and y. Then by Taylor's theorem.

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) + \cdots \quad (1)$$

Note:-1

Putting $x=a$ and $y=b$ in Taylor's series.

$f(a+h, b+k)$

$$= f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] + \left[\frac{h^2}{2!} f_{xx}(a,b) + h f_{xy}(a,b) + \frac{k^2}{2!} f_{yy}(a,b)\right] + \ldots$$

Note:-2 In note 1,

Putting $a+h = x$ and $b+k = y$

So that $h = x-a$, and $k = y-b$.

we have,

$$f(x,y) = f(a,b) + \left((x-a) f_x(a,b) + (y-b) f_y(a,b)\right)$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

Note :- 3 Putting $a=0$, $b=0$ in Note 2, we have

$$f(x,y) = f(0,0) + \left(x f_x(0,0) + y f_y(0,0)\right)$$
$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

is known as Maclaurin's series for two variables.

Ex:-1 Obtain the Taylor's series expansion $x^3+y^3+xy^2$ in terms of powers of $(x-1)$ and $(y-2)$ upto third degree terms.

Soln:-

Given that $(x-a) = (x-1)$
and $(y-b) = (y-2)$

i.e) $a=1$, $b=2$.

$f(x,y) = x^3+y^3+xy^2$; $f(1,2) = 1^3+2^3+1(2^2) = 1+8+4 = 13$

$f_x(x,y) = 3x^2+y^2$; $f_x(1,2) = 3(1)^2+(2)^2 = 3+4 = 7$.

$f_y(x,y) = 3y^2+2xy$, ; $f_y(1,2) = 3(2)^2+2(1)(2) = 12+4 = 16$.

$f_{xx}(x,y) = 6x$; $f_{xx}(1,2) = 6$

$f_{yy}(x,y) = 6y + 2x$; $f_{yy}(1,2) = 6(2) + 2(1)$
$= 12 + 2 = 14$

$f_{xy}(x,y) = 2y$; $f_{xy}(1,2) = 2(2) = 4.$

$f_{xxx}(x,y) = 6.$ $f_{xxx}(1,2) = 6.$

$f_{xyy}(x,y) = 2$ $f_{xyy}(1,2) = 2$

$f_{xxy}(x,y) = 2y$ $f_{xxy}(1,2) = 0$

$f_{yyy}(x,y) = 6.$ $f_{yyy}(1,2) = 6$

By Taylor's series at $(1,2)$,
we have,

$$f(x,y) = f(1,2) + \left[(x-1) f_x(1,2) + (y-2) f_y(1,2)\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,2) + 2(x-1)(y-2) f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)\right]$$

$$+ \frac{1}{3!}\left[(x-1)^3 f_{xxx} + 3(x-1)^2 (y-2) f_{xxy} + 3(x-1)(y-2)^2 f_{xyy} + (y-2)^3 f_{yyy}\right]$$

$+ \dots$

$$f(x,y) = 13 + 7(x-1) + 16(y-2) + \frac{1}{2!}\left[6(x-1)^2 + 2(x-1)(y-2)(4) + 14(y-2)^2\right]$$
$$+ \frac{1}{3!}\left[6(x-1)^3 + 3(x-1)^2(y-2)(0) + 3(x-1)(y-2)^2(2) + 6(y-2)^3\right] + \dots$$
$$= 13 + 7(x-1) + 16(y-2) + \frac{1}{2!}\left[6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2\right]$$
$$+ \frac{1}{3!}\left[6(x-1)^3 + 6(x-1)(y-2)^2 + 6(y-2)^3\right] + \dots$$

Ex:-2 Find the Taylor's series expansion of function of $f(x,y) = \sqrt{1+x+y^2}$ in powers of $x-1$ and y upto second degree terms.

Soln: Given that $x-1 = x-a$ and $y-0 = y-b$.

i.e) $a = 1$ $b = 0$.

$f(x,y) = (1+x+y^2)^{1/2}$. $f(1,0) = (1+1+0)^{1/2} = \sqrt{2}$.

$f_x(x,y) = \frac{1}{2}(1+x+y^2)^{-1/2}\cdot(1)$; $f_x(1,0) = \frac{1}{2}(2)^{-1/2} = \frac{1}{2\sqrt{2}}$.

$f_y(x,y) = \frac{1}{\cancel{2}}(1+x+y^2)^{-1/2}\cdot\cancel{2}y$; $f_y(1,0) = 0$.

$f_{xx}(x,y) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x+y^2)^{-3/2}(1)$
$= -\frac{1}{4}(1+x+y^2)^{-3/2}$; $f_{xx}(1,0) = -\frac{1}{4}\frac{1}{(2)^{3/2}} = -\frac{1}{4}\left(\frac{1}{2\sqrt{2}}\right) = -\frac{1}{8\sqrt{2}}$.

$f_{yy}(x,y) = \left[\left[-\frac{1}{2}(1+x+y^2)^{-3/2} \cdot 2y\right][y] + (1+x+y^2)^{-1/2}(1)\right]$

$f_{yy}(1,0) = \cancel{2}(2)^{-1/2}$
$= \frac{1}{\sqrt{2}}$

$f_{xy}(x,y) = y\left(-\frac{1}{2}\right)(1+x+y^2)^{-3/2}$

$f_{xy}(1,0) = 0.$

$f(x,y) = f(1,0) + \left[(x-1) f_x(1,0) + (y-0) f_y(1,0)\right]$
$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,0) + (x-1)(y-0) f_{xy}(1,0) + (y-0)(y-0) f_{yy}(1,0)\right] + \ldots$

$= \sqrt{2} + \left((x-1)\frac{1}{2\sqrt{2}} + y(0)\right)$
$+ \frac{1}{2!}\left[(x-1)^2\left(\frac{-1}{8\sqrt{2}}\right) + y^2\left(\frac{1}{\sqrt{2}}\right)\right] + \ldots$

Ex 1-3 Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ using Taylor's series upto third degree terms.

Soln: Given that $x+2 = x-(-2)$
$= x-a$

$y-1 = y-b \Rightarrow a = -2$
$b = 1.$

$f(x,y) = x^2y^2 + 2x^2y + 3xy^2$

$f(x,y) = x^2y^2 + 2x^2y + 3xy^2$; $f(-2,1)$
$= (-2)^2(1)^2 + 2(-2)^2(1) + 3(-2)(1)^2$
$= 4 + 8 - 6$
$= 6$

$f_x(x,y) = 2xy^2 + 4xy + 3y^2$;

$f_x(-2,1) = 2(-2)(1)^2 + 4(-2)(1) + 3(1)^2$
$= -4 - 8 + 3 = -9$

$f_y(x,y) = 2x^2y + 2x^2 + 6xy$

$f_y(-2,1) = 2(-2)^2(1) + 2(-2)^2 + 6(-2)(1)$
$= 8 + 8 - 12$
$= 4.$

$f_{xx}(x,y) = 2y^2 + 4y$; $f_{xx}(-2,1) = 2(1)^2 + 4(1)$
$= 6.$

$f_{yy}(x,y) = 2x^2 + 6x$; $f_{yy}(-2,1) = 2(-2)^2 + 6(-2)$
$= 8 - 12 = -4.$

$f_{xy}(x,y) = 4xy + 4x + 6y$;

$f_{xy}(-2,1) = 4(-2)(1) + 4(-2) + 6(1)$
$= -8 - 8 + 6$
$= -10$

$f_{xxx}(x,y) = 0$; $f_{xxx}(-2,1) = 0$

$f_{yyy}(x,y) = 0$; $f_{yyy}(-2,1) = 0$

$f_{xyy}(x,y) = 4x + 6$; $f_{xyy}(-2,1) = 4(-2) + 6$
$= -8 + 6 = -2$

$f_{xxy}(x,y) = 4y + 4$; $f_{xxy}(-2,1) = 4(1) + 4$
$= \cancel{-8 + 4 = -4}$
$= 4 + 4$
$= 8$

By taylor's series at $(-2,1)$ we have

$f(x,y) = f(-2,1) + ((x+2) f_x(-2,1) + (y-1) f_y(-2,1))$

$+\frac{1}{2!}\left[(x+2)^2 f_{xx}(-2,1) + 2(x+2)(y-1) f_{xy}(-2,1) + (y-1)^2 f_{yy}(-2,1)\right]$

$+\frac{1}{3!}\left[(x+2)^3 f_{xxx}(-2,1) + 3(x+2)^2(y-1) f_{xxy}(-2,1) + 3(x+2)(y-1)^2 f_{xyy}(-2,1) + (y-1)^3 f_{yyy}(-2,1)\right] + \ldots$

~~$\frac{1}{3!}\left[(x+2)^3(0) + 3(x+2)\right)$~~

$= 6 + \left[(x+2)(-9) + (y-1)(4)\right]$

$+\frac{1}{2!}\left[(x+2)^2(+6) + 2(x+2)(y-1)(-10) + (y-1)^2(-4)\right]$

$+\frac{1}{3!}\left[(x+2)^3(0) + 3(x+2)^2(y-1)(+8) + 3(x+2)(y-1)^2(-2) + (y-1)^3(0)\right] + \ldots$

$= 6 - 9(x+2) + 4(y-1)$

$+\frac{1}{2!}\left(6(x+2)^2 - 20(x+2)(y-1) - 4(y-1)^2\right)$

$+\frac{1}{3!}\left(24(x+2)^2(y-1) - 6(x+2)(y-1)^2\right)$

$+ \ldots$

Maxima and Minima for the functions of two Variables.

Defn: A function $f(x)$ has a maximum at C if $f'(c)=0$ and $f''(C)$ should be negative.

A function $f(x)$ has a minimum at C if $f'(C)=0$ and $f''(C)$ should be positive.

A function $f(x,y)$ has a maximum value at (c,d) if $f(c+l, d+m) - f(c,d)$ is positive where l and m are sufficiently small values.

If $f(x,y)$ reaches a maximum or minimum at (c,d), then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at (c,d).

Procedure to find the maxima and minima of $f(x,y)$

Step-1 Find the respective derivatives of the function $f(x,y)$ such as $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Step-2 Equate both the derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to zero.

(i.e) $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ and find the solution. Let the solution be (c,d).

Step:-3 Compute the respective derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ at (c,d).

Step:-4 Let A be $\frac{\partial^2 f}{\partial x^2}$, B be $\frac{\partial^2 f}{\partial x \partial y}$ and C be $\frac{\partial^2 f}{\partial y^2}$.

Step:-5 Find the value of $AC-B^2$

Step:-6 Then we make the conclusion.

(i) If $AC-B^2$ is positive and A is negative then $f(x,y)$ has a maximum at (c,d)

(ii) If $AC-B^2$ is positive and A is positive then $f(x,y)$ has a minimum at (c,d)

(iii) If $AC-B^2$ is negative then $f(x,y)$ has neither a maximum nor a minimum at (c,d). Such a point is called a Saddle point.

(iv) If $AC-B^2=0$, then nothing is known and further investigation is required.

(v) $f(c,d)$ is not an extremum if $AC-B^2<0$

Definition:

A function $f(x,y)$ is said to be stationary at (c,d) of $f(c,d)$ is said to be a stationary value of $f(x,y)$ if $f_x(c,d) = f_y(c,d) = 0$.

Note:- 1

1) If $AC-B^2 > 0$ then $A \neq 0$ and $C \neq 0$.

2) Every extremum value is a stationary value but a stationary value need not be an extremum.

ex:-1 Examine $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$ for extreme values

Soln:

Given $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$

$$\frac{\partial f}{\partial x} = 3x^2+3y^2-30x+72$$

$$\frac{\partial f}{\partial y} = 6xy-30y$$

$$A = \frac{\partial^2 f}{\partial x^2} = 6x-30, \quad B = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$C = \frac{\partial^2 f}{\partial y^2} = 6x-30.$$

The stationary points are given by $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$3x^2+3y^2-30x+72=0$ and $6xy-30y=0$

$3(x^2+y^2-10x+24)=0$ and $6y(x-5)=0$
→ ① → ②

from equation ②, $y=0$ or $x=5$

when $y=0 \Rightarrow$ ① $\Rightarrow x^2-10x+24=0$

$(x-4)(x-6)=0$

$x=4$ and $x=6$.

When $x=5$ ① $\Rightarrow y^2-1=0$

$y=\pm 1$.

The Stationary points are

$(5,1)$, $(5,-1)$, $(4,0)$ and $(6,0)$

At $(5,\pm 1)$,

$AC-B^2=(6x-30)^2-36y^2$

$=-36<0$.

But $A=0$ at $(5,\pm 1)$

We can't decide about maxima and minima

At $(4,0)$

$AC-B^2=(6x-30)^2-36(0)^2$

$=(24-30)^2-0$

$=(-6)^2=36>0$

At $(4,0)$ $A=6x-30$

$=24-30=-6<0$

At $(6,0)$: $AC-B^2 = 16 > 0$

and $A = 6(6)-30 = 6 > 0$

$\therefore$ function attains minimum at $(6,0)$ and its value is

$f(6,0) = (6)^3 + 0 - 15(6)^2 - 0 + 72(6)$

$= 216 - 540 + 432$

$= 108$

$\therefore$ function attains maximum at $(4,0)$ and its value is

$f(4,0) = (4)^3 + 0 - 15(4)^2 - 0 + 72(4)$

$= 112$

Eg: 2 Find the maxima and minima of $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Soln: Given

$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\frac{\partial f}{\partial x} = 4x^3 + 0 - 4x + 4y - 0$

$= 4x^3 - 4x + 4y$

$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$

$A = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$, $B = \frac{\partial^2 f}{\partial x \partial y} = -4$

$C = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$.

The stationary points are given by, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$4x^3 - 4x + 4y = 0$ and $4y^3 + 4x - 4y = 0$

$\cancel{4x(x^2-1)}$

$4(x^3 - x + y) = 0$ —— (1) $4(y^3 - y + x) = 0$ —— (2)

(1) + (2) $x^3 + y^3 = 0$

$(x+y)(x^2 - xy + y^2) = 0$

$x + y = 0$ or $x^2 - xy + y^2 = 0$

$x = -y$ or $x^2 - xy + y^2 = 0$

Put $x = -y$ in (1)

$(-y)^3 - (-y) + y = 0$

$-y^3 + 2y = 0$

$y^3 - 2y = 0$

$y(y^2 - 2) = 0 \Rightarrow y = 0$ or $y^2 = 2$

$y = \pm\sqrt{2}$

If $y = 0$ (1) $\Rightarrow x = 0$,

If, $y = +\sqrt{2}$, $x = -\sqrt{2}$

If $y = -\sqrt{2}$, $x = \sqrt{2}$

$\therefore (0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$

are the stationary points.

At $(0,0)$

$AC - B^2 = (12(0)^2 - 4)(12(0)^2 - 4) - (-4)^2$

$= (-4)(-4) - (16)$

$= 0.$

At $(\sqrt{2}, -\sqrt{2})$

$$AC - B^2 = (12(\sqrt{2})^2 - 4)(12(-\sqrt{2})^2 - 4) - (-4)^2$$
$$= (24-4)(24-4) - 16$$
$$= 400 - 16$$
$$= 384 > 0$$

$\therefore$ At $(\sqrt{2}, -\sqrt{2})$,

$$A = (12(\sqrt{2})^2 - 4)$$
$$= 24 - 4 = 20 > 0$$

$\therefore f$ has minimum at $(\sqrt{2}, -\sqrt{2})$.

Minimum Value

$f(\sqrt{2}, -\sqrt{2})$

$$= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 4 - 8 - 4$$

Minimum Value $= -8$

At $(-\sqrt{2}, \sqrt{2})$.

$$AC - B^2 = (12(-\sqrt{2})^2 - 4)(12(\sqrt{2})^2 - 4) - (-4)^2$$
$$= (24-4)(24-4) - 16$$
$$= 400 - 16 = 384 > 0$$

At $(-\sqrt{2}, \sqrt{2})$

$$A = 12(-\sqrt{2})^2 - 4 = 24 - 4 = 20 > 0.$$

$\therefore f$ has minimum at $(-\sqrt{2}, \sqrt{2})$.

$$f(-\sqrt{2}, \sqrt{2}) = (-\sqrt{2})^4 + (+\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(+\sqrt{2})^2$$
$$= 4 + 4 - 4 - 8 - 4 = -8$$

Eg:3 Find the maximum or minimum values of $f(x,y) = 3x^2 - y^2 + x^3$

Soln:

Given that $f(x,y) = 3x^2 - y^2 + x^3$

$$\frac{\partial f}{\partial x} = 6x + 3x^2$$

$$\frac{\partial f}{\partial y} = -2y$$

$$A = \frac{\partial^2 f}{\partial x^2} = 6 + 6x$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$C = \frac{\partial^2 f}{\partial y^2} = -2$$

The stationary points are given by $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

$6x + 3x^2 = 0$ and $-2y = 0$

$3x(2+x) = 0$ and $y = 0$

$\Rightarrow x = 0,\ x = -2$

The stationary points are $(0,0)$ $(-2,0)$

At the point $(0,0)$

$$AC - B^2 = (6+6x)(-2) - 0$$

$$= -12 - 12x$$

At the point $(0,0)$,

$$AC-B^2 = -12 < 0$$

The point $(0,0)$ is neither a maximum nor minimum.

At the point $(-2,0)$:

$$AC-B^2 = -12-12(-2)$$
$$= -12+24$$
$$= 12 > 0$$

At $(-2,0)$

But $\frac{\partial^2 f}{\partial x^2} = A = 6+6(-2)$
$$= 6-12 = -6 < 0$$

$\therefore$ The point $(-2,0)$ is the maximum point.

$\therefore$ maximum Value is

$$f(x,y) = 12-0-8$$
$$= 4.$$

Method of Lagrangian Multiplier.

Suppose we require to find the maximum and minimum values of $f(x,y,z)$ where x, y, z are subject to constraint equation $g(x,y,z)=0$.

We define a function

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

Where λ is called the Lagrange multiplier which is independent of x, y, z.

The necessary conditions for maximum or minimum are

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

Solving the four equations for four unknowns λ, x, y, z we obtain the points x, y, z.

A point may be maxima or minima or neither which is decided by the physical consideration.

This method is also applicable when we have more than one constrained equation connecting the variable.

<u>Eg:1</u> Find the dimensions of the rectangular box without a top of maximum capacity, whose surface area is 108 sq.cm.

<u>Soln:</u> Let the given surface area is

$$g(x, y, z) = xy + 2xz + 2yz = 108.$$

The volume is $v = xyz = f(x,y,z)$.

let us consider the Lagrangian function as

$$F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$$

$$F(x,y,z) = xyz + \lambda(xy + 2xz + 2yz - 108) \quad \text{--- (1)}$$

$$\frac{\partial F}{\partial x} = yz + \lambda(y+2z)$$

$$\frac{\partial F}{\partial y} = xz + \lambda(x+2z)$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2x+2y)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(y+2z) = 0 \quad \text{--- (2)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(x+2z) = 0 \quad \text{--- (3)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda(2x+2y) = 0 \quad \text{--- (4)}$$

Solving equations (2), (3) & (4) we get,

$$(2)(x) - (3)(y) \Rightarrow \lambda(2zx - 2zy) = 0$$

$$2\lambda z(x-y) = 0$$

$$\lambda \neq 0,\ z \neq 0,\ x - y = 0 \Rightarrow x = y.$$

$$(2)x - (4)z \Rightarrow \lambda(xy + 2xz - 2xz - 2yz) = 0$$

$$\lambda y(x - 2z) = 0$$

$$y \neq 0;\ z = x/2$$

∴ Substitute these values in $xy+2zx+2yz=108$

$$x^2+x^2+x^2=108$$

$$3x^2=108$$

$$x^2=\frac{108}{3}=36$$

$$x=6.$$

$$\therefore y=6,\ z=3.$$

∴ The dimensions of the box having maximum capacity is length $=6$, Breadth $=6$, Height $=3$.

Ex:-2 Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square meter.

Soln: Let x, y, z be the length, breadth and height of the box.

The surface area $= xy+2yz+2zx = 432$

Volume $= xyz$.

Let us consider the Lagrangian function as

$$f(x,y,z) = (xyz) + \lambda(xy+2yz+2zx-432).$$

$\frac{\partial f}{\partial x} = yz + \lambda(y+2z)$.

$\frac{\partial f}{\partial y} = zx + \lambda(x+2z)$.

$\frac{\partial f}{\partial z} = xy + \lambda(2x+2y)$.

Let $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, and $\frac{\partial f}{\partial z} = 0$.

$yz + \lambda(y+2z) = 0$

$-\lambda = \frac{yz}{(y+2z)}$ —— (2)

$zx + \lambda(x+2z) = 0$

$-\lambda = \frac{zx}{(x+2z)}$ —— (3)

$xy + \lambda(2x+2y) = 0$

$\Rightarrow -\lambda = \frac{xy}{2x+2y}$ —— (4)

from (2) & (3) we get

$\frac{yz}{(y+2z)} = \frac{zx}{x+2z} \Rightarrow x = y$.

from (3) & (4) we get.

$\frac{zx}{(x+2z)} = \frac{xy}{2x+2y} \Rightarrow 2z = y$.

Equation (1) becomes,

$xy + 2yz + 2zx = 432$

$(2z)(2z) + 2(2z)(z) + (2z)(2z) = 432$

$4z^2 + 4z^2 + 4z^2 = 432$

$12z^2 = 432$

$z^2 = 36$

$\therefore z = 6.$

$x = 12, \ y = 12, \ z = 6.$

Therefore the dimensions of the box are 12, 12, 6.

The maximum volume

$= xyz = 12 \times 12 \times 6$

$= 864 \ cm^3.$

Eg:-3 Find the shortest and longest distance from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$.

Soln: Let $P(x, y, z)$ be any point on the sphere.

The distance from the point $(1, 2, -1)$ to the sphere is

$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2.$

$f = (x-1)^2 + (y-2)^2 + (z+1)^2.$

Let $g = x^2 + y^2 + z^2 - 24.$

Let the auxiliary function F be $F = f + \lambda g$.

$$F = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2+y^2+z^2-24)$$

$\frac{\partial f}{\partial x} = 0 \Rightarrow 2(x-1) + \lambda 2x = 0$

$\Rightarrow x-1 = -\lambda x$

$\Rightarrow \lambda = -\left(\frac{x-1}{x}\right)$ —— ②

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2(y-2) + 2y\lambda = 0$

$\Rightarrow \lambda = \left(-\left(\frac{y-2}{\lambda}\right)\right)$ —— ③

$\frac{\partial f}{\partial z} = 0 \Rightarrow 2(z+1) + 2z\lambda = 0$

$\Rightarrow \lambda = -\left(\frac{z+1}{z}\right)$ —— ④

from equations ②, ③ and ④.

we get.

$$\frac{(x-1)}{x} = \frac{y-2}{y} = \frac{z+1}{z}.$$

from ①st, ②nd,

$(x-1) \times y = x(y-2)$

$xy - y = xy - 2x.$

from ②nd, ③rd $y = 2x.$

$(y-2)z = y(z+1) \Rightarrow -2z = y.$

$\Rightarrow -2z = 2x$

$x = -z$

$\therefore 2x = y = -2z$

Now substitute these values is

$x^2 + y^2 + z^2 - 24 = 0$

$x^2 + (2x)^2 + (-x)^2 - 24 = 0$

$x^2 + 4x^2 + x^2 - 24 = 0$

$6x^2 = 24$

$x^2 = 4$

$x = \pm 2.$

$y = \pm 4, \quad z = \pm 2.$

Substituting these values is

$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$

$d^2 = (2-1)^2 + (4-2)^2 + (2+1)^2$

$= (1)^2 + (2)^2 + (3)^2$

$= 1 + 4 + 9$

$= 14.$

$d = \sqrt{14}.$

$d^2 = (-2-1)^2 + (-4-2)^2 + (-2+1)^2$

$= (-3)^2 + (-6)^2 + (-1)^2$

$= 9 + 36 + 1$

$= 46$

$d = \sqrt{46}$

Shortest distance $= \sqrt{14}$

longest distance $= \sqrt{46}.$

Unit - IV

Integral Calculus

Defn: The area A of a region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n$$
$$= \lim_{n\to\infty} \left(f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \right)$$

We can get the same value for the left end points

$$A = \lim_{n\to\infty} L_n$$
$$= \lim_{n\to\infty} \left(f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x \right)$$

The definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Provided that this limit exists and gives the same value for all possible points. If it exists, then f is integrable on $[a,b]$.

Theorem: 1 If f is continuous on $[a,b]$ or if f has only a finite number of discontinuities, then f is integrable on (a,b). The definite integral $\int_a^b f(x)\,dx$ exists.

Theorem: 2 If f is integrable on $[a,b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Properties of definite integral

(i) $\int_a^b c\,dx = c(b-a)$, where c is any constant.

(ii) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(iii) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant.

(iv) $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.

(v) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(vi) If $f(x) \geq 0$ for $a \leq x \leq b$,
then $\int_a^b f(x)\,dx \geq 0$

(vii) If $f(x) \geq g(x)$ for $a \leq x \leq b$,
then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

(viii) If $m \leq f(x) \leq M$ for $a \leq x \leq b$,
then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Eg:-1 What is wrong with the following Calculation

$$\int_{-1}^{3} \left(\frac{1}{x^2}\right) dx = \left(\frac{x^{-1}}{-1}\right)_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}.$$

Soln: The Calculation is wrong because the answer is negative but $f(x) = \frac{1}{x^2} \geq 0$ and it says that $\int_a^b f(x)\,dx \geq 0$ when $f \geq 0$.

The fundamental theorem of Calculus applies only to Continuous functions.

Hence we Cannot apply because $f(x) = \frac{1}{x^2}$ is not Continuous.

on $[-1,3]$. Here f has an infinite discontinuity at $x=0$.

So $\int_{-1}^{3} \left(\frac{1}{x^2}\right) dx$ does not exist.

Eg:-2 What is wrong with the equation $\int_{-1}^{2} \frac{4}{x^3}\, dx = \left[\frac{-2}{x^2}\right]_{-1}^{2} = \frac{3}{2}$.

Soln: The function $f(x) = \frac{1}{x^3}$ is not continuous on $[-1,2]$. The function $f(x)$ has an infinite discontinuity at $x=0$.

$\therefore \int_{-1}^{2} \frac{4}{x^3}\, dx$ does not exist.

The fundamental theorem of Calculus

Suppose f is continuous on $[a,b]$

(i) If $g(x) = \int_a^x f(t)\, dt$, then $g'(x) = f(x)$.

(ii) $\int_a^b f(x)\, dx = F(b) - F(a)$,

where F is any anti-derivative of f

i) $F' = f$.

Indefinite integrals

1) $\int c\, f(x)\, dx = c \int f(x)\, dx$ where c is a Constant.

2) $\int k\, dx = kx + c$, where k is a Constant.

3) $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

4) $\int \frac{1}{x}\, dx = \log x + c$

5) $\int e^x\, dx = e^x + c$

6) $\int a^x\, dx = \frac{a^x}{\log a} + c$

7) $\int \sin x\, dx = -\cos x + c$

8) $\int \cos x\, dx = \sin x + c$

9) $\int \sec^2 x\, dx = \tan x + c$

10) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

11) $\int \sec x \tan x\, dx = \sec x + c$

12) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$.

13) $\int \frac{1}{x^2+1}\, dx = \tan^{-1}(x) + c$
$= -\cot^{-1}(x) + c$

14) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1}(x) + c$
$= -\cos^{-1}(x) + c$

) $\int \sinh x \, dx = \cosh x + c$

) $\int \cosh x \, dx = \sinh x + c$

) $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

18) $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$

19) $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

20) $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + c$

21) $\int \frac{dx}{\sqrt{x(x^2-1)}} = \sec^{-1}(x) + c$
$= -\operatorname{cosec}^{-1}(x) + c$

22) $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + c$

23) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}(x) + c$

24) $\int \frac{dx}{x^2-1} = \tanh^{-1}(x) + c$
$= \coth^{-1}(x) + c$

Ex:-1 find the general indefinite integral $\int 10x^4 - 2\sec^2 x \, dx$.

Soln:

$\int 10x^4 - 2\sec^2 x \, dx$

$= 10\int x^4 \, dx - 2\int \sec^2 x \, dx$

$= 10\left(\frac{x^5}{5}\right) - 2\tan x + c$

$= 2x^5 - 2\tan x + c$

Eg:-2 Evaluate $\int_1^9 \left(\frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\right) dt$

Soln:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt = \int_1^9 \left(\frac{2t^2}{t^2} + \frac{t^2\sqrt{t}}{t^2} - \frac{1}{t^2}\right) dt$$

$$= \int_1^9 \left(2 + \sqrt{t} - 1/t^2\right) dt$$

$$= \int_1^9 \left(2 + t^{1/2} - t^{-2}\right) dt$$

$$= \left(2t + \frac{t^{3/2}}{3/2} - \frac{t^{-1}}{-1}\right)_1^9$$

$$= \left(2t + \frac{2}{3} t^{3/2} + 1/t\right)_1^9$$

$$= \left((2\times9) + \frac{2}{3}(9)^{3/2} + 1/9\right) - \left((2\times1) + \frac{2}{3}(1)^{3/2} + 1/1\right)$$

$$= \left(18 + \frac{2}{3}(9)(3) + 1/9\right) - \left(2 + \frac{2}{3} + 1\right)$$

$$= 18 + 18 + 1/9 - 2 - \frac{2}{3} - 1$$

$$= 34 + 1/9 - 6/9 - 3$$

$$= 31 - 5/9$$

$$= \frac{274}{9}$$

$\frac{31}{9}$

$\frac{279}{5}$

Methods of Integration

(i) Substitution Rule
(ii) Integration by Parts
(iii) Integration by method of partial fractions
(iv) Successive reduction method.

Integrals of the functions containing linear functions of x.

For this type, let us consider $ax+b = t$ then $a\,dx = dt$

$$\Rightarrow dx = dt/a.$$

$$\int f(ax+b)\,dx = \int f(t)\,\frac{1}{a}\,dt$$

$$= \frac{1}{a}\int f(t)\,dt.$$

Ex 1: Evaluate $\int \frac{x^3}{\sqrt{1-x^8}} dx$

Soln: Let us consider $x^4 = u$.

then $4x^3 dx = du$.

$$\therefore \int \frac{x^3}{\sqrt{1-x^8}} dx = \int \frac{du}{4\sqrt{1-u^2}}$$

$$= \frac{1}{4} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{4} \sin^{-1}(u) + C.$$

$$= \frac{1}{4} \sin^{-1}(x^4) + C.$$

Eg: 2 Evaluate $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$.

Put $t = \sin^{-1} x$

then $dt = \frac{1}{\sqrt{1-x^2}} dx$.

$$\therefore \int \frac{\sin^{-1}x}{\sqrt{1-x^2}} dx = \int t\, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1}(x))^2}{2} + C.$$

Ex 1.3 Evaluate $\int \operatorname{cosec} x\, dx$.

Soln: Let $\int \operatorname{cosec} x\, dx = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} dx$

Put $\operatorname{cosec} x + \cot x = t$

then $(-\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x)\, dx = dt$

$\Rightarrow -\operatorname{cosec} x (\cot x + \operatorname{cosec} x)\, dx = dt$

$\Rightarrow \operatorname{cosec} x (\cot x + \operatorname{cosec} x)\, dx = -dt$

$$\int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} dx$$

$$= \int \frac{-dt}{t}$$

$$= -\int \frac{dt}{t} = -\log t$$

$$= -\log(\operatorname{cosec} x + \cot x).$$

Ex:-4 Evaluate $\int \frac{\tan x}{\sec x + \cos x}\,dx$.

Soln:

$$\int \frac{\tan x}{\sec x + \cos x}\,dx = \int \frac{\frac{\sin x}{\cos x}}{\left(\frac{1}{\cos x} + \cos x\right)}\,dx.$$

$$= \int \frac{\sin x}{\cancel{\cos x}\left(\frac{1+\cos^2 x}{\cancel{\cos x}}\right)}\,dx.$$

$$= \int \frac{\sin x}{1+\cos^2 x}\,dx$$

Put $u = \cos x$, $du = -\sin x\,dx$.

$-du = \sin x\,dx$.

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2}$$

$$= -\int \frac{du}{1+u^2}$$

$$= -\tan^{-1}(u) + C$$

$$= -\tan^{-1}(\cos x) + C.$$

Ex:5 Evaluate $\int_0^{\pi/2} \frac{dx}{1+\tan x}$.

Soln:

$$\int \frac{dx}{1+\tan x} = \int \frac{dx}{1+\frac{\sin x}{\cos x}}.$$

$$= \int \frac{\cos x}{(\cos x + \sin x)}\,dx.$$

$$\int_0^{\pi/2} \frac{dx}{1+\tan x} = \frac{1}{2}\int_0^{\pi/2} \frac{2\cos x \cdot dx}{(\sin x+\cos x)}$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{(\cos x+\sin x)+(\cos x-\sin x)}{(\sin x+\cos x)}\,dx.$$

$$= \frac{1}{2}\int_0^{\pi/2} dx + \frac{1}{2}\int_0^{\pi/2} \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$$

$$= \frac{1}{2}(x)_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \frac{d(\sin x+\cos x)}{\sin x+\cos x}\,dx.$$

$$= \left(\frac{x}{2}\right)_0^{\pi/2} + \frac{1}{2}\left(\log|\sin x+\cos x|\right)_0^{\pi/2}$$

$$= \left(\frac{\pi/2}{2}-0\right) + \frac{1}{2}\left(\log(1+0)-\log(0+1)\right)$$

$$= \frac{\pi}{4}$$

Ex:6 Evaluate

$$\int x^3\sqrt{x^2+1}\,dx.$$

Soln:

Let $u = x^2+1 \Rightarrow x^2 = u-1$.

$du = 2x\,dx$.

$x\,dx = \frac{du}{2}$.

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2 \cdot x\sqrt{x^2+1}\,dx$$

$$= \int (u-1)\sqrt{u}\,\frac{du}{2}.$$

$= \frac{1}{2}\int (u^{3/2} - u^{1/2})\, du$

$= \frac{1}{2}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right] + c$

$= \frac{\cancel{2}}{\cancel{2}}\left[\frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}\right] + c$

$= \left(\frac{(1+x^2)^{5/2}}{5} - \frac{(1+x^2)^{3/2}}{3}\right) + c.$

Integrals of symmetric functions.

Suppose f is continuous on $[-a, a]$

(i) If f is even $[f(-x) = f(x)]$,

then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$

(ii) If f is odd $(f(-x) = -f(x))$,

then $\int_{-a}^{a} f(x)\, dx = 0.$

Eg:1 Evaluate $\int_{-\pi/4}^{\pi/4} (x^3 + x^4 \tan x)\, dx.$

Soln: let $f(x) = x^3 + x^4 \tan x$

Replacing x by $-x$ we get.

$$f(-x) = (-x)^3 + (-x)^4 \tan(-x).$$

$$= -x^3 - x^4 \tan x.$$

$$f(-x) = -(x^3 + x^4 \tan x)$$

$$= -f(x).$$

$\therefore f(x)$ is an odd function.

$$\int_{-\pi/4}^{\pi/4} (x^3 + x^4 \tan x)\, dx = 0.$$

Eg:2 Evaluate $\int_{-\pi/4}^{\pi/4} (\tan^2 x \sec^2 x)\, dx.$

Soln:

Let $f(x) = \tan^2 x \sec^2 x$

Replacing x by $-x$ we get

$$f(-x) = \tan^2(-x) \sec^2(-x)$$

$$= \tan^2(x) \sec^2(x)$$

$$f(-x) = f(x).$$

The integrand $f(x)$ is an even function and by the properties of the symmetric functions we have

$$\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^2 x \, dx = 2\int_0^{\pi/4} \tan^2 x \sec^2 x \, dx$$

let $u = \tan x$

$du = \sec^2 x \, dx$

When $x = 0$ $u = \tan 0 = 0$

$x = 0 \Rightarrow u = 0$

When $x = \pi/4$, $u = \tan \pi/4 = 1$

$x = \pi/4 \Rightarrow u = 1$.

$$2\int_0^{\pi/4} \tan^2 x \sec^2 x \, dx = 2\int_0^1 u^2 du = 2\left(\frac{u^3}{3}\right)_0^1 = \frac{2}{3}.$$

Integration by Parts

$$\int u \, dv = uv - \int v \, du.$$

This formula is called the formula for integration by parts.

Eg:-1 Evaluate $\int e^{ax}(\cos bx)\,dx$ by using integration by parts.

Soln:-

let $I = \int e^{ax}(\cos bx)\,dx$.

Where a and b are constants and are not equal to zero.

let $u = e^{ax}$ $\quad dv = \cos bx\,dx$

$du = ae^{ax}dx$ $\quad v = \frac{\sin bx}{b}$

We know that

$$\int u\,dv = uv - \int v\,du$$

$$I = e^{ax} \times \frac{\sin(bx)}{b} - \int \frac{\sin bx (a e^{ax})dx}{b}$$

$$I = \frac{1}{b} e^{ax} \sin(bx) - \frac{a}{b}\int e^{ax}\sin(bx)\,dx$$

Again assuming

$u = e^{ax}$, $\cancel{du = ae^{ax}dx}$.

$\cancel{du = ae^{ax}dx}$,

$u = e^{ax}$ $\quad dv = \sin bx\,dx$

$du = ae^{ax}dx$ $\quad v = \frac{-\cos(bx)}{b}$

$$I = \frac{1}{b} e^{ax}\sin(bx) - \frac{a}{b}\left(-\frac{e^{ax}\cos(bx)}{b} - \int -\frac{\cos(bx)}{b} \times a\, e^{ax}\,dx\right.$$

$$I = \frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax} \cos(bx)$$

$$- \frac{a^2}{b^2} \int e^{ax} \cos(bx)\, dx$$

$$= \frac{1}{b} e^{ax} \sin(bx) + \frac{a}{b^2} e^{ax} \cos(bx)$$

$$- \frac{a^2}{b^2} I$$

$$I + \frac{a^2}{b^2} I = \frac{e^{ax}}{b} \left(\frac{a}{b} \cos(bx) + \sin(bx) \right)$$

$$I \left(1 + \frac{a^2}{b^2}\right) = \frac{e^{ax}}{b^2} \left(a\cos(bx) + b\sin(bx) \right)$$

$$I = \frac{e^{ax}}{\left(1 + \frac{a^2}{b^2}\right) b^2} \left(a\cos(bx) + b\sin(bx) \right)$$

$$= \frac{e^{ax}}{\frac{a^2+b^2}{b^2} \times b^2} \left(a\cos(bx) + b\sin(bx) \right)$$

$$= \frac{e^{ax}}{a^2+b^2} \left(a\cos(bx) + b\sin(bx) \right).$$

Eg:-2 Evaluate $\int e^{-ax} \sin(bx)\, dx$. by using integration by parts.

Soln:

let $I = \int e^{-ax} \sin(bx)\, dx$ where a and b are Constants and are not equal to zero.

let $u = e^{-ax}$ $du = -a e^{-ax} dx$

$dv = \sin bx \, dx$.

$v = -\frac{\cos bx}{b}$

$$I = -e^{-ax} \frac{\cos(bx)}{b} - \int -\frac{\cos(bx)}{b} (-a e^{-ax}) dx$$

$$= -\frac{1}{b} \cdot e^{-ax} \cos bx - \frac{a}{b} \int e^{-ax} \cos(bx) dx$$

Again assuming

$u = e^{-ax}$, $dv = \cos(bx) \, dx$.

$du = -a e^{-ax}$ $v = \frac{\sin bx}{b}$.

$$I = -\frac{1}{b} e^{-ax} \cos(bx) - \frac{a}{b} \left(e^{-ax} \frac{\sin(bx)}{b} - \int \frac{\sin bx}{b} (-a e^{-ax} dx) \right)$$

$$= -\frac{1}{b} e^{-ax} \cos bx - \frac{a}{b^2} e^{-ax} \sin(bx) - \frac{a^2}{b^2} \left(\int e^{-ax} \sin(bx) dx \right)$$

$$= -\frac{1}{b} e^{-ax} \cos bx - \frac{a}{b^2} e^{-ax} \sin bx - \frac{a^2}{b^2} I$$

$$\left(1+\frac{a^2}{b^2}\right)I = \frac{-e^{-ax}}{b}\left(+\frac{a}{b}\sin(bx)+\cos(bx)\right)$$

$$\left(\frac{a^2+b^2}{b^2}\right)I = \frac{-e^{-ax}}{b}\left(\frac{a\sin(bx)+b\cos(bx)}{b}\right)$$

$$= \frac{-e^{-ax}(b^2)}{b^2(a^2+b^2)}(a\sin(bx)+b\cos(bx))$$

$$= \frac{-e^{-ax}}{(a^2+b^2)}(a\sin(bx)+b\cos(bx))$$

Eg:3 Evaluate $\int\left(\frac{\log x}{x}\right)^2 dx$ by using integration by parts.

Soln: Let $u=(\log x)^2$ and $dv = \frac{1}{x^2}dx$.

$du = 2\log x \times \frac{1}{x}dx$, $v=\frac{-1}{x}$.

$$\int\left(\frac{\log x}{x}\right)^2 dx = -\frac{(\log x)^2}{x} - \int -\frac{1}{x}\left(\frac{2}{x}\right)\log x\, dx$$

$$= \frac{-(\log x)^2}{x} + 2\int\frac{1}{x^2}(\log x)dx.$$

Again by using integration by parts on the second terms of RHS.

$u = \log x$, $\quad dv = \frac{1}{x^2}\,dx$.

$du = \frac{1}{x}\,dx$, $\quad v = -\frac{1}{x}$

$$\int \left(\frac{\log x}{x}\right)^2 dx = -\frac{(\log x)^2}{x} + 2\left(-\frac{1}{x}\log x - \int \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx\right)$$

$$= -\frac{(\log x)^2}{x} - \frac{2}{x}\log x + 2\int \frac{1}{x^2}\,dx$$

$$= -\frac{(\log x)^2}{x} - \frac{2}{x}\log x + 2\left(-\frac{1}{x}\right) + c$$

$$= -\frac{(\log x)^2}{x} - \frac{2}{x}\log x - \frac{2}{x} + c.$$

Eg:-4 Establish a reduction formula for $I_n = \int \sin^n x\,dx$ ~~by~~ Hence find $\int_0^{\pi/2} \sin^n x\,dx$.

Soln:

Let us Consider

$$I_n = \int \sin^n x\,dx.$$

$$= \int \sin^{n-1} x \cdot \sin x\,dx$$

Let $u = \sin^{n-1} x$. $\quad dv = \sin x\,dx$

$du = (n-1)\sin^{n-2} x \cos x\,dx$ $\quad v = -\cos x$

$$I_n = \sin^{n-1}x(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2}x \cos x \, dx$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)\int \cos^2 x \sin^{n-2}x \, dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x \, dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int (\sin^{n-2}x - \sin^n x) \, dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x \, dx - (n-1)\int \sin^n x \, dx$$

$$= -\cos x \sin^{n-1}x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = -\cos x \sin^{n-1}x + (n-1) I_{n-2}$$

$$\boxed{I_n (n-1+1) = -\cos x \sin^{n-1}x + (n-1) I_{n-2}}$$

$$I_n = -\frac{\cos x \sin^{n-1}x}{n} + \frac{(n-1)}{n} I_{n-2}.$$

$$\int_0^{\pi/2} \sin^n x \, dx = \left(\frac{-\cos x \sin^{n-1}x}{n} + \frac{n-1}{n} I_{n-2} \right)_0^{\pi/2}$$

First term of RHS will be ~~vanishes~~ for both upper and the lower limit

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx.$$

$$= \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \frac{1}{2} \times I_0$$

If n is even,

$$I_0 = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \pi/2$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}.$$

If n is odd,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

$$= \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \cdots \frac{2}{3} \times I_1.$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx = (-\cos x)_0^{\pi/2}$$

$$= \left(-\cos \tfrac{\pi}{2} - (\cos 0)\right)$$

$$= (0 - (-1)) = 1.$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} & \text{If } n \text{ is odd.} \end{cases}$$

Eg: 5 Evaluate $\int_0^{\pi/4} x \tan^2 x \, dx$.

Soln: Let $u = x$, $dv = \tan^2 x \, dx$

$du = dx \quad v = .$

$$\int_0^{\pi/4} x (\tan^2 x) \, dx$$

$$= \int_0^{\pi/4} x (\sec^2 x - 1) \, dx$$

$(1 + \tan^2 x = \sec^2 x$

$\sec^2 x - \tan^2 x = 1)$

$$= \int_0^{\pi/4} x \sec^2 x \, dx - \int_0^{\pi/4} x \, dx$$

Let $u = x \quad dv = \sec^2 x \, dx$

$du = dx \quad v = \tan x.$

$$\int_0^{\pi/4} x \tan^2 x \, dx = (x \tan x)_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx - \left(\frac{x^2}{2}\right)_0^{\pi/4}$$

$$= \frac{\pi}{4} - (\log \sec x)_0^{\pi/4} - \left(\frac{(\pi/4)^2}{2}\right)$$

$$= \frac{\pi}{4} - (\log \sqrt{2}) - \frac{\pi^2}{32}.$$

Eg:5 Evaluate $\int_0^{\pi/4} x \tan^2 x \, dx$.

Soln: let $u = x$, $dv = \tan^2 x \, dx$
$du = dx$ $v = $.

$$\int_0^{\pi/4} x (\tan^2 x) \, dx$$

$$= \int_0^{\pi/4} x (\sec^2 x - 1) \, dx$$

$(1+\tan^2 x = \sec^2 x)$
$\sec^2 x - \tan^2 x = 1)$

$$= \int_0^{\pi/4} x \sec^2 x \, dx - \int_0^{\pi/4} x \, dx$$

let $u = x$ $dv = \sec^2 x \, dx$
$du = dx$ $v = \tan x$.

$$\int_0^{\pi/4} x \tan^2 x \, dx = (x \tan x)_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx - \left(\frac{x^2}{2}\right)_0^{\pi/4} + c$$

$$= \frac{\pi}{4} - (\log \sec x)_0^{\pi/4} - \left(\frac{(\pi/4)^2}{2}\right)$$

$$= \frac{\pi}{4} - (\log \sqrt{2}) - \frac{\pi^2}{32}.$$

Ex:6
Evaluate $\int_0^{\pi/2} \cos^5 x \, dx$

Soln: $\int_0^{\pi/2} \cos^5 x \, dx = \int_0^{\pi/2} \cos^4 x \cos x \, dx$

$$= \int_0^{\pi/2} (\cos^2 x)^2 \cos x \, dx$$

$$= \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x \, dx$$

Let $u = \sin x$ $\quad du = \cos x \, dx$

when $x = 0$, $u = 0$

when $x = \pi/2$, $u = 1$

$$\therefore \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x \, dx$$

$$= \int_0^1 (1-u^2)^2 \, du$$

$$= \int_0^1 (1+u^4-2u^2) \, du$$

$$= \left(u + \frac{u^5}{5} - \frac{2u^3}{3} \right)_0^1$$

$$= \left(1 + \frac{1}{5} - \frac{2}{3} \right)$$

$$= \frac{15+3-10}{15} = \frac{8}{15}$$

Ex:-7 Evaluate $\int \sin^4 x \, dx$.

Soln: $\int \sin^4 x \, dx = \int (\sin^2 x)^2 dx$.

$$= \int \left(\frac{1-\cos 2x}{2} \right)^2 dx$$

$$= \frac{1}{4}\int (1+\cos^2 2x - 2\cos 2x)\,dx$$

Here $\cos^2 2x = \left(\frac{1+\cos 4x}{2}\right)$

$$\therefore \int \sin^4 x\,dx = \frac{1}{4}\int \left(1+\left(\frac{1}{2}+\frac{\cos 4x}{2}\right) - 2\cos 2x\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2}+\frac{\cos 4x}{2} - 2\cos 2x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \frac{1}{2}\frac{\sin 4x}{4} - \frac{2\sin 2x}{2}\right) + C$$

$$= \frac{1}{4}\left(\frac{3x}{2} + \frac{\sin 4x}{8} - \sin 2x\right) + C$$

Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\ -\pi/2 \le \theta \le \pi/2$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\ -\pi/2 < \theta < \pi/2$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\ 0 \le \theta \le \pi/2$ or $\pi \le \theta \le 3\pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

Ex-1. Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ by using trigonometric substitution

Soln:

let $I = \int \frac{1}{\sqrt{a^2-x^2}} dx$.

Put $x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$.

$dx = a\cos\theta \, d\theta$.

$\therefore I = \int \frac{1}{\sqrt{a^2-a^2\sin^2\theta}} a\cos\theta \, d\theta$.

$= \int \frac{1}{\sqrt{a^2(1-\sin^2\theta)}} a\cos\theta \, d\theta$.

$= \int \frac{1}{\sqrt{a^2\cos^2\theta}} a\cos\theta \, d\theta$.

$= \int \frac{\cancel{a\cos\theta}}{\cancel{a\cos\theta}} d\theta$.

$= \theta + c$

$= \sin^{-1}\left(\frac{x}{a}\right) + c$. $\left(\begin{array}{l} x = a\sin\theta \\ \frac{x}{a} = \sin\theta \\ \theta = \sin^{-1}\left(\frac{x}{a}\right) \end{array} \right)$.

Ex:-2

Evaluate $\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$

Soln:

let $I = \int_{\sqrt{2}/3}^{2/3} \frac{1}{x^5\sqrt{9x^2-1}} dx$

Put $x = \frac{1}{3}\sec\theta$. $dx = \frac{1}{3}\sec\theta\tan\theta\, d\theta$

Then $\sec\theta = 3x$

$\theta = \sec^{-1}(3x)$

When $x = \sqrt{2}/3$, $\theta = \sec^{-1}\left(3\times\frac{\sqrt{2}}{3}\right)$

$= \sec^{-1}(\sqrt{2})$

$\theta = \pi/4$.

When $x = 2/3$: $\theta = \sec^{-1}\left(3\times\frac{2}{3}\right)$

$= \sec^{-1}(2)$

$\theta = \pi/3$

$$I = \int_{\sqrt{2}/3}^{2/3} \frac{1}{x^5\sqrt{9x^2-1}}\, dx$$

$$= \int_{\pi/4}^{\pi/3} \frac{\frac{1}{3}\sec\theta\tan\theta\, d\theta}{\left(\frac{1}{3}\sec\theta\right)^5\sqrt{9\left(\frac{1}{3}\sec\theta\right)^2-1}}$$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta\, d\theta}{\left(\frac{3}{3^5}\right)\sec^5\theta\sqrt{\sec^2\theta-1}}$$

$$= 3^4\int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\tan^2\theta}\times\sec^5\theta}$$

$$= 3^4\int_{\pi/4}^{\pi/3} \frac{\sec\theta\,\cancel{\tan\theta}}{\cancel{\tan\theta}\times\sec^5\theta}\, d\theta$$

$$= 3^4 \int_{\pi/4}^{\pi/3} \frac{1}{\sec^4\theta} \, d\theta$$

$$= 3^4 \int_{\pi/4}^{\pi/3} \cos^4\theta \, d\theta$$

$$= 3^4 \int_{\pi/4}^{\pi/3} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{3^4}{2^2} \int_{\pi/4}^{\pi/3} 1 + \cos^2 2\theta + 2(\cos 2\theta)) \, d\theta$$

$$= \frac{81}{4} \int_{\pi/4}^{\pi/3} (1 + \cos^2 2\theta + 2(\cos 2\theta)) \, d\theta$$

$$= \frac{81}{4} \int_{\pi/4}^{\pi/3} \left(1 + \left(\frac{1+\cos 4\theta}{2}\right) + 2(\cos 2\theta)\right) d\theta$$

$$= \frac{81}{4} \int_{\pi/4}^{\pi/3} \left(\frac{3}{2} + \frac{1}{2}\cos 4\theta + 2\cos\theta\right) d\theta$$

$$= \frac{81}{4\times 2} \int_{\pi/4}^{\pi/3} (3 + \cos 4\theta + 4(\cos 2\theta)) \, d\theta$$

$$= \cancel{\frac{81}{8} \int_{\pi/4}^{\pi/3} 3\theta} \quad = \cancel{\frac{81}{8} \int_{\pi/4}^{\pi/3}}$$

$$= \frac{81}{8}\left(3\theta + \frac{\sin 4\theta}{4} + 4\,\frac{\sin 2\theta}{2}\right)_{\pi/4}^{\pi/3}$$

$$= \frac{81}{8}\left[\left(3\left(\frac{\pi}{3}\right) + \frac{1}{4}\sin\frac{4\pi}{3} + 2\sin\frac{2\pi}{3}\right) - \left(\frac{3\pi}{4} + \frac{1}{4}\sin\pi + 2\sin\frac{\pi}{2}\right)\right]$$

$$= \frac{81}{8}\left(\frac{3\pi}{3} + \frac{1}{4}\sin\frac{4\pi}{3} + 2\sin\frac{2\pi}{3} - \frac{3\pi}{4} - \frac{1}{4}\sin\pi - 2\sin\frac{\pi}{2}\right).$$

$$= \frac{81}{8}\left(\pi + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{3\pi}{4}\right) - 0 - 2(1)\right)$$

$$= \frac{81}{8}\left(\frac{\pi}{4} + \frac{7\sqrt{3}}{8} - 2\right).$$

Strategy for evaluating integrals of the form.

(i) $\int \sin mx \cos nx \, dx$.

(ii) $\int \sin mx \sin nx \, dx$

(iii) $\int \cos mx \cos nx \, dx$.

(i) $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$.

(ii) $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

(iii) $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$.

Ex:4 Evaluate $\int \sin 4x \cos 5x \, dx$.

Soln: We know that

$$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$$

$$\int \sin 4x \cos 5x \, dx = \frac{1}{2}\int (\sin(4x-5x) + \sin(4x+5x))\,dx$$

$$= \frac{1}{2}\int (\sin(-x) + \sin 9x)\,dx$$

$$= \frac{1}{2}\int ((-\sin x) + \sin 9x)\,dx$$

$$= \frac{1}{2}\left(-(-\cos x) - \frac{\cos 9x}{9}\right) + C$$

$$= \frac{1}{2}\left(\cos x - \frac{\cos 9x}{9}\right) + C$$

Integrals of the form

a) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Divide the given expression under the square root by the numerical value of the coefficient of x^2 and complete the square of the terms which contain x and then the reduced integral can be evaluated.

b) $\int \frac{(px+q)}{\sqrt{ax^2+bx+c}}\,dx$

Write down the numerator in the form

$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$

where A and B are constants.

The values of A and B can be found by equating the coefficient of x and the constant term.

Then the integral can be written as

$$\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}} = A\int \frac{\frac{d}{dx}(ax^2+bx+c)\,dx}{\sqrt{ax^2+bx+c}} + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= 2A\sqrt{ax^2+bx+c} + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

c) $\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}\,(x-k)}$

The Substitution $x-k = \frac{1}{t}$ will reduce the expression $\frac{1}{(x-k)\sqrt{ax^2+bx+c}}$ to the form $\frac{1}{\sqrt{Ax^2+Bx+C}}$ then it Canbe integrated.

Ex :-1 ((a) form)

Evaluate $\displaystyle\int \frac{dx}{\sqrt{2-3x+x^2}}$

Soln:- Here x^2-3x+2 can be written as

$$x^2-3x+2 = (x)^2 - 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2$$

$$= \left(x-\frac{3}{2}\right)^2 - \frac{9}{4} + 2$$

$$= \left(x-\frac{3}{2}\right)^2 - \frac{1}{4}.$$

$$\int \frac{dx}{\sqrt{x^2-3x+2}} = \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= \int \frac{dx}{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

We know that.

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right).$$

$$\therefore \int \frac{dx}{\sqrt{x^2-3x+2}} = \cosh^{-1}\left(\frac{x-\frac{3}{2}}{\frac{1}{2}}\right)+C$$

$$= \cosh^{-1}\left(2\left(x-\tfrac{3}{2}\right)\right)+C$$

$$= \cosh^{-1}\left(2x-3\right)+C.$$

Ex:-2 ((b) form).

Evaluate $\int \frac{x}{\sqrt{x^2+x+1}}\,dx.$

Soln: Let us Consider,

$$x = A\frac{d}{dx}(x^2+x+1)+B$$

$$x = A(2x+1)+B$$

Equating the Coefficient of x on both sides.

$$1 = 2A$$

$$\Rightarrow \boxed{A=\frac{1}{2}}$$

Equating the constant terms on both sides.

$$0 = A + B$$

$$\Rightarrow B = -A$$

$$\boxed{B = -\frac{1}{2}}$$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{A(2x+1)+B}{\sqrt{x^2+x+1}}\,dx$$

$$= A\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + (B)\int \frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$= \frac{1}{2}\,2\left(\sqrt{x^2+x+1}\right) - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+2(x)(\frac{1}{2})+1+(\frac{1}{2})^2-(\frac{1}{2})^2}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\int \frac{dx}{\sqrt{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Example 1.3 : form (b)

Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}}\, dx$

Soln:

Let us consider

$2x+5 = A \frac{d}{dx}(x^2-2x+10) + B$

$2x+5 = A(2x-2) + B$

Equating the coefficient of x on both sides,

$2 = 2A \Rightarrow \boxed{A=1}$

Equating the constant terms on both sides, we get

$5 = -2A + B$

$5 = -2(1) + B$

$\boxed{B = 7}$

$\int \frac{2x+5}{\sqrt{x^2-2x+10}}\, dx$

$= \int \frac{A \frac{d}{dx}(x^2-2x+10) + B}{\sqrt{x^2-2x+10}}\, dx$

$= A \int \frac{2x-2}{\sqrt{x^2-2x+10}}\, dx + B \int \frac{dx}{\sqrt{(x)^2-2(x)+1-1+10}}$

$= A \int \frac{2x-2}{\sqrt{x^2-2x+10}}\, dx + B \int \frac{dx}{\sqrt{(x-1)^2+9}} + c$

$$= 2\sqrt{x^2-2x+10} + 7\int \frac{dx}{\sqrt{(x-1)^2+(3)^2}}$$

$$= 2\sqrt{x^2-2x+10} + 7\sinh^{-1}\left(\frac{x-1}{3}\right) + C$$

Ex:-4 form (c)

Evaluate the integral

$$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

Soln: Let us put $x+1 = 1/t$

$dx = -1/t^2 \, dt$

$$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \int \frac{-1/t^2 \, dt}{\frac{1}{t}\sqrt{(1/t-1)^2 + (1/t-1)+1}}$$

$$= \int \frac{-1/t^2 \, dt}{\frac{1}{t}\sqrt{1/t^2 - 2/t + 1 + 1/t - 1 + 1}}$$

$$= -\int \frac{dt}{t\sqrt{\frac{1}{t^2} - 1/t + 1}}$$

$$= -\int \frac{dt}{t\sqrt{\frac{1-t+t^2}{t^2}}}$$

$$= -\int \frac{dt}{\sqrt{t^2 - t + 1}}$$

$$= -\int \frac{dt}{\sqrt{(t)^2 - 2(t)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}}$$

$$= -\int \frac{dt}{\sqrt{\left(t - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\sinh^{-1}\left(\frac{t - \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= -\sinh^{-1}\left(\frac{\frac{1}{1+x} - \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$\because x+1 = \frac{1}{t}$

$t(x+1) = 1$

$tx + t - 1 = 0$

$t(x+1) = 1$

$t = \frac{1}{1+x}$

$$= -\sinh^{-1}\left(\frac{\left(1 - \frac{1}{2}(1+x)\right)2}{(1+x)\sqrt{3}}\right) + C$$

$$= -\sinh^{-1}\left(\frac{\left(\frac{1}{2} - \frac{x}{2}\right)2}{\sqrt{3}(1+x}\right) + C$$

$$= -\sinh^{-1}\left(\frac{1-x}{\sqrt{3}(1+x)}\right) + C$$

Integrals of the form

(i) $\int \sqrt{ax^2+bx+c}\, dx$

In this form, we divide the expression under the square root by the numerical value of the coefficient of x^2 complete the square of the terms which contain x and then integrate.

ii) $\int (px+q)(\sqrt{ax^2+bx+c})\, dx$.

In this form, write down the term $(px+q) = A \frac{d}{dx}(ax^2+bx+c) + B$ where A and B are constants. The values of A and B can be easily determined by equating the coefficient of x and the constant terms.

Then

$$\int (px+q)\sqrt{ax^2+bx+c}\, dx$$

$$= A\int \sqrt{ax^2+bx+c}\ .\frac{d}{dx}(ax^2+bx+c)$$

$$+ B\int \sqrt{ax^2+bx+c}\, dx.$$

$= \frac{2}{3} A (ax^2 + bx + c)^{3/2} + B \int \sqrt{ax^2+bx+c}\, dx$

Then the integral $\int \sqrt{ax^2+bx+c}\, dx$ can be easily evaluated.

Formulas:

(i) $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$

(iii) $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{x}{a}\right) + C$

Ex 1-1 Evaluate $\int \sqrt{1+x-2x^2}\, dx$.

Soln:

$I = \int \sqrt{1+x-2x^2}\, dx$

$I = \sqrt{2} \int \sqrt{\frac{1}{2} + \frac{x}{2} - x^2}\, dx$

$= \sqrt{2} \int \sqrt{\frac{1}{2} - \left(x^2 - 2(x)\left(\frac{1}{2\times 2}\right) + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right)}$

$= \sqrt{2} \int \sqrt{\frac{1}{2} - \left(\left(x - \frac{1}{4}\right)^2\right) + \frac{1}{16}}$

$= \sqrt{2} \int \sqrt{\frac{1}{2} + \frac{1}{16} - \left(x - \frac{1}{4}\right)^2}$

$= \sqrt{2} \int \sqrt{\frac{9}{16}}$

$= \sqrt{2} \int \sqrt{\left(\frac{3}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2}\, dx$

$= \sqrt{2} \left(\frac{1}{2}\left(x-\frac{1}{4}\right)\sqrt{\left(\frac{3}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2} + \frac{1}{2}\left(\frac{9}{16}\right)\sin^{-1}\left(\frac{x-\frac{1}{4}}{\frac{3}{4}}\right) \right) + c$

Ex:-2

Evaluate $\int (3x-2)\sqrt{x^2+x+1}\, dx$.

Soln:

Let $I = \int (3x-2)\sqrt{x^2+x+1}\, dx$

let $3x-2 = A\frac{d}{dx}(x^2+x+1)+B$

$3x-2 = A(2x+1)+B$

Equating the coefficients of x, we get

$3 = 2A$

$\Rightarrow \boxed{A = 3/2}$

$-2 = A+B \Rightarrow -2 = 3/2 + B$

$-2 - 3/2 = B$

$\boxed{-7/2 = B}$

$I = \frac{3}{2}\int \sqrt{x^2+x+1}\, \frac{d}{dx}(x^2+x+1)\, dx - \frac{7}{2}\int \sqrt{x^2+x+1}\, dx$

$= \frac{3}{2} \times \frac{2}{3}(x^2+x+1)^{3/2}$

$- \frac{7}{2} \int \sqrt{x^2 + 2(x)(\frac{1}{2}) + (\frac{1}{2})^2 - (\frac{1}{2})^2 + 1} \, dx$

$(x^2+x+1)^{3/2} - \frac{7}{2} \int \sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}} \, dx$

$= (x^2+x+1)^{3/2} - \frac{7}{2} \int \sqrt{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \, dx$

$= (x^2+x+1)^{3/2} - \frac{7}{2} \left(\frac{1}{2} (x+\frac{1}{2}) \sqrt{x^2+x+1} - \frac{1}{2} \cdot \frac{3}{4} \sinh^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) \right) + C$

$= (x^2+x+1)^{3/2} - \frac{7}{8} \left((2x+1)\sqrt{x^2+x+1} \right) - \frac{21}{16} \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

Integrals of the form

(2) $\int \frac{1}{(ax^2+bx+c)} \, dx$

Divide the denominator by the coefficient of x^2 and complete the square of the term containing x reduces to the integral of the form

$\int \frac{1}{(a^2+x^2)} \, dx$

b) $\int \frac{px+q}{(ax^2+bx+c)} dx$

Here write down

$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$,

where A and B are constants.

Then A and B can be determined by equating the coefficient of x and the constant terms.

Then $\int \frac{px+q}{(ax^2+bx+c)} dx$

$= A\int \frac{\frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)} + B\int \frac{dx}{(ax^2+bx+c)}$

$= A\log(ax^2+bx+c) + B\int \frac{dx}{(ax^2+bx+c)}$

Then the integral $\int \frac{dx}{(ax^2+bx+c)}$ can be integrated as in (a).

Ex:-1 Evaluate

$\int \frac{1}{(4x^2-4x+2)} dx$

Soln:- Let us consider

$4x^2-4x+2 = 4\left(x^2-x+\frac{1}{2}\right)$

$$4x^2-4x+2 = 4\left((x)^2-2(x)(1/2)+(1/2)^2-(1/2)^2+1/2\right).$$
$$= 4\left(\left(x-1/2\right)^2+1/4\right).$$

$$\therefore \int \frac{dx}{4x^2-4x+2}$$
$$= \frac{1}{4}\int \frac{dx}{\left(\left(x-1/2\right)^2+(1/2)^2\right)}$$
$$= \frac{1}{4}\cdot\frac{1}{(1/2)}\tan^{-1}\left(\frac{x-1/2}{1/2}\right)+c.$$
$$= \frac{1}{2}\tan^{-1}(2x-1)+c$$

Ex:-2 Evaluate $\int \frac{2x+3}{x^2+x+1}dx.$

Sol: Let us consider,
$$2x+3 = A\frac{d}{dx}(x^2+x+1)+B.$$
$$2x+3 = A(2x+1)+B.$$

By solving the above equation we have $A=1$ and $B=2$.

$$\int \frac{2x+3}{x^2+x+1}dx = \int \frac{\frac{d}{dx}(x^2+x+1)}{x^2+x+1}dx$$
$$+2\int \frac{1}{(x^2+x+1)}dx.$$

$$= \log(x^2+x+1) + 2\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$= \log(x^2+x+1) + 2\int \frac{1}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \log(x^2+x+1) + 2\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \log(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

Integrals of the form.

a) $\int \frac{dx}{(Ax^2+B)\sqrt{Cx^2+D}}$ and

b) $\int \frac{dx}{(ax^2+bx+c)\sqrt{Ax^2+Bx+C}}$.

a) To evaluate the integral $\int \frac{dx}{(Ax^2+B)\sqrt{Cx^2+D}}$ make use of the substitution that $x=\frac{1}{t}$ or $\frac{Cx^2+D}{Ax^2+B}=t^2$,

By this substitution, the integral will reduce to anyone of the standard forms.

b) To evaluate the integral $\int \frac{dx}{(ax^2+bx+c)\sqrt{Ax^2+Bx+C}}$ use the substitution $\frac{Ax^2+Bx+C}{ax^2+bx+c} = t^2$

Ex:-1 $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Soln:

Put $x = 1/t$

Then $dx = -\frac{1}{t^2}dt$,

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{-dt/t^2}{\left(1+1/t^2\right)\left(\sqrt{1-1/t^2}\right)}$$

$$= -\int \frac{-dt \times t}{\cancel{t^2}\left(\frac{t^2+1}{\cancel{t^2}}\right)\sqrt{t^2-1}}$$

$$= -\int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

Put $t^2-1 = u^2$.

Differentiating $2t\,dt = 2u\,du$.

$\Rightarrow t\,dt = u\,du$.

$$= -\int \frac{u\,du}{(u^2+2)\,u}$$

$$= -\int \frac{u\,du}{(u^2+2)u}$$

$$= -\int \frac{du}{u^2+(\sqrt{2})^2}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) + C$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{\left(\frac{1}{x}\right)^2-1}}{\sqrt{2}}\right) + C$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x\sqrt{2}}\right) + C$$

Integrals of the form

a) $\int \sqrt{(x-\alpha)(\beta-x)}\,dx$

b) $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ and

c) $\int \sqrt{\frac{x-\alpha}{(\beta-x)}}\,dx$ where $\beta > \alpha$

To evaluate the integrals of the above forms, we can use the substitution

$x = \alpha \cos^2$

Ex: 11 Evaluate $\int \sqrt{(x-3)(7-x)}\, dx$

Soln:

Let $x = 3\cos^2\theta + 7\sin^2\theta$

Then $dx = (-6\cos\theta\sin\theta + 14\sin\theta\cos\theta)\, d\theta$

$= 8\sin\theta\cos\theta\, d\theta.$

$x-3 = 3\cos^2\theta + 7\sin^2\theta - 3$

$= 7\sin^2\theta - 3(\cos^2\theta - 1)$

$= 7\sin^2\theta - 3\sin^2\theta.$

$= 4\sin^2\theta.$

$7-x = 7 - 3\cos^2\theta - 7\sin^2\theta$

$= \cancel{7(\sin^2\theta)}$

$= 7(1-\sin^2\theta) - 3\cos^2\theta$

$= 7\cos^2\theta - 3\cos^2\theta$

$= 4\cos^2\theta.$

$\therefore \int \sqrt{(x-3)(7-x)}\, dx$

$= \int \sqrt{4\sin^2\theta \, 4\cos^2\theta}\; 8\sin\theta\cos\theta\, d\theta$

$= 32 \int \sin\theta\cos\theta \times \sin\theta\cos\theta\, d\theta.$

$= 32 \int \sin^2\theta\cos^2\theta\, d\theta$

$= 32 \int (\sin\theta\cos\theta)^2 d\theta.$

$$= \frac{32}{8} \int \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$= \frac{32}{4} \int \sin^2 2\theta \, d\theta$$

$$= 8 \int \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$= \frac{\cancel{8}^{4}}{\cancel{2}} \int (1-\cos 4\theta) \, d\theta$$

$$= 4 \int (1-\cos 4\theta) \, d\theta$$

$$= 4 \int d\theta - \cos 4\theta \, d\theta$$

$$= 4\theta - \frac{4\sin 4\theta}{4} + C$$

$$= 4\theta - \sin 4\theta + C$$

$$= 4\theta - 2\sin 2\theta \cos 2\theta + C$$

$$= 4\theta - 4\sin\theta\cos\theta\,(2\cos^2\theta - 1) + C$$

$$= 4\sin^{-1}\left(\sqrt{\frac{x-3}{4}}\right) - \sqrt{(x-3)(7-x)} \left\{\frac{7-x}{2} - 1\right) + C$$

$$\cancel{= 4\sin^{-1}\left(\frac{\sqrt{x-3}}{2}\right)\left(\frac{5-x}{2}\right)}$$

$$= 4\sin^{-1}\left(\sqrt{\frac{x-3}{4}}\right) - \left(\frac{5-x}{2}\right)\sqrt{(x-3)(7-x)} + C$$

The denominator of the integrand is of first degree in $\cos x$ and $\sin x$.

When the denominator of the integrand is of first degree in $\cos x$ and $\sin x$,

put $t = \tan x/2$

Then $dt = \frac{1}{2} \sec^2 (x/2)\, dx$

$= \frac{1}{2} \left(1 + \tan^2 (x/2)\right) dx.$

$= \frac{1}{2} (1+t^2)\, dx.$

$\therefore dx = \dfrac{2\,dt}{1+t^2}$

We have $\sin x = \dfrac{2\tan x/2}{1+\tan^2 x/2}$

$= \dfrac{2t}{1+t^2}$ and

$\cos x = \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}$

$= \dfrac{1-t^2}{1+t^2}$

Ex:1 Evaluate $\int_0^{\pi} \frac{dx}{5+4\cos x}$

Soln:

Let $t = \tan x/2$.

Therefore, $dx = \frac{2\,dt}{1+t^2}$.

$\cos x = \frac{1-t^2}{1+t^2}$

When $x=0$, $t=0$ and

When $x=\pi$, $t=\infty$

$$\int_0^{\pi} \frac{dx}{5+4\cos x} = \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^{\infty} \frac{2dt}{(1+t^2)5+4-4t^2}$$

$$= \int_0^{\infty} \frac{2\,dt}{5+5t^2+4-4t^2}$$

$$= \int_0^{\infty} \frac{2\,dt}{9+t^2}$$

$$= 2\int_0^{\infty} \frac{dt}{t^2+(3)^2}$$

$$= \left(2 \times \frac{1}{3} \tan^{-1} t/3\right)_0^{\infty}$$

$$= 2/3 \cdot \pi/2 = \pi/3.$$

Integral of the form

$$\int \frac{(ax^2+b)\,dx}{x^4+cx^2+1}$$

We have $ax^2+b = \frac{a+b}{2}(x^2+1) + \frac{a-b}{2}(x^2-1)$.

Hence the integral can be written as the sum of the two integrals $I_1 + I_2$, where

$$I_1 = \frac{a+b}{2}\int \frac{(x^2+1)\,dx}{x^4+cx^2+1}$$ and

$$I_2 = \frac{a-b}{2}\int \frac{(x^2-1)\,dx}{x^4+cx^2+1}.$$

The integral I_1 can be written as $\frac{a+b}{2}\int \frac{(1+1/x^2)\,dx}{(x^2+c+1/x^2)}$ and this integral can be evaluated by the substitution $x-1/x = t$.

Similarly, the integral I_2 can be written as $\frac{a-b}{2}\int \frac{(1-1/x^2)\,dx}{(x^2+c+1/x^2)}$ and this integral can be evaluated by the substitution $x+1/x = t$

Eg:11 Evaluate $\int \frac{dx}{1+x^4}$

Soln: Let $I = \int \frac{dx}{1+x^4}$

The given integral can be written as

$$\int \frac{dx}{1+x^4} = \int \frac{1/x^2}{\left(x^2+1/x^2\right)}\,dx$$

$$= \frac{1}{2}\int \frac{2/x^2}{\left(x^2+1/x^2\right)}\,dx$$

$$= \frac{1}{2}\int \left(\frac{\left(1+1/x^2\right)}{x^2+1/x^2} - \frac{\left(1-1/x^2\right)}{\left(x^2+1/x^2\right)} \right) dx$$

$$= \frac{1}{2}\int \frac{1+1/x^2}{x^2+1/x^2}\,dx - \frac{1}{2}\int \frac{1-1/x^2}{x^2+1/x^2}\,dx$$

$$= \frac{1}{2}\int \frac{1+1/x^2}{\left(x-1/x\right)^2+2}\,dx - \frac{1}{2}\int \frac{1-1/x^2}{\left(x+1/x\right)^2-2}\,dx$$

Put $u = x - 1/x$ in the first integral and $v = x + 1/x$ in the second integral.

Therefore we get

$$\int \frac{dx}{1+x^4} = \frac{1}{2}\int \frac{1+1/x^2}{\left(x-1/x\right)^2+2}\,dx$$

$$-\frac{1}{2}\int \frac{1-1/x^2}{\left(x+1/x\right)^2-2}\,dx$$

$$= \frac{1}{2}\int \frac{du}{\left(x-1/x\right)^2+2} - \frac{1}{2}\int \frac{dv}{v^2-}$$

$$= \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} - \frac{1}{2}\int \frac{dv}{v^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\left(\frac{1}{2\sqrt{2}}\right)\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + c$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + c$$

Integration of Rational functions by partial fractions (The denominator $Q(x)$ is a product of

Ex:-1 Evaluate

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$ by using partial fraction.

Soln:- Since the degree of the numerator is less than the degree of the denominator, we don't need to divide.

Let us factor the denominator as,

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$
$$= x(2x^2+4x-x-2)$$
$$= x(2x(x+2)-(x+2))$$
$$= x((x+2)(2x-1))$$

Writing the function using partial fraction, we have

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(2x-1)(x+2)}$$
$$= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$\Rightarrow x^2+2x-1$
$= A(2x-1)(x+2)+Bx(x+2)$
$+C(x(2x-1))$

x^2+2x-1
$= A(2x^2+3x-2)+B(x^2+2x)$
$+C(2x^2-x)$.

Equating the coefficient of x^2, we get,

$1 = 2A+B+2C \rightarrow (1)$

Equating the Coefficient of x. we get

$2 = 3A+2B-C \rightarrow (2)$

Equating the Constants, we get.

$-1 = -2A \Rightarrow \boxed{A = 1/2}$

$(1) \Rightarrow 1 = 1+B+2C$
$\Rightarrow B+2C = 0$

$(2) \Rightarrow 2 = 3/2+2B-C$
$\Rightarrow 2B-C = 2-3/2$
$\Rightarrow 2B-C = 1/2 \;(\times 2)$
$B+2C = 0$

$2B-C = 1/2$
$(-)\; 2B+4C = 0$
$-5C = 1/2$
$\boxed{C = -1/10}$

$3B = 1/2 \Rightarrow B = 1/6$

$B + 2C = 0 \Rightarrow B - \frac{2}{10} = 0$

$\cancel{B = \frac{1}{6}}$ $\boxed{B = \frac{1}{5}}$

$C = -\frac{1}{10}$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\int \frac{dx}{x} + \frac{1}{5}\int \frac{dx}{2x-1} - \frac{1}{10}\int \frac{dx}{x+2}$$

$$= \frac{1}{2}\log|x| + \frac{1}{5\times 2}\log|2x-1| - \frac{1}{10}\log|x+2| + C$$

$$= \frac{1}{2}\log|x| + \frac{1}{10}\log|2x-1| - \frac{1}{10}\log|x+2| + C$$

Q(x) is a product of linear factors, some of which are repeated.

Ex:1 Evaluate $\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$ by using partial fraction.

Soln: Since the degree of the numerator is higher than the degree of the denominator, first applying the long division, we get.

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1 + \frac{4x}{x^3-x^2-x+1}$$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ & \overset{(-)\quad(+)\quad(+)\quad(-)}{} \\ \hline & x^3-x^2+3x+1 \\ & x^3-x^2-x+1 \\ & \overset{(-)\quad(+)\quad(+)\quad(-)}{} \\ \hline & 4x \end{array}$$

Factorizing the denominator x^3-x^2-x+1

$$\begin{array}{r|rrrr} 1 & 1 & -1 & -1 & 1 \\ & 0 & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

$$\begin{aligned}(x^3-x^2-x+1) &= (x^2-1)(x-1) \\ &= (x+1)(x-1)(x-1) \\ &= (x+1)(x-1)^2 \\ &= (x+1)(x-1)^2\end{aligned}$$

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)}$$

$$= \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$4x = A(x^2-1) + B(x+1) + C(x^2-2x+1)$

Equating the coefficient of x^2, x and constant terms, we get

$0 = A + C \Rightarrow \boxed{A = -C}$

$4 = B - 2C \Rightarrow B = 2C + 4$

$0 = -A + B + C$

$0 = C + 2C + 4 + C$

$\Rightarrow 4C = -4$

$\boxed{C = -1}$

$\boxed{A = 1}$ $\boxed{B = 2}$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$$

$$= \int (x+1)\,dx + \int \frac{dx}{(x-1)} + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x+1}$$

$$= \frac{(x+1)^2}{2} + \log|x-1| + 2\left(\frac{-1}{x-1}\right) - \log|x+1| + C$$

$= \frac{(x+1)^2}{2} + \log\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + c.$

Q(x) Contains a repeated irreducible quadratic factor.

Evaluate

Ex:-1 $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$

Soln: Let us use the partial fraction for the given function as.

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$-x^3+2x^2-x+1$

$= A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)(x)$

$= A(x^4+2x^2+1) + (Bx+C)(x^3+x) + (Dx+E)(x)$

$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$

If we equate the Coefficients, we get the system.

$A+B=0$

$\boxed{C=-1}$

$2A+B+D=2$

$C+E=-1$

$\boxed{A=1}$

$\Rightarrow \boxed{B=-1}$

$\Rightarrow \boxed{E=0}$

$\Rightarrow \boxed{D=1.}$

Therefore the given integral is written as

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$$

$$= \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right) dx$$

$$= \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\right) dx$$

$$= \log|x| - \frac{1}{2}\log(x^2+1) - \tan^{-1}(x) + \int \frac{1}{2}\frac{2x\,dx}{(x^2+1)^2}$$

$$= \log|x| - \frac{1}{2}\log(x^2+1) - \tan^{-1}(x) + \frac{1}{2}\int \frac{d(x^2)}{(x^2+1)^2}$$

$= \log|x| - \frac{1}{2}\log(x^2+1) - \tan^{-1}(x)$

$+\frac{1}{2}\left(\frac{-1}{(x^2+1)}\right) + C.$

Eg:-2

Evaluate the integral

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\, dx.$$

Soln: Let us use the partial fraction for the given function as.

$$\frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Multiply by $(x-1)^2(x^2+1)$.

$x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$ (x^2-2x+1)

$= A(x^3-x^2+x-1) + B(x^2+1) + (Cx+D)(x^2-2x+1)$

$= (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D)$

$0 = A+C \Rightarrow \boxed{A=-C}$

$-A+B-2C+D = 1$ — ①

— ②

$A+C-2D = -2$

$-A+B+D = -1 \Rightarrow B+D = -1+A$

① $\Rightarrow -\cancel{A}-1+\cancel{A}-2C = 1$

$\Rightarrow -2C = 2 \Rightarrow \boxed{C=-1}$

$\boxed{C=-1} \Rightarrow \boxed{A=1}$

(2) $\Rightarrow 1-1-2D=-2$

$-2D=-2$

$\boxed{D=1}$

$B+D=-1+A$

$\Rightarrow B+1=-1+1$

$\Rightarrow \boxed{B=-1}$

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \int\left(\frac{1}{(x-1)} \frac{-1}{(x-1)^2} + \frac{(-x+1)}{(x^2+1)}\right)dx$$

$$= \log|x-1| - \frac{1}{}\int \frac{d(x-1)}{(x-1)^2} - \frac{1}{2}\int \frac{2x\,dx}{x^2+1} + \int\frac{1}{(x^2+1)}dx$$

$$= \log|x-1| + \frac{1}{(x-1)} - \frac{1}{2}\int\frac{d(x^2+1)}{(x^2+1)} + \tan^{-1}(x)$$

$$= \log|x-1| + \frac{1}{(x-1)} - \frac{1}{2}\log|x^2+1| + \tan^{-1}(x) + C$$

Improper Integrals

Type: I Infinite integrals.

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$ provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^{\infty} f(x)\,dx$ provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called Convergent if the corresponding limit exists and divergent if the limit does not exist.

(C) The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is defined as

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Where a is any real number. It is said to Converge if both terms Converge and diverge if either term diverges.

Ex:1 Evaluate $\int_1^{\infty} \frac{1}{x}\, dx$ and determine whether the integral is convergent or divergent.

Soln:

We have $\int_1^{\infty} \frac{1}{x}\, dx$

$$= \lim_{t \to \infty} \int_1^{t} \frac{dx}{x}$$

$$= \lim_{t \to \infty} \left(\log|x|\right)_1^{t}$$

$$= \lim_{t \to \infty} (\log t - \log 1)$$

$$= \lim_{t \to \infty} (\log t)$$

$$= \infty.$$

The limit does not exist as a finite number and so the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$ is divergent.

Ex:2 Evaluate $\int_2^{\infty} \frac{1}{x^2}\, dx$ and determine whether the integral is convergent or divergent.

Soln:

We have $\int_2^{\infty} \frac{1}{x^2}\, dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x^2}\, dx$

$$= \lim_{t \to \infty} \left(\frac{-1}{x}\right)_2^{t}$$

$$= \lim_{t\to\infty}\left(\left(\frac{-1}{t}\right)+\frac{1}{2}\right)$$

$$= \frac{1}{2}.$$

$\therefore$ The limit exists as a finite number and so the integral $\int_2^\infty \frac{1}{x^2}\,dx$ is Convergent.

Ex:-3 Evaluate $\int_4^\infty \frac{1}{\sqrt{x}}$ and determine whether it is convergent or divergent.

Soln:

We have $\int_4^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_4^t \frac{1}{\sqrt{x}}\,dx.$

$$= \lim_{t\to\infty}\left(2\sqrt{x}\right)_4^t$$

$$= \lim_{t\to\infty}\left(2\sqrt{t}-2\sqrt{4}\right)$$

$$= \infty.$$

The limit does not exist as a finite number so the improper improper integral $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent.

Ex:-4 Determine whether integral $\int_1^\infty \frac{\ln x}{x}\,dx$ is convergent or divergent. Evaluate it, if it is convergent.

Soln:
$$\int_1^\infty \frac{\log x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\log x}{x}\,dx$$

$u = \log x$, $\cancel{dv = \frac{1}{x}dx}$

$du = \frac{1}{x}\,dx$ $\quad \cancel{v = \log x}$

$$= \lim_{t\to\infty}\int_1^t u\,du$$

$$= \lim_{t\to\infty}\left(\frac{u^2}{2}\right)_1^t$$

$$= \lim_{t\to\infty}\left(\frac{(\log x)^2}{2}\right)_1^t$$

$$= \lim_{t\to\infty}\left(\frac{(\log t)^2}{2} - 0\right)$$

$= \infty$

It is divergent.

Ex:5 Evaluate $\int_3^\infty \frac{dx}{(x-2)^{3/2}}$ and determine whether it is convergent or divergent.

Soln:
$$\int_3^\infty \frac{dx}{(x-2)^{3/2}}$$

let $u = x-2$

$du = dx$

When $x=3$, $u=1$

$x=\infty$, $u=\infty$

$$\int_3^{\infty} \frac{dx}{(x-2)^{3/2}} = \int_1^{\infty} \frac{du}{(u)^{3/2}}$$

$$= \int_1^{\infty} (u^{-3/2})\, du$$

$$= \left(\frac{u^{-3/2+1}}{-3/2+1}\right)_1^{\infty}$$

$$= \left(\frac{u^{-1/2}}{-1/2}\right)_1^{\infty}$$

$$= \left(-\frac{2}{u^{1/2}}\right)_1^{\infty}$$

$$= -2\left(\frac{1}{u^{1/2}}\right)_1^{\infty}$$

$$= -2(0-1) = 2.$$

It is Convergent.

Ex:-6 For what values of p is $\int_1^{\infty} \frac{1}{x^p}\, dx$ Convergent (P-test)

Soln: We know that if $p=1$, then the integral is divergent. Let us assume that $p \neq 1$.

Then $\int_1^{\infty} \frac{1}{x^p}\, dx$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\, dx$$

$$= \lim_{t\to\infty} \left(\frac{x^{-p+1}}{-p+1}\right)_1^t$$

$$= \lim_{t\to\infty} \left(\frac{1}{(1-p)\, x^{p-1}}\right)_1^t$$

$$= \lim_{t\to\infty} \left(\frac{1}{1-p}\right)\left(\frac{1}{t^{p-1}} - 1\right)$$

If $p>1$, then $p-1>0$.

As $t\to\infty$, $t^{p-1}\to\infty$

and $\frac{1}{t^{p-1}} \to 0$.

$$\therefore \int_1^{\infty} \frac{1}{x^p}\, dx = \frac{1}{p-1} \text{ if } p>1.$$

If $\therefore$ the integral converges when $p>1$.

On the other hand, if $p<1$, then $p-1<0$ and so $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t\to\infty$.

and the integral diverges.

Discontinuous integrands.

Definition of an improper integral of Type-2

a) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx.$$

if this limit exists (as a finite number).

b) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx.$$

if this limit exists (as a finite number)

The improper integral $\int_a^b f(x)\,dx$ is called convergent, if the corresponding limit exists and divergent if the limit does not exist.

c) If f has a discontinuity at c, where $a<c<b$, then the improper integral $\int_a^b f(x)\,dx$ is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$ It is said to converge if both terms converge and diverge if either term diverge

Ex: 1 Evaluate $\int_{-1}^{1} \frac{dx}{x}$

Soln:
We first note that the given integral is improper because $f(x) = \left(\frac{1}{x}\right)$ has the vertical asymptote $x = 0$,

We have $\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x}$

$$\int_{0}^{1} \frac{dx}{x} = \lim_{t \to 0^+} \int_{t}^{1} \frac{dx}{x}$$

$$= \lim_{t \to 0^+} \left(\log |x|\right)_{t}^{1}$$

$$= \lim_{t \to 0^+} \left(\log 1 - \log |t|\right)$$

$$= \infty$$

It follows that $\int_{-1}^{1} \frac{dx}{x}$ is divergent.

A Comparison Test for improper integrals.

Comparision Test:- Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

a) If $\int_a^\infty f(x)\,dx$ is Convergent, then $\int_a^\infty g(x)\,dx$ is Convergent.

b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Ex:1.1 Does $\int_1^\infty \frac{1}{e^x+x^2}\,dx$ Converge or not ?

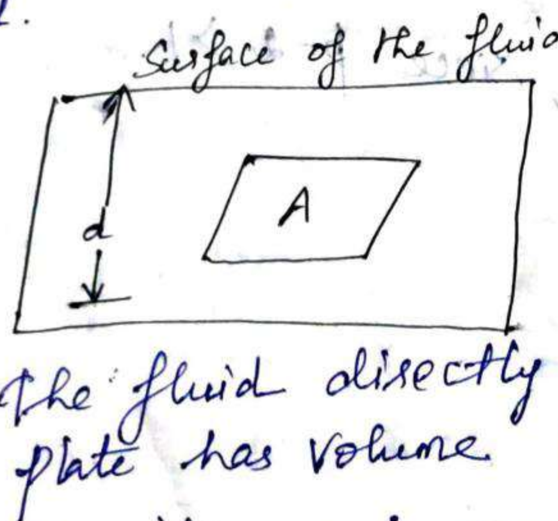
Soln: The integral $\int_1^\infty \frac{dx}{e^x+x^2}$ is Convergent, because $\frac{1}{x^2} > \frac{1}{e^x+x^2} > 0$ and $\int_1^\infty \frac{dx}{x^2}$ is Convergent by the p-test, Since $p = 2 > 1$.

Applications of Integral Calculus

Hydrostatic pressure:

Sea divers realize that water pressure increases as they dive deeper into the sea. This significant feature happens because the weight of the water above them increases.

Let us assume that a thin horizontal plate with area A square meters is submerged in a fluid of density "ρ" kilograms per cubic meter at a depth "d" meters below the surface of the fluid.



The fluid directly above the plate has volume $V = Ad$

Then its mass is $m = \rho V$
$= \rho A d$

The force exerted by the fluid on the plate is

$$F = mg = \rho g A d,$$

where "g" is the acceleration due to gravity.

The pressure "p" on the plate is defined as

$$p = \frac{F}{A} = \rho g d$$

The SI unit for measuring pressure is Newtons per square meter is called Pascal.

We know that the density of water is given by.

$$\rho = 1000 \ Kg/m^3.$$

The pressure at the bottom of a swimming pool 2 meter deep is given by.

$$p = \rho g d = 1000 \ Kg/m^3 \times 9.8 \times m/s^2 \times 2 \text{ meter}$$

$$= 19{,}600 \ Pa$$

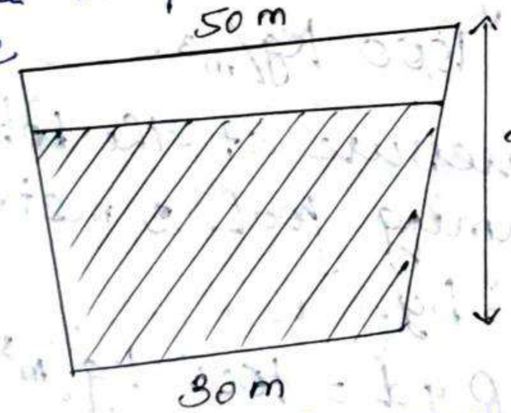
$$= 19.6 \ KPa \quad (1 \ N/m^2 = 1 \ Pa)$$

The principle of fluid pressure is that, at any point in a liquid the pressure is the

same in all directions. Therefore, a diver has the same pressure on nose and both ears. Hence the pressure in any direction at a depth "d" in a fluid with mass density "ρ" is given by $P = \rho g d$
$= \delta d$.

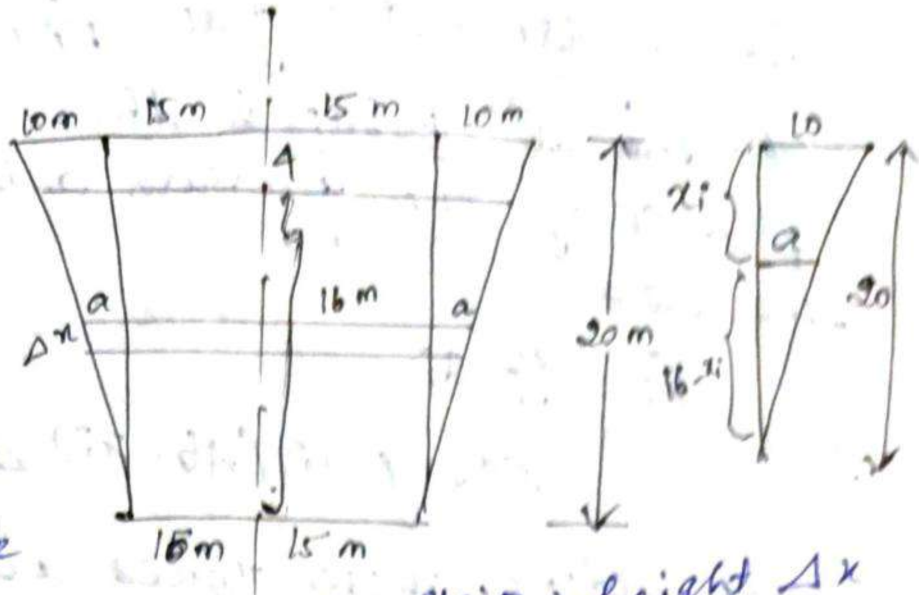
This relationship helps us to determine the hydrostatic force against a vertical plate or wall or dam in a fluid.

Ex:4 A dam has the shape of the trapezoid as shown here



The height is 20 m and the width is 50 m at the top and 30 m at the bottom. Find the force on the dam due to hydrostatic pressure if the water level is 4 m from the top of the dam.

Soln:



let the i^{th} horizontal strip → height Δx

From similar triangle of figure.

$$\frac{a}{16-x_i} = \frac{10}{20}$$

$$a = \frac{1}{2}(16-x_i) = 8 - \frac{x_i}{2}$$

$$w_i = 2(15+a)$$

$$= 2(15+8-x_i/2)$$

$$= 46 - x_i$$

If A_i is the area of the i^{th} strip, then

$$A_i \approx w_i \Delta x = (46-x_i)\Delta x.$$

If Δx is very small, then the pressure P_i on the i^{th} strip is almost constant and we can use the equation

$$P = \rho g d = 8d$$

to write $P_i \approx 1000 g x_i$

The hydrostatic force F_i acting on the i^{th} strip is the product of the pressure and the area.

$$F_i = P_i A_i$$

$$= 1000\, g\, x_i (46 - x_i)\, \Delta x.$$

By adding these forces and taking limit as $n \to \infty$, we obtain the total hydrostatic force of the dam.

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} 1000\, g\, x_i (46 - x_i) \Delta x$$

$$= \int_0^{16} 1000\, g\, x \cdot (46 - x)\, dx.$$

$$= 1000\, g \int_0^{16} (46x - x^2)\, dx$$

$$(\because g = 9.8)$$

$$= (1000)(9.8) \left(\frac{46x^2}{2} - \frac{x^3}{3} \right)_0^{16}$$

$$= 9800 \left(23x^2 - \frac{x^3}{3} \right)_0^{16}$$

$$= 9800 \left(23(16)^2 - \frac{1}{3}(16)^3 \right)$$

$$= 9800 (5888 - 1365.333)$$

$$= 44322136.6$$

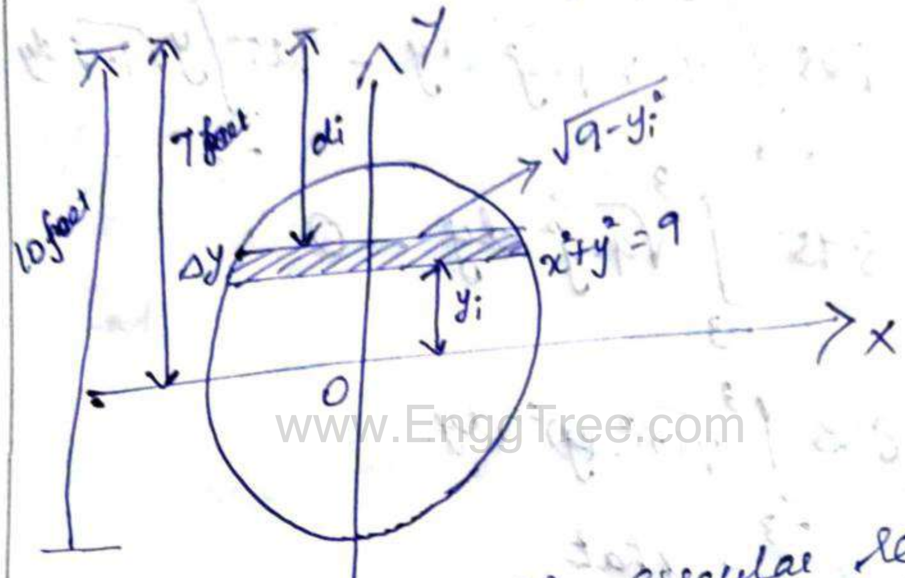
$$= 4.43 \times 10^7\, N.$$

Ex: 2 Find the hydrostatic force on one end of cylindrical drum with radius 3 feet if the drum is submerged in water 10 feet deep.

Soln:

The Circle has a simple equation.

$$x^2+y^2=9.$$



We divide the circular region into horizontal strips of equal width.

From the equation of the circle, the length of the i^{th} strip is given by $2\sqrt{9-(y_i)^2}$.

Its area is $A_i = 2\sqrt{9-(y_i)^2}\,\Delta y$

the pressure on this strip is given by $\delta d_i = 62.7\,(7-y_i)$.

$\therefore$ The force on the strip is

$\delta d_i A_i = (62.7)(7-y_i)\,2\sqrt{9-(y_i)^2}\,\Delta y.$

The total force is obtained by adding all forces on all the strips and taking the limit as

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 62.7(7-y_i)\,2\sqrt{9-(y_i)^2}\,\Delta y$$

$$= 125\int_{-3}^{3} (7-y)\sqrt{9-y^2}\,dy$$

$$= 125\int_{-3}^{3}(7)\sqrt{9-y^2}\,dy - 125\int_{-3}^{3} y\sqrt{9-y^2}\,dy$$

$$= 875\int_{-3}^{3}\sqrt{9-y^2}\,dy - 0.$$

$$= 875\int_{-3}^{3}\sqrt{(3)^2-(y)^2}\,dy.$$

We know that

$$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$$

$$= 875\left(\frac{1}{2}y\sqrt{9-y^2} + \frac{3^2}{2}\sin^{-1}\left(\frac{y}{3}\right)\right)_{-3}^{3}$$

$$= 875\left(\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(1)\right)$$

$$- \left(-\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(-1)\right)$$

$$= 875\left(\frac{9}{2}\left(\frac{\pi}{2}\right) - \frac{9}{2}\left(-\frac{\pi}{2}\right)\right)$$

$$= 875\left(9\left(\frac{\pi}{2}\right)\right) = 875\times 9\times\frac{22}{2}$$

$$= 875\times 9\times 11 = 12375 \text{ lb}$$

Moments and Centre of Mass.

Eg:1 Find the moments and centre of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1,1)$, $(2,-1)$ and $(3,2)$ respectively.

Soln: We know that the moments of the system about y-axis and x-axis respectively are given by

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and $$M_x = \sum_{i=1}^{n} m_i y_i$$

It is given that the masses are 3, 4 and 8 and the corresponding points are $(-1,1)$, $(2,-1)$, $(3,2)$

$$M_y = \sum_{i=1}^{n} m_i x_i = 3(-1)+4(2)+8(3)$$
$$= -3+8+24$$
$$= 29.$$

$$M_x = \sum_{i=1}^{n} m_i y_i = 3(1)+4(-1)+8(2)$$
$$= 3-4+16$$
$$= 15.$$

$$m = 3+4+8 = 15$$

$\bar{x} = \frac{M_y}{m} = \frac{29}{15} = 1.9333$ and

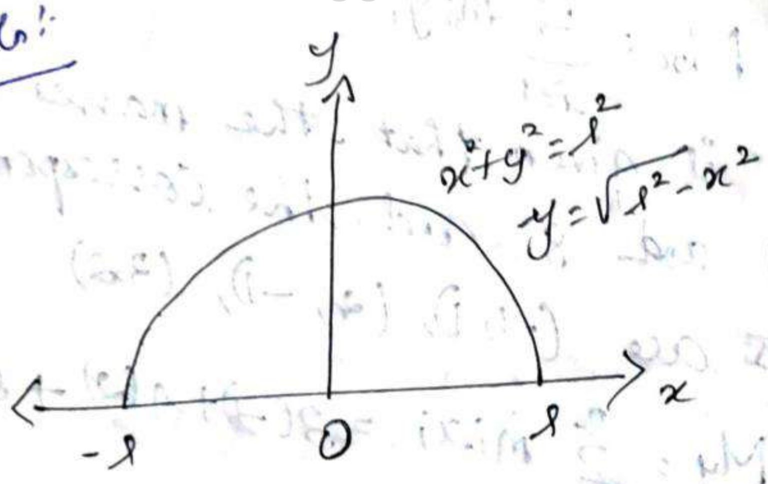
$\bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1.000$

Then the Centre of mass is $(1.9333, 1.000)$

Centroid and Symmetry.

Ex: 1 Find the Centre of mass of a semicircular plate of radius r.

Soln:



Let us Consider

$$x^2+y^2=r^2$$

$$y^2=r^2-x^2.$$

$$f(x)=y=\sqrt{r^2-x^2}$$

and $a=-r, b=r$.

Here there is no need to use the formula to Calculate $\bar{x}$, because, by the symmetry principle, the centre of mass must lie on the y-axis. Hence $\bar{x} = 0$.

The Area of the semicircle is $A = \frac{1}{2}\pi r^2$

Now $\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}(f(x))^2 \, dx$.

$= \frac{1}{\frac{1}{2}\pi r^2}\int_{-r}^{r} \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx$

$= \frac{1}{\pi r^2}\int_{-r}^{r} (r^2 - x^2)\, dx$

$= \frac{2}{\pi r^2}\int_{0}^{r} (r^2 - x^2)\, dx$ ($\because r^2 - x^2$ is an even function)

$= \frac{2}{\pi r^2}\left(r^2 x - \frac{x^3}{3}\right)_0^r$

$= \frac{2}{\pi r^2}\left(\left(r^3 - \frac{r^3}{3}\right) - 0\right)$

$= \frac{2}{\pi r^2}\left(\frac{2r^3}{3}\right)$

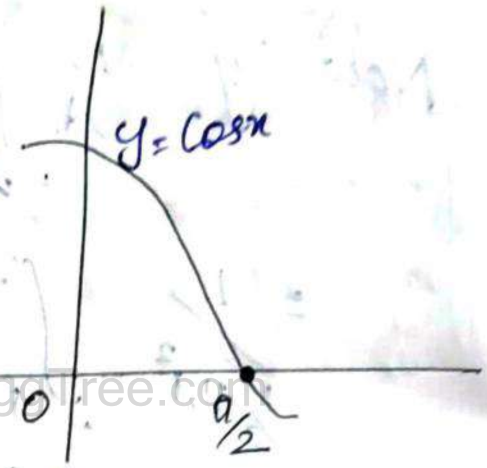
$\bar{y} = \frac{4}{3}\frac{r}{\pi}$

$\bar{x} = 0, \bar{y} = \frac{4}{3}\frac{r}{\pi}$.

$\therefore$ The centre of mass is located at the point $\left(0, \frac{1}{3}\frac{\pi}{a}\right)$

Ex 1.2 Find the centroid of the region bounded by the curves $x=0$, $x=\pi/2$, $y=0$ and $y=\cos x$.

Soln:



The area of the region

$$A = \int_0^{\pi/2} \cos x \, dx$$

$$= \left(\sin x\right)_0^{\pi/2}$$

$$= \sin \pi/2 - \sin 0$$

$$= 1 - 0$$

$$A = 1.$$

We know that the centroid of the region is given by $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left(f(x)\right)^2 dx.$$

Now, $\bar{x} = \frac{1}{A} \int_a^b x f(x)\, dx$

$= 1 \int_0^{\pi/2} x \cos x \, dx.$

$u = x \quad dv = \cos x \, dx$

$du = dx \quad v = \sin x.$

$\bar{x} = (x \sin x)_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx$

$= \pi/2 - (-\cos x)_0^{\pi/2}$

$= \pi/2 + (0 - 1)$

$= \pi/2 - 1$

$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} (f(x))^2 \, dx.$

$= \frac{1}{2} \int_0^{\pi/2} \cos^2 x \, dx.$

$= \frac{1}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx$

$= \frac{1}{4} \left(x + \frac{\sin 2x}{2} \right)_0^{\pi/2}.$

$= \frac{1}{4} \left(\pi/2 + 0 - 0 \right)$

$= \pi/8.$

The Centroid is $(\pi/2 - 1, \pi/8)$

<u>Note:</u> If the region R lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ then the centroid of R is $(\bar{x}, \bar{y})$,

where

$$\bar{x} = \frac{1}{A}\int_a^b x\,(f(x) - g(x))\,dx$$

and $$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right]dx$$

<u>Ex 1.3</u> Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

<u>Soln:</u> Let us consider

$f(x) = x$ and $g(x) = x^2$

$a = 0,\ b = 1$

$$\text{Area} = A = \int_0^1 (x - x^2)\,dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \left(\frac{1}{6}\right).$$

$$\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))\,dx$$

$$= \frac{1}{(1/6)}\int_0^1 x(x - x^2)\,dx$$

$$= 6\int_0^1 (x^2 - x^3)\,dx$$

$$= 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= 6\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 6\left(\frac{1}{12}\right)$$

$$\bar{x} = \frac{1}{2}$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left((f(x))^2 - (g(x))^2\right)dx.$$

$$= \frac{1}{1/6}\int_0^1 \frac{1}{2}(x^2 - x^4)\,dx$$

$$= \frac{6}{2}\int_0^1 (x^2 - x^4)\,dx$$

$$= 3\left(\frac{x^3}{3} - \frac{x^5}{5}\right)_0^1$$

$$= 3\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= 3\left(\frac{2}{15}\right) = \frac{2}{5}$$

$\therefore$ The centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$.

Eg: Evaluate the integral $\int_0^1 \frac{1}{(1+\sqrt{x})^4} dx$.

Soln:

$$\int_0^1 \frac{1}{(1+\sqrt{x})^4} dx$$

let $1+\sqrt{x} = t$

$\sqrt{x} = t-1$

$\frac{1}{2\sqrt{x}} dx = dt$

$dx = 2\sqrt{x}\, dt \Rightarrow 2(t-1) dt$

when $x=0,\ t=1,$

$x=1,\ t=2$

$$\int_0^1 \frac{dx}{(1+\sqrt{x})^4} = \int_1^2 \frac{2(t-1)\,dt}{t^4}$$

$$= 2\left[\int_1^2 \frac{t}{t^4} dt - \int_1^2 \frac{1}{t^4} dt\right)$$

$$= 2\int_1^2 t^{-3} dt - \int_1^2 t^{-4} dt)$$

$$= 2\left[\left(\frac{t^{-2}}{-2}\right)_1^2 - \left(\frac{t^{-3}}{-3}\right)_1^2\right]$$

$$= 2\left(\frac{-1}{2}\left(\frac{1}{4}-1\right) - \left(\frac{-1}{3}\right)\left(\frac{1}{8}-1\right)\right.$$

$$= -1\left(\frac{-3}{4}\right) + \frac{2}{3}\left(\frac{-7}{8}\right)$$

$$= \frac{3}{4} - \frac{\cancel{14}^{7}}{\cancel{24}_{12}} = \frac{9-7}{12} = \frac{2}{12} = \frac{1}{6}.$$

Unit - V

Double integration in Cartesian Co-ordinates

Problems in double integration (Cartesian Coordinates).

Ex: 1 Evaluate $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$.

Soln:

$$\int_0^1 \int_1^2 x(x+y)\,dy\,dx = \int_0^1 \int_1^2 (x^2+xy)\,dy\,dx$$

$$= \int_0^1 \left(x^2y + \frac{xy^2}{2}\right)_1^2 dx$$

$$= \int_0^1 \left(\left(2x^2+\frac{4x}{2}\right) - \left(x^2+\frac{x}{2}\right)\right) dx$$

$$= \int_0^1 \left(x^2+\frac{3x}{2}\right) dx$$

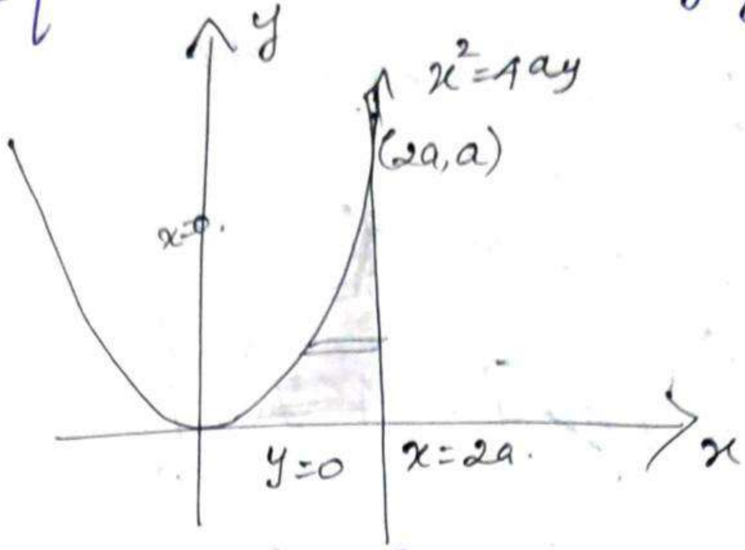
$$= \left(\frac{x^3}{3}+\frac{3x^2}{4}\right)_0^1$$

$$= \left(\frac{1}{3}+\frac{3}{4}\right) = \frac{4+9}{12} = \frac{13}{12}.$$

Ex: 2 Evaluate $\iint xy\,dx\,dy$, where R is the region bounded by x-axis $x=2a$, and $x^2=4ay$.

Soln:

The region, (the area) of which is required is shown in figure



Given $x=2a$, $x^2=4ay$

$\Rightarrow$ $(2a)^2=4ay$

$4a^2=4ay$

$y=a$

$(2a,a) \rightarrow$ The point of intersection.

y varies from $y=0$ to $y=a$.

To find the limits of x, draw a strip parallel to x-axis.

On this strip, x varies from $x=2\sqrt{ay}$ to $x=2a$.

Required integral

$$\iint xy\,dx\,dy = \int_0^a \int_{2\sqrt{ay}}^{2a} xy\,dx\,dy$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_{2\sqrt{ay}}^{2a} dy$$

$$= \frac{1}{2}\int_0^a \left(4a^2 - 4ay\right) dy$$

$$= \frac{1}{2}\int_0^a \left(4ay^2 - 4ay^2\right) dy$$

$$= \frac{1}{2}\left(\frac{4a^2y^2}{2} - \frac{4ay^3}{3}\right)_0^a$$

$$= \frac{1}{2}\left(2a^4 - \frac{4}{3}a^4\right)$$

$$= \frac{a^4}{3}.$$

Ex:-3 find the value of

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy.$$

Soln:- $\displaystyle\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$

$$= \int_0^\infty \frac{e^{-y}}{y} (x)_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (y)\, dy$$

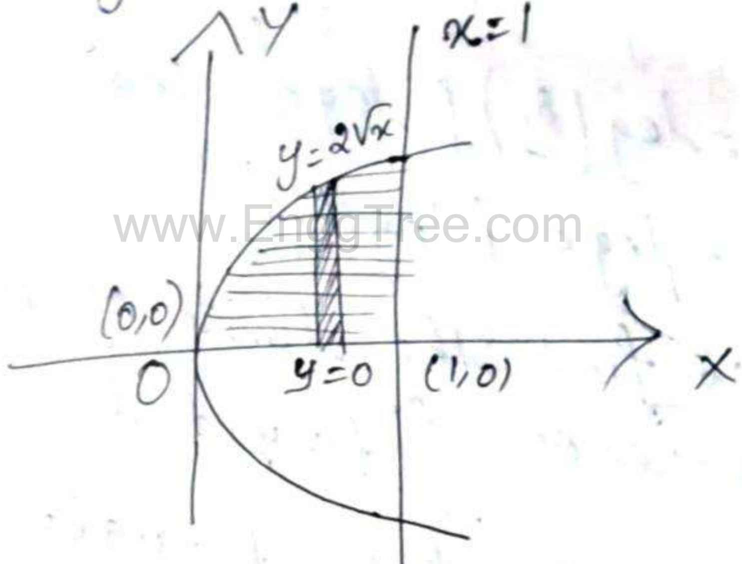
$$= \int_0^\infty e^{-y}\, dy = -\left(e^{-y}\right)_0^\infty$$

$$= -\left(e^{-\infty} - 1\right) = -(0-1)$$

$$= 1.$$

Ex:-4 Find the limits of integration in the double integral $\iint f(x,y)\,dx\,dy$ where R is in the first quadrant and bounded by $x=1$, $y=0$, $y^2=4x$.

Soln: Let us draw the region of integration given by $y=0$, $x=1$, $y^2=4x$



The limits of integration are $x=0$ to $x=1$.
and $y=0$ to $y=2\sqrt{x}$.

Ex:-5 Evaluate $\int_1^a \int_2^b \frac{dx\,dy}{xy}$.

Soln: $\int_1^a \int_2^b \frac{dx\,dy}{xy}$

$$= \int_1^a \left(\frac{1}{y} \log x \right)_2^b dy$$

$$= \int_1^a \frac{1}{y} (\log b - \log 2)\, dy$$

$$= \int_1^a \frac{1}{y} \log(b/2)\, dy$$

$$= \log(b/2) \int_1^a \frac{dy}{y}$$

$$= \log(b/2) \left(\log y \right)_1^a$$

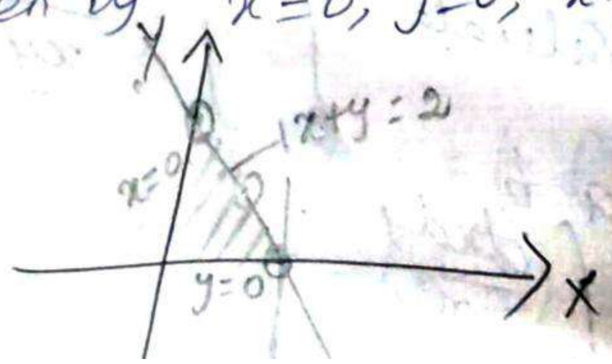
$$= \log(b/2) \left(\log a - \log 1 \right)$$

$$= \log a \left(\log(b/2) \right)$$

Ex:-6 find the limits of the integration $\iint_R f(x,y)\, dx\, dy$ where R is the triangle bounded by $x=0$, $y=0$, $x+y=2$.

Soln:-

Let us draw the region of integration given by $x=0$, $y=0$, $x+y=2$.



x	-1	0	1	2
y	3	2	1	0

The limits of the integration are $x=0$ to $x=1$

$y=0$ to $y=2-x$.

Ex:7 Evaluate $\int_0^{\log 8 = \ln 8} \int_0^{\log y = \ln y} e^{x+y}\,dx\,dy$

Soln:

$$\int_0^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy.$$

$$= \int_0^{\log 8}\int_0^{\log y} e^x e^y\,dy\,dx.$$

$$= \int_0^{\log 8} e^x \left(e^y\right)_0^{\log y} dx.$$

$$\int_0^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy$$

$$= \int_0^{\log 8}\int_0^{\log y} e^x \cdot e^y\,dx\,dy$$

$$= \int_0^{\log 8}\int_0^{\log y} e^y e^x\,dx\,dy$$

$$= \int_0^{\log 8} \left(e^y e^x\right)_0^{\log y} dy$$

$$= \int_0^{\log 8} (e^y e^{\log y})\, dy$$

$$= \int_0^{\log 8} (e^y y)\, dy$$

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

$$= (ye^y)_0^{\log 8} - \int_0^{\log 8} e^y dy$$

$$= \log 8 (e^{\log 8}) - (e^y)_0^{\log 8}$$

$$= 8 \log 8 - (e^{\log 8} - 1)$$

$$= 8 \log 8 - (8 - 1)$$

$$= 8 \log 8 - 7.$$

Problems in double integration (Polar Coordinates)

Ex:-1 Evaluate $\int_0^{\pi/2} \int_0^{\sin\theta} r\, d\theta\, dr$

Soln:- $\int_0^{\pi/2} \int_0^{\sin\theta} r\, d\theta\, dr$

$$= \int_0^{\pi/2} \int_0^{\sin\theta} r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin^2\theta)\, d\theta \quad \left(\because \int_0^{\pi/2} \sin^n\theta = \frac{n-1}{n} \cdots \frac{1}{2}\cdot\frac{\pi}{2} \text{ if } n \text{ is even}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

Ex:2 Evaluate $\int_0^{\pi}\int_0^{a} r\,dr\,d\theta$

Soln:

$$\int_0^{\pi}\int_0^{a} r\,dr\,d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^a d\theta$$

$$= \int_0^{\pi}\left(\frac{a^2}{2}\right) d\theta$$

$$= \frac{a^2}{2}(\theta)_0^{\pi}$$

$$= \frac{a^2}{2} \times \pi = \frac{\pi a^2}{2}$$

Change of order of integration.

Procedure to evaluate the double integration by changing its order of integration.

* Using the limits of the given double integration sketch the region of integration.

* Find the intersecting points from the curves and mark them.

* If the limit of the inner integral is a function of x we have to change the limit of the inner integral as a function of y.

* If the limit of the inner integral is a function of y, we have to change the limit of the inner integral as a function of x

* After changing the order of integration, if the integration is first w.r.to x then consider the horizontal strip and find the new limits.

* After changing the order of integration, if the integration is w.r.to y, then consider the vertical strip and find the new limits.

* Evaluate the given double integral as usual.

Note: when all limits are Constants. we can change the order of integration as we like.

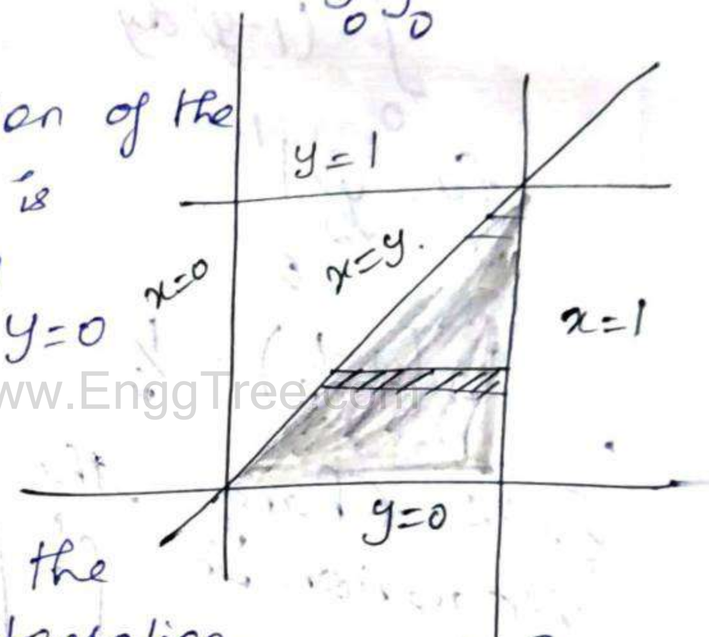
Example: $\int_1^2 \int_2^3 xy\, dy\, dx = \int_2^3 \int_1^2 xy\, dx\, dy$

Example:-1

Change the order the integration and evaluate $\int_0^1 \int_0^x dy\, dx$.

Soln:

The region of the integration is bounded by $x=0$, $x=1$, $y=0$, $y=x$.



To change the order of integration, The first integration is with respect to x and the second integration is with respect to y. Since the first integration is with respect to x, we have to Consider the horizontal strip.

x Varies from $x=y$ to $x=1$

y Varies from $y=0$ to $y=1$

Hence by changing the order we get

$$\int_0^1 \int_0^x dy\, dx = \int_0^1 \int_y^1 dx\, dy$$

$$= \int_0^1 (x)_y^1\, dy$$

$$= \int_0^1 (1-y)\, dy$$

$$= \left(y - \frac{y^2}{2}\right)_0^1$$

$$= \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

__Ex:2__ Change the order of integration in $\int_0^1 \int_y^{2-y} xy\, dx\, dy$ and hence evaluate it.

__Soln:__ The region of integration is bounded by $y = 0, y = 1, x = y, x = 2-y$

y varies from $y = 0$ to $y = 1$

x varies from $x = y$ to $x = 2-y$

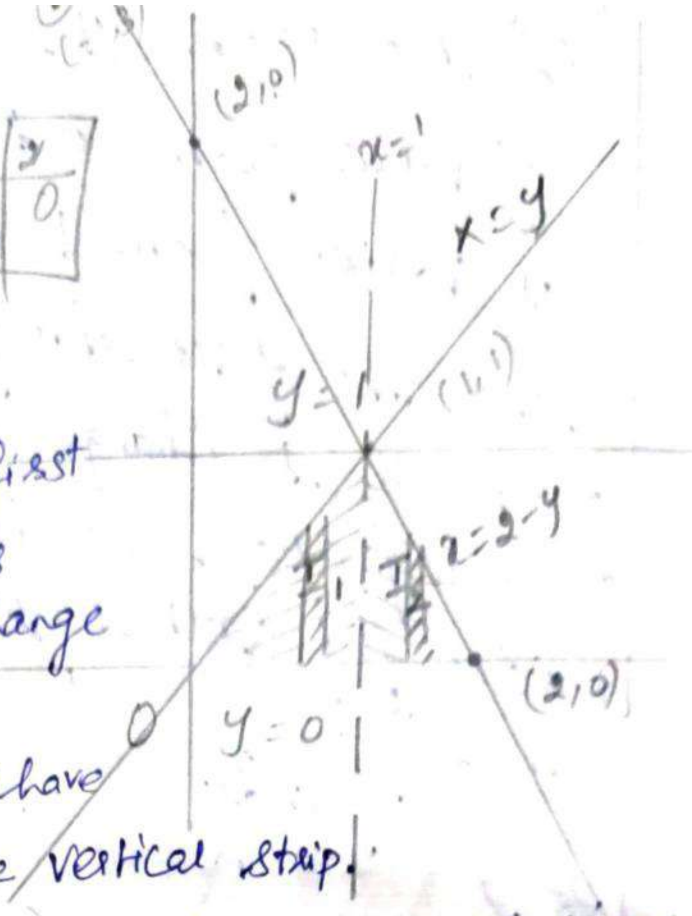
Now,

$$I = I_1 + I_2$$

$$= \int_0^1 \int_y^1 xy\, dx\, dy + \int_0^1 \int_1^{2-y} xy\, dx\, dy$$

$x = 2 - y.$

x	-1	0	1	2
y	3	2	1	0



In I_1,

Since the first integration is w.r.to x, to change the order of integration, we have to consider the vertical strip.

Here y varies from $y=0$ to $y=x$
x varies from $x=0$ to $x=1$.

Hence $\int_0^1 \int_y^1 xy \, dx\, dy = \int_0^1 \int_0^x xy \, dy\, dx$

$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^x dx$

$= \int_0^1 \left(x \frac{x^2}{2}\right) dx.$

$= \frac{1}{2}\int_0^1 x^3 \, dx$

$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^1$

$= \left(\frac{x^4}{8}\right)_0^1 = \frac{1}{8}(1-0)$

$= \frac{1}{8}$

~~In I_2~~

In the region I_2

y varies from 0 to 1

x varies from 1 to $2-y$.

Since the first integration is w.r.to x, to change the order of integration, we have to consider the vertical strip.

$$\cancel{\int_0^1 \int_1^{2-y} xy\,dx\,dy} =$$

Here y varies from $y=0$ to $y=2-x$

x varies from $x=1$ to $x=2$.

$$\int_0^1 \int_1^{2-y} xy\,dx\,dy$$

$$= \int_1^2 \int_0^{2-x} xy\,dy\,dx$$

$$= \int_1^2 \left(\frac{xy^2}{2}\right)_0^{2-x} dx$$

$$= \int_1^2 \left(\frac{x(2-x)^2}{2}\right) dx$$

$$= \int_1^2 \frac{x(4+x^2-4x)}{2}\,dx$$

$$= \frac{1}{2}\int_1^2 (4x + x^3 - 4x^2)\, dx$$

$$= \frac{1}{2}\left(\frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3}\right)_1^2$$

$$= \frac{1}{2}\left(2(4) + 4 - \frac{32}{3} - 2 + \frac{4}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(12 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right)$$

$$= \frac{1}{2}\left(10 - \frac{28}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{120 - 112 - 3}{12}\right)$$

$$= \frac{1}{2}\left(\frac{5}{12}\right) = \frac{5}{24}.$$

$$\underline{I} = I_1 + I_2 = \frac{1}{8} + \frac{5}{24} = \frac{3+5}{24} = \frac{8}{24} = \frac{1}{3}.$$

<u>Eg: 3</u> Change the order of integration in the integral $\int_0^a \int_0^{2\sqrt{ax}} x^2\, dy\, dx$ and evaluate.

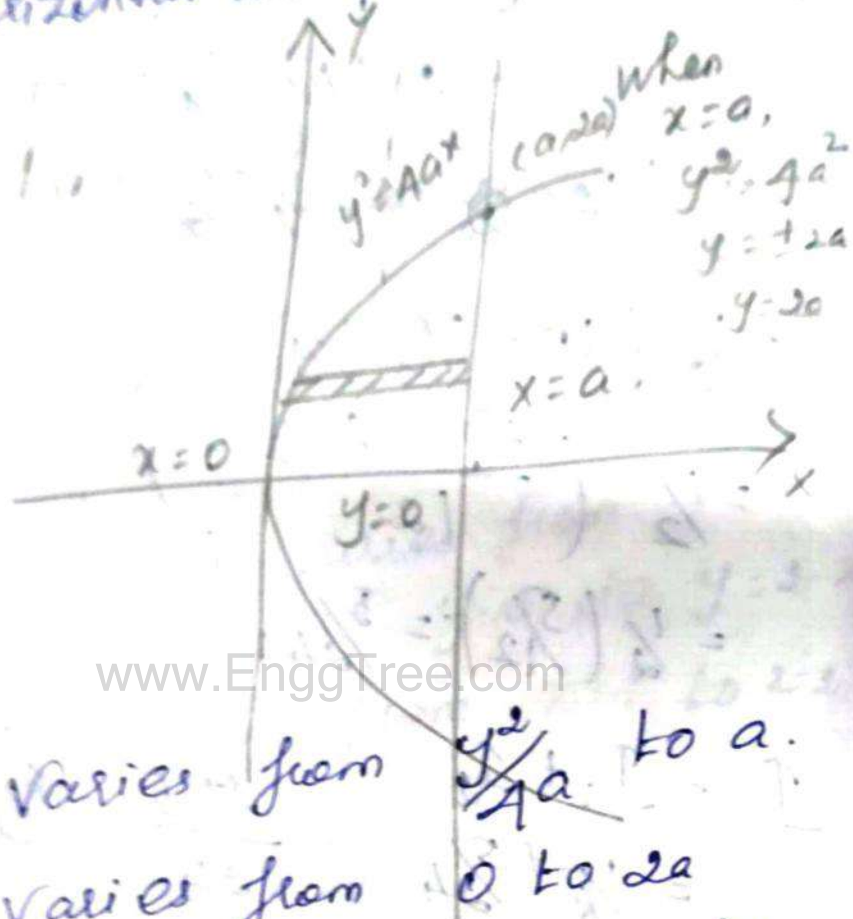
<u>Soln:</u> Given the limit of y is from 0 to $2\sqrt{ax}$.

The limit of x from 0 to a.

The limit of y from 0 to $2\sqrt{ax}$.

After ~~the~~ changing the order of integration, the first integration is w.r.to x,

$\therefore$ we have to consider the horizontal strip.



x varies from $y^2/4a$ to a.

y varies from 0 to $2a$

Hence by changing the order, we get

$$\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dy \, dx$$

$$= \int_0^{2a} \int_{y^2/4a}^{a} x^2 \, dx \, dy$$

$$= \int_0^{2a} \left(\frac{x^3}{3}\right)_{y^2/4a}^{a} dy$$

$$= \int_0^{2a} \left(\frac{a^3}{3} - \frac{y^6}{64a^3 \times 3} \right) dy$$

$$= \left(\frac{a^3}{3} y - \frac{y^7}{7(192a^3)} \right)_0^{2a}$$

$\frac{192}{1344}$

$$= \left(\frac{a^3}{3} 2a - \frac{128 a^7}{1344 a^3} \right)$$

$$= \left(\frac{2a^4}{3} - \frac{2}{21} a^4 \right)$$

$$= \left(\frac{14a^4 - 2a^4}{21} \right) = \frac{12a^4}{21}$$

$$= \left(\frac{4a^4}{7} \right)$$

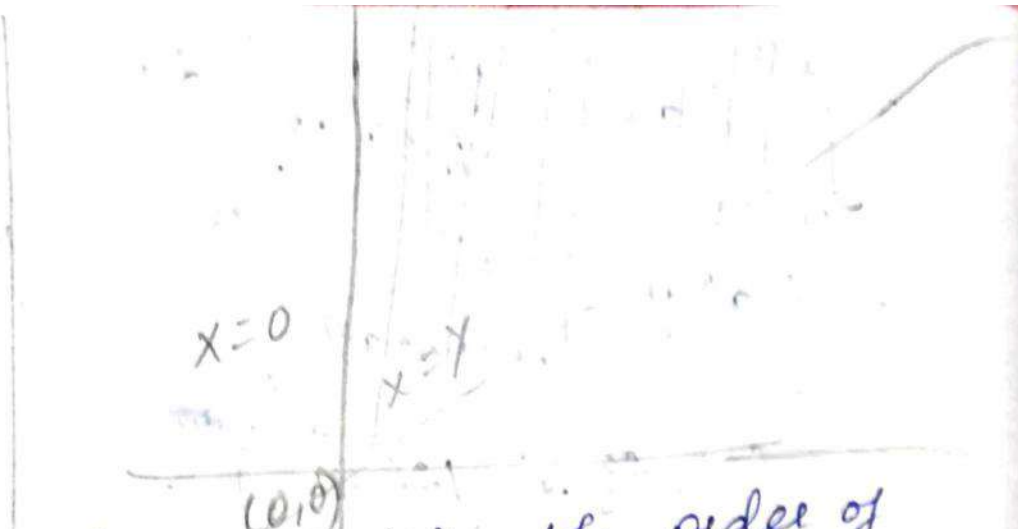
$$\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dy\, dx = \int_0^{2a} \int_{y^2/4a}^{a} x^2 \, dx\, dy$$

$$= 4a^4/7$$

Eg: 4 Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy\, dx$ and hence evaluate it

Soln: Given y varies from x to ∞

x varies from 0 to ∞.



After Changing the order of integration, the first integration is with respect to x.
we have to consider the horizontal strip.

x varies from $x=0$ to $x=y$

y varies from $y=0$ to $y=\infty$.

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \left(\frac{e^{-y}}{y}\right) dx\, dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y}\, x\right)_0^y dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y}\, y\right) dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = -\left(e^{-\infty} - e^{0}\right)$$

$$= -(0-1)$$

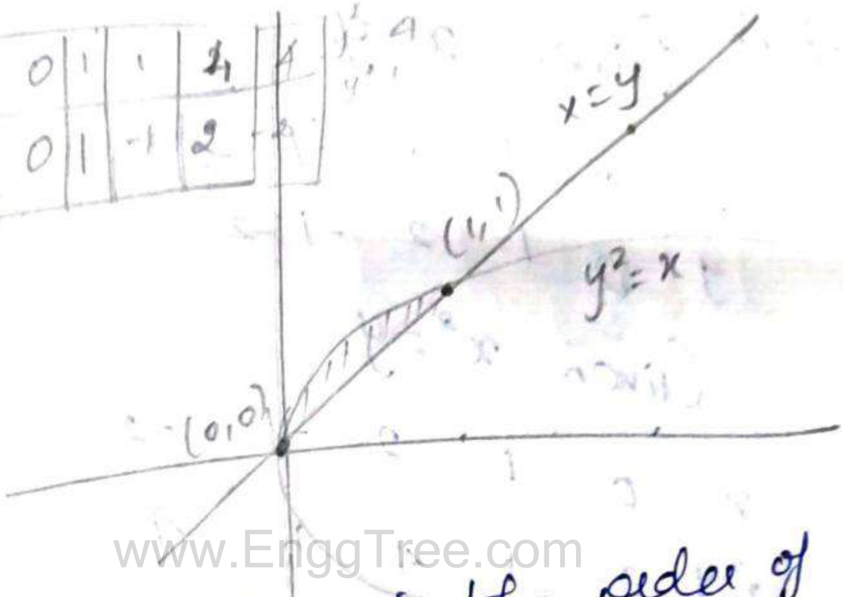
$$= 1.$$

Eg: 4. Change the order of integration in $\int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy$.

Soln: Given the limit of y is $y=0$ to $y=1$

limit of x is $x=y^2$ to $x=y$.

x	0	1	1	4	4
y	0	1	-1	2	-2



After changing the order of integration, the first integration is with respect to y, we have to consider the vertical strip.

x varies from 0 to 1

y varies from $y=x$ to $y=\sqrt{x}$.

$$\int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy = \int_0^1 \int_x^{\sqrt{x}} f(x,y)\,dy\,dx.$$

Finding an Area

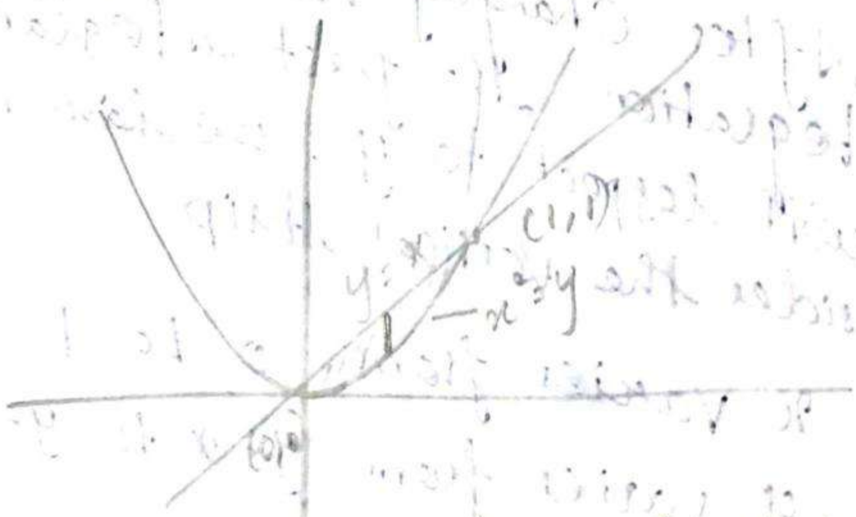
Eg:-1 Evaluate $\iint xy(x+y)\,dx\,dy$ over the region bounded by $x^2=y$ and $y=x$.

Soln: Given $x=y$.

x	0	1	2	-1	-2
y	0	1	2	-1	-2

Given $x^2=y$.

x	0	1	2	-1	-2
$x^2=y$	0	1	4	1	4



x varies from $x=0$ to $x=1$

y varies from $y=x^2$ to $y=x$.

The required area

$$= \int_0^1 \int_{x^2}^{x} xy(x+y)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)\, dy\, dx$$

$$= \int_0^1 \left(x^2 \frac{y^2}{2} + x\frac{y^3}{3} \right)_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left(\frac{5}{6} \frac{x^5}{5} - \frac{x^7}{14} - \frac{x^8}{24} \right)_0^1$$

$$= \left(\frac{1}{6} - \frac{1}{14} - \frac{1}{24} \right)$$

$$\begin{array}{r|l} 6 & 6, 14, 24 \\ \hline 4 & 1, 14, 4 \\ \hline 14 & 1, 14, 1 \\ \hline & 1, 1, 1 \end{array}$$

$$= \left(\frac{56 - 24 - 14}{336} \right)$$

$$= \frac{\cancel{18}^{3}}{\cancel{336}_{56}} = \frac{3}{56} \text{ sq units}$$

<u>Eg: 2</u> Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = x$ by double integration.

<u>Soln:</u> The region, the area of which is required is bounded by the parabolas $(y-0)^2 = -(x-4)$.

and $y^2 = x$ and is shown in the figure

Given $y^2 = 4 - x$

$y = \pm \sqrt{4-x}$.

x	0	2	4
$y = \pm\sqrt{4-x}$	± 2	$\pm\sqrt{2}$	0

Given $y^2 = x$.

$\Rightarrow y = \pm x$.

x	0	1	2
$y = \pm\sqrt{x}$	0	± 1	$\pm\sqrt{2}$

The required area $= 2 \int_0^{\sqrt{2}} \int_{y^2}^{4-y^2} dx\, dy$.

$$\text{Area} = 2\int_0^{\sqrt{2}} (x)_{y^2}^{4-y^2}\, dy$$

$$= 2\int_0^{\sqrt{2}} (4-y^2-y^2)\, dy$$

$$= 2\left(4y - 2\frac{y^3}{3}\right)_0^{\sqrt{2}}$$

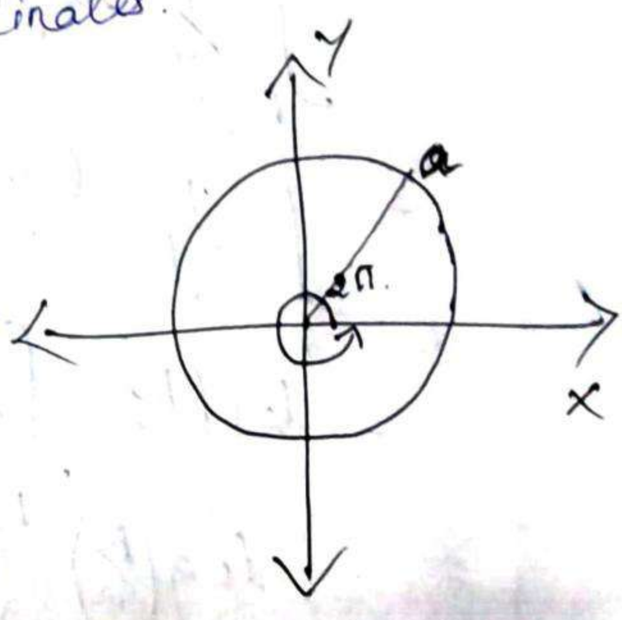
$$= 2\left(4\sqrt{2} - \frac{4\sqrt{2}}{3}\right)$$

$$= 2\left(\frac{2(4\sqrt{2})}{3}\right)$$

$$= \frac{16}{3}\sqrt{2} \text{ Sq. units}$$

Eg:-3 Find the area of a Circle of radius "a" by double integration in polar Coordinates.

Soln:



$$\text{Area} = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2}\right)_0^a d\theta$$

$$= \frac{a^2}{2}(\theta)_0^{2\pi}$$

$$= \frac{2\pi a^2}{2} = \pi a^2.$$

Ex:4 Evaluate $\int_1^a \int_2^b \frac{dx\,dy}{xy}$

Soln:

$$\int_1^a \int_2^b \frac{dx\,dy}{xy} = \int_1^a \left(\frac{-1}{x^2}\right)_2^b \left(\frac{1}{y}\right) dy$$

$$= \int_1^a \left(\left(\frac{-1}{b^2} + \frac{1}{4}\right)\left(\frac{1}{y}\right)\right) dy$$

Ex:4 Evaluate $\int_1^a \int_2^b \left(\frac{dx}{x}\right)\left(\frac{dy}{y}\right)$

Soln:

$$\int_1^a \int_2^b \left(\frac{dx}{x}\right)\left(\frac{dy}{y}\right)$$

$$= \int_1^a \left(\frac{1}{y} \log x\right)_2^b$$

$$= \int_1^a \frac{1}{y} (\log b - \log 2)\, dy$$

$$= \int_1^a \frac{1}{y} \log\left(\frac{b}{2}\right) dy$$

$$= \log\left(\frac{b}{2}\right) \int_1^a \frac{1}{y}\, dy$$

$$= \log\left(\frac{b}{2}\right) (\log y)_1^a$$

$$= \log\left(\frac{b}{2}\right) (\log a - \log(1))$$

$$= \log\left(\frac{b}{2}\right) \log a$$

CHANGE OF VARIABLES BETWEEN CARTESIAN AND POLAR COORDINATES.

Ex:-1 Evaluate by Changing to polar Coordinates $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx dy$.

Soln: The limit of y is from 0 to a

i.e) $y=0$ to $y=a$

The limit of x is from y to a

i.e) $x=y$ to $x=a$.

To Change Cartesian Coordinate to polar Coordinate

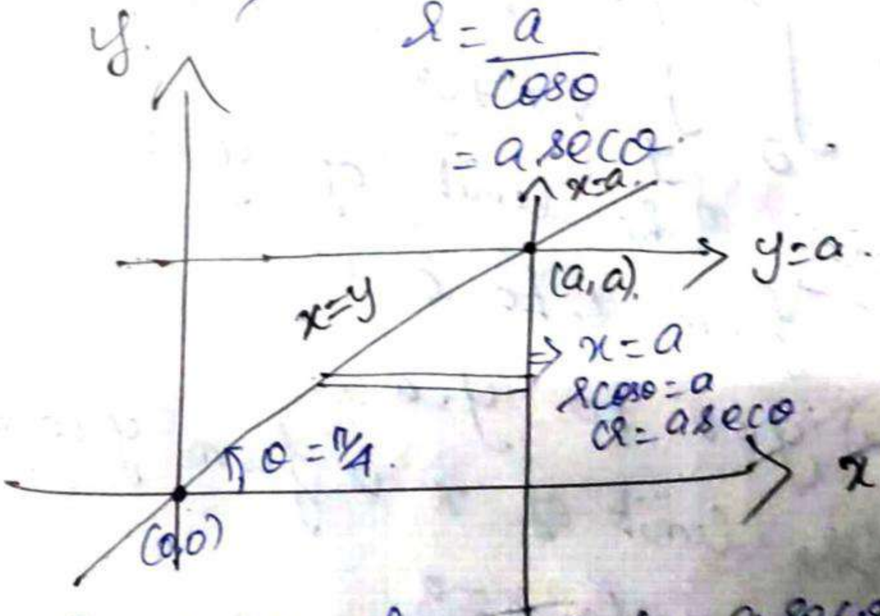
put $x=r\cos\theta$, $y=r\sin\theta$,

$dxdy = r\,dr\,d\theta$

$x=a \Rightarrow r\cos\theta = a$

$r = \frac{a}{\cos\theta}$

$= a\sec\theta$



r varies from 0 to $a\sec\theta$

θ varies from 0 to $\pi/4$.

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos\theta}{r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/4} \left(r\cos\theta\right)_0^{a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} a\sec\theta \times \frac{1}{\sec\theta}\,d\theta$$

$$= a\int_0^{\pi/4} d\theta = a(\theta)_0^{\pi/4}$$

$$= a\frac{\pi}{4}$$

Ex:2 By changing to polar coordinates find the value of $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$.

Soln: The limit of y is from 0 to a

ie) $y=0$ to $y=a$.

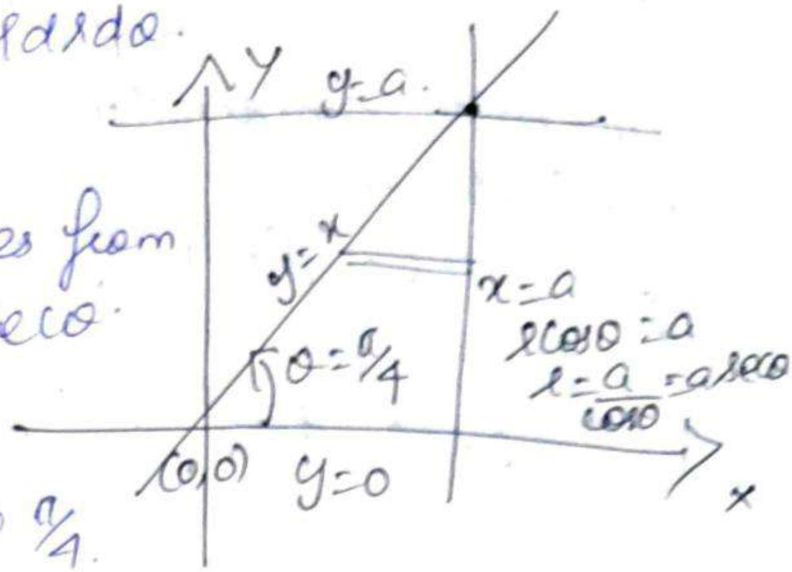
The limit of x is from y to a

ie) $x=y$ to $x=a$.

To change cartesian coordinate to polar coordinate

Put $x = r\cos\theta$, $y = r\sin\theta$,

$dx\,dy = r\,dr\,d\theta$.



r varies from 0 to $a\sec\theta$.

θ varies from 0 to $\pi/4$.

$$\therefore \int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{1/2}}\,dx\,dy$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta\, r\,dr\,d\theta}{r} \qquad \left(\because x^2+y^2 = r^2,\ \sqrt{x^2+y^2} = r\right)$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left(\frac{r^3}{3}\right)_0^{a\sec\theta}(\cos^2\theta)\,d\theta.$$

$$= \int_0^{\pi/4}\frac{a^3\sec^3\theta}{3}\times\frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4}\sec\theta\,d\theta$$

$$= \frac{a^3}{3}\left(\log(\sec\theta+\tan\theta)\right)_0^{\pi/4}$$

$$I = \frac{a^3}{3}\left(\log(\sqrt{2}+1) - 0\right)$$

$$I = \frac{a^3}{3}(\log\sqrt{2}+1).$$

Ex:-3 Express $\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}} dx dy$ in polar coordinates and then evaluate it.

Soln: The limit of y is from 0 to a

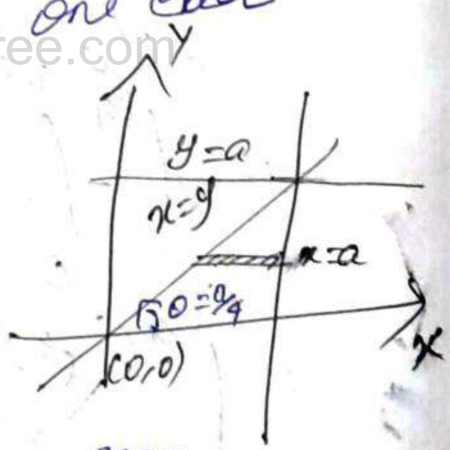
The limit of x is from y to a

To change Cartesian coordinate to polar coordinate put

$x = r\cos\theta,\ y = r\sin\theta,$

$dxdy = r\,dr\,d\theta.$

From the figure one end is at $r=0$.



$x = a$

$\Rightarrow r\cos\theta = a$

$\Rightarrow r = \frac{a}{\cos\theta}.$

r varies from 0 to $a\sec\theta$

θ varies from 0 to $\pi/4$

$$\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}} dx\,dy$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta \cdot r\,dr\,d\theta}{(r^2)^{3/2}}$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} (\cos^2\theta \, dr \, d\theta)$$

$$= \int_0^{\pi/4} (r)_0^{a\sec\theta} \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/4} (a\sec\theta) \times \frac{1}{\sec^2\theta} \, d\theta$$

$$= a\int_0^{\pi/4} \frac{1}{\sec\theta} \, d\theta$$

$$= a\int_0^{\pi/4} \cos\theta \, d\theta$$

$$= a\,(\sin\theta)_0^{\pi/4}$$

$$= a\left(\frac{1}{\sqrt{2}}\right) = \frac{a}{\sqrt{2}}.$$

<u>Ex: 4</u> By changing into polar coordinates evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2} \, dy\,dx$

<u>Soln:</u>

Here $x = 0$ to $x = 2$.

$y = 0$ to $y = \sqrt{2x - x^2}$

$\Rightarrow y^2 = 2x - x^2$ Centre

$(h,k) = (-g,-f)$

$y^2 + x^2 - 2x = 0$. $x^2+y^2+2gx+2fy+c=0$

Radius $= \sqrt{g^2+f^2-c}$ $(-(-1),0)$ $= (1,0)$

$= 1$ $= \sqrt{1+(0)-0}$

To Change Cartesian Coordinates to polar Coordinates

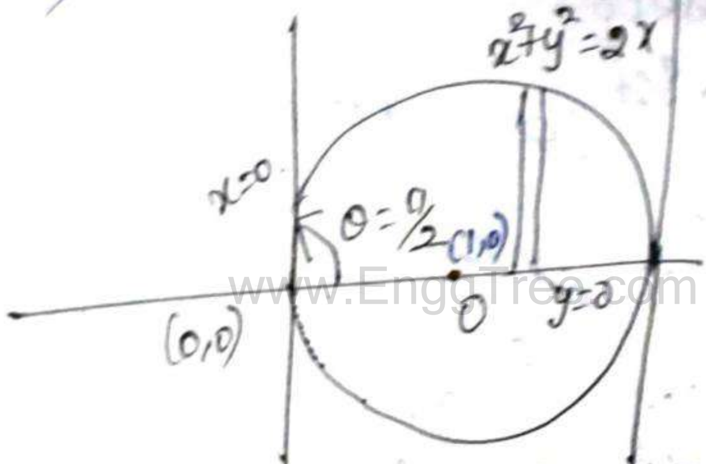
Put $x = r\cos\theta$, $y = r\sin\theta$

In this, the polar equation of the circle is

$r^2\cos^2\theta + r^2\sin^2\theta - 2r\cos\theta = 0$

$r^2(1) = 2r\cos\theta$

$\Rightarrow r = 2\cos\theta$



In polar Coordinates the region of integration is bounded by $r=0$, $r=2\cos\theta$

$\theta = 0$, $\theta = \pi/2$

Then put $x = r\cos\theta$,

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\,dy\,dx$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2} \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \cancel{\int_0^{2\cos\theta}} (\cos\theta)(r)_0^{2\cos\theta} \, d\theta$$

$$= \int_0^{\pi/2} \cancel{\int_0^{2a}} (\cos\theta)(2\cos\theta) \, d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta \, d\theta$$

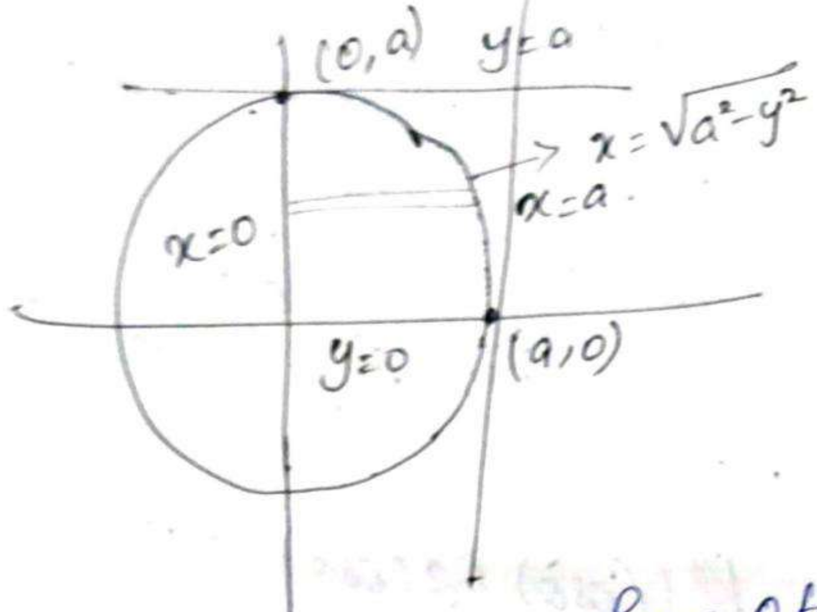
$$= \cancel{2}\left(\frac{\pi}{\cancel{2}} \cdot \frac{1}{2}\right)$$

$$\underline{I} = \pi/2$$

Double integration (Cartesian Coordinate).

Eg:-1 Evaluate $\iint xy \, dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$

Soln: From the diagram, we have the following limits



The limit y varies from 0 to a

x varies from 0 to $\sqrt{a^2-y^2}$

(i) $x=0$ to $x=\sqrt{a^2-y^2}$

$$\iint xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_0^a \left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} y\,dy$$

$$= \int_0^a \left(\frac{a^2-y^2}{2}\, y\right) dy$$

$$= \frac{1}{2}\int_0^a \left(a^2 y - \frac{y^3}{3}\right)_0^a dy$$

$$= \frac{1}{2}\left(a^2\frac{y^2}{2} - \frac{y^4}{4}\right)_0^a$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

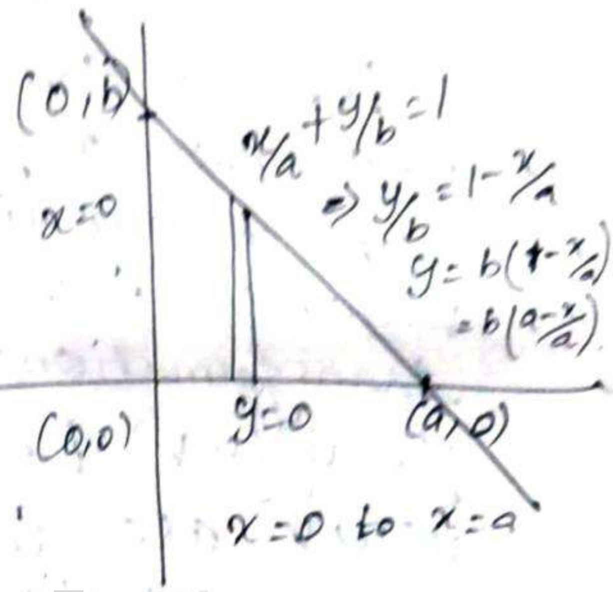
$$= \frac{1}{2}\left(\frac{2a^4-a^4}{4}\right) = \frac{a^4}{8}$$

Ex: 2 Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant bounded by $x/a + y/b = 1$

Soln

$$\iint xy\,dx\,dy$$

$$= \int_0^a \int_0^{b\left(\frac{a-x}{a}\right)} xy\,dy\,dx$$



$$= \int_0^a x\left(\frac{y^2}{2}\right)_0^{b\left(\frac{a-x}{a}\right)} dx$$

$$= \int_0^a x\left(\frac{b^2(a-x)^2}{a^2\,2}\right) dx$$

$$= \frac{b^2}{a^2} \times \frac{1}{2} \int_0^a (a^2 + x^2 - 2ax)\,x\,dx$$

$$= \frac{1}{2}\frac{b^2}{a^2} \int_0^a (a^2x + x^3 - 2ax^2)\,dx$$

$$= \frac{1}{2}\frac{b^2}{a^2} \left(\frac{a^2x^2}{2} + \frac{x^4}{4} - \frac{2ax^3}{3}\right)_0^a$$

$$= \frac{1}{2}\frac{b^2}{a^2} \left(\frac{a^4}{2} + \frac{a^4}{4} - \frac{2a^4}{3}\right)$$

$$= \frac{1}{2}\left(\frac{b^2}{a^2}\right)\left(\frac{6a^4 + 3a^4 - 8a^4}{12}\right) = \frac{b^2}{2a^2}\left(\frac{a^4}{12}\right)$$

$$= \frac{b^2a^2}{24}$$

Change of Variables in double integration

Let us Consider the integral $I = \iint_{R_{xy}} f(x,y)\, dx\, dy$.

Let us Consider the Variables x, y be changed into newly introduced variables u, v by the standard transformation $x = \phi(u,v)$, $y = \psi(u,v)$. Here $\phi(u,v)$ and $\psi(u,v)$ are Continuous and have derivatives in R'_{UV} in the UV-plane corresponding to R_{xy} in the XY-plane.

$$\therefore \iint_{R_{XY}} f(x,y)\, dx\, dy = \iint_{R'_{UV}} f\left(\phi(u,v), \psi(u,v)\right) |J|\, du\, dv$$

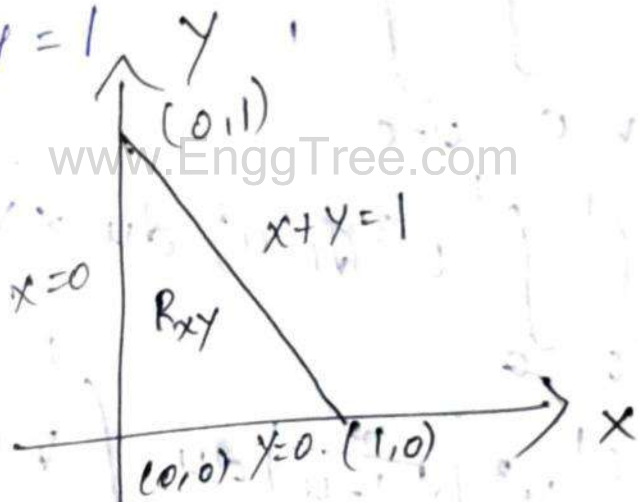
Where $J = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$.

Note: As we know, the Cartesian Co-ordinates (x,y) are Changed into polar Coordinates (r,θ) by introducing the transformation $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

$$\therefore \iint\limits_{R_{xy}} f(x,y)\,dx\,dy = \iint\limits_{R'_{r,\theta}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Eg:-1 Evaluate $\iint\limits_{D} xy\sqrt{1-x-y}\,dx\,dy$ where D is the region bounded by $x=0$, $y=0$ and $x+y=1$ using the transformation $x+y=u$, $y=uv$

Soln: It is given that D is the region bounded by $x=0$, $y=0$ and $x+y=1$



The region of the integration is bounded by $(0,0)$, $(1,0)$ and $(0,1)$

The given transformation is

$x+y=u$ and $y=uv$

At $y=0$, $uv=0 \Rightarrow u=0$ or $v=0$

At $x=0$, $u=y$ (or) $v=1$

At $x+y=1$, $u=1$

Now

$x = u - y = u - uv$

$y = uv$

$$J\left(\frac{x, y}{u, v}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\therefore I = \int_{v=0}^{1} \int_{u=0}^{1} u(1-v)\, uv\, (\sqrt{1-u})\, |J|\, du\, dv$$

$$= \int_0^1 \int_0^1 u^3 (\sqrt{1-u})\, du \times v(1-v)\, dv$$

$$= \int_0^1 u^3 (\sqrt{1-u})\, du \int_0^1 v(1-v)\, dv$$

let us consider $1-u = t$

$\Rightarrow -du = dt$

$1-u = t$

u	t
0	1
1	0

$$I = \int_1^0 (1-t)^3 \sqrt{t}\, (-dt) \int_0^1 v(1-v)\, dv$$

$= \int_0^1 (1-t)^3\, t^{1/2}\, dt \qquad \int_0^1 (v - v^2)\, dv$

$= \int_0^1 t^{1/2}(1 - 3t + 3t^2 - t^3)\, dt \qquad \left(\frac{v^2}{2} - \frac{v^3}{3}\right)_0^1$

$= \int_0^1 \left(t^{1/2} - 3t^{3/2} + 3t^{5/2} - t^{7/2}\right) dt \qquad \left(\frac{1}{2} - \frac{1}{3}\right)$

$= \left(\frac{t^{1/2+1}}{1/2+1} - \frac{3t^{3/2+1}}{3/2+1} + \frac{3t^{5/2+1}}{5/2+1} - \frac{t^{7/2+1}}{7/2+1}\right)_0^1$

$= \left(\frac{1}{3/2} - \frac{3}{5/2} + \frac{3}{7/2} - \frac{1}{9/2}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$

$= \left(\frac{2}{3} - \frac{6}{5} + \frac{6}{7} - \frac{2}{9}\right)\left(\frac{1}{6}\right)$

$= \left(\frac{630 - \overset{1134+810}{} - 210}{945}\right)\left(\frac{1}{6}\right)$

$= \left(\frac{96}{945}\right)\left(\frac{1}{6}\right)$

$= \frac{16}{945}$

Triple Integration

Ex:-1 Evaluate $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} xy\,dx\,dy\,dz$

Soln:-

$$\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} xy\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{2} xy\,(z)_1^2\,dx\,dy$$

$$= \int_0^1\int_0^2 (2-1)\,xy\,dx\,dy$$

$$= \int_0^1\int_0^2 xy\,dx\,dy$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^2 dx$$

$$= \int_0^1 x\,(2)\,dx$$

$$= \int_0^1 2x\,dx$$

$$= 2\left(\frac{x^2}{2}\right)_0^1$$

$$= (1-0)$$

$$= 1.$$

Ex:-2 $\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^4 \sin\varphi \, dr \, d\varphi \, d\theta$

Soln:

$$I = \int_0^{2\pi}\int_0^{\pi} \sin\varphi \left(\frac{r^5}{5}\right) d\varphi \, d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi} (-\cos\varphi)_0^{\pi} \, d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi} (-\cos\pi + \cos 0) \, d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi} (-(-1)+1) \, d\theta$$

$$= \frac{2a^5}{5}\int_0^{2\pi} d\theta$$

$$= \frac{2a^5}{5}(\theta)_0^{2\pi}$$

$$= \frac{4a^5\pi}{5}$$

Ex:-3 Evaluate $\iiint xyz \, dx \, dy \, dz$ over the first octant of $x^2+y^2+z^2=a^2$.

Soln:

From the given equation $x^2+y^2+z^2$

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a.

$$\iiint xyz\,dx\,dy\,dz$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\cdot(a^2-x^2-y^2)\,dy\,dx.$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (xya^2 - x^3y - xy^3)\,dy\,dx.$$

$$= \frac{1}{2}\int_0^a \left(\frac{xy^2a^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{x}{2}(a^2-x^2)a^2 - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4}\right) dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{x}{2}(a^2-x^2)(a^2-x^2) - \frac{x(a^2-x^2)^2}{4}\right) dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{x}{4}(a^2-x^2)^2\right) dx.$$

$$= \frac{1}{8}\int_0^a x(a^4+x^4-2a^2x^2)\,dx$$

$$= \frac{1}{8}\int_0^a (a^4x+x^5-2a^2x^3)\,dx$$

$$= \frac{1}{8}\left(\frac{a^4x^2}{2}+\frac{x^6}{6}-\frac{2a^2x^4}{4}\right)_0^a$$

$$= \frac{1}{8}\left(\frac{a^6}{2}+\frac{a^6}{6}-\frac{2a^6}{4}\right)$$

$$= \frac{1}{8}\left(\frac{6a^6+2a^6-6a^6}{\cancel{48}\,12}\right)$$

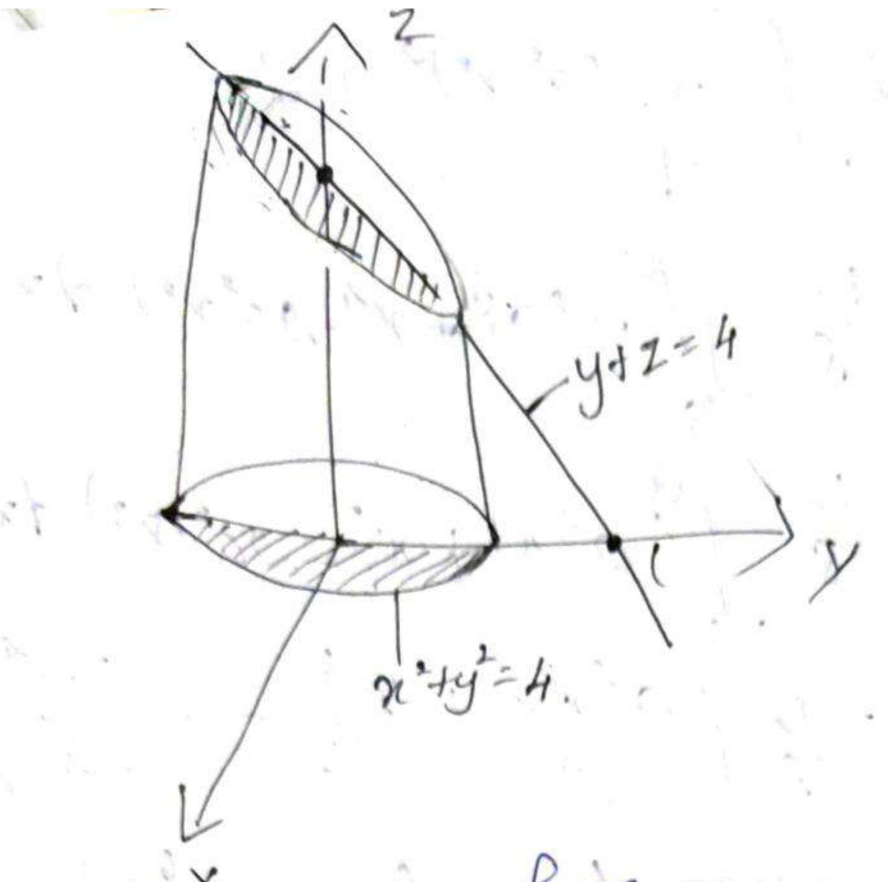
$$\begin{array}{r|l} 2 & 2,6,4 \\ 2 & 1,3,2 \\ 3 & 1,3,1 \\ & 1,1,1 \end{array}$$

$$= \frac{2a^6}{(8)(12)} = \frac{a^6}{(6)(8)}$$

$$= \frac{a^6}{48}.$$

Ex:-4 Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

Soln: let $V = \iiint_V dz\,dy\,dx.$



Here z varies from

$z=0$ to $z=4-y$

x varies from $x=-2$ to $x=2$

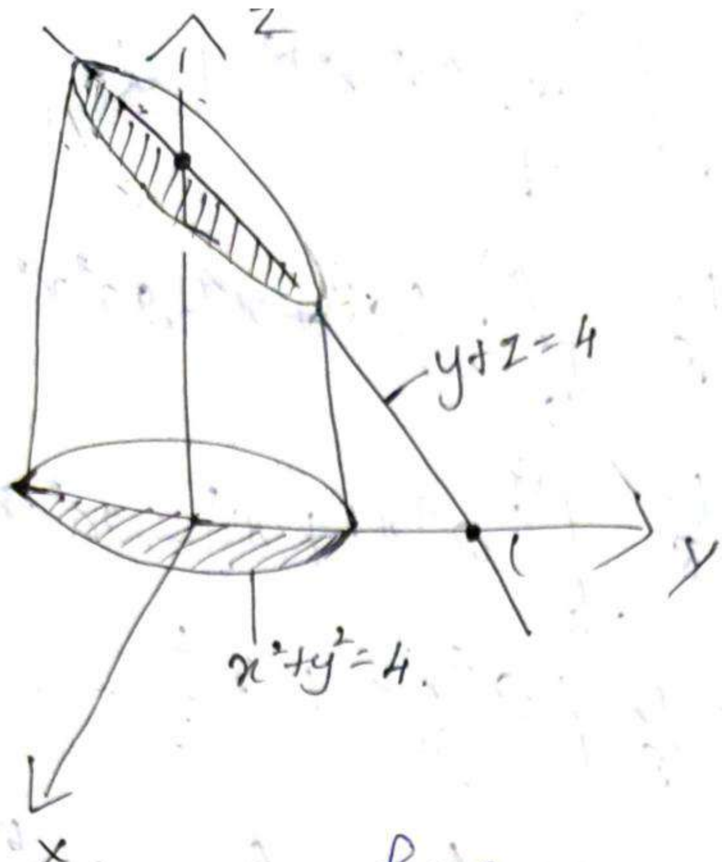
y varies from $y=-\sqrt{4-x^2}$ to $y=\sqrt{4-x^2}$.

$$V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(z)_0^{4-y}\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\,dx$$

$$=\int_{-2}^{2}\left(4y-\frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$



Here z varies from

$z = 0$ to $z = 4 - y$

x varies from $x = -2$ to $x = 2$

y varies from $y = -\sqrt{4-x^2}$ to

$y = \sqrt{4-x^2}$.

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-y} dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (z)_0^{4-y}\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy\,dx.$$

$$= \int_{-2}^{2} \left(4y - \frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left(4\left(\sqrt{4-x^2} - 4 - \frac{x^2}{2}\right) - \left(-4\sqrt{4-x^2} - 4 - \frac{x^2}{2}\right)\right) dx$$

$$= \int_{-2}^{2} \left(4\sqrt{4-x^2} + 4\sqrt{4-x^2}\right) dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx$$

$$= 8\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 8\times 2\int_{0}^{2} \sqrt{4-x^2}\, dx.$$

$$= 16\left(\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2}\right)_0^2$$

$$= 16\left(\frac{4}{2}\sin^{-1}(1) + \frac{1}{2}\sqrt{4-4}\right)$$

$$= 16\left(\frac{2\pi}{2}\right) = 16\pi \text{ Cubic units.}$$

Note:- 1. Cylindrical and rectangular Cartesian Coordinates are related by

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dx\,dy\,dz = r\,dz\,dr\,d\theta$

Then $\iiint_V f(x,y,z)\,dx\,dy\,dz$

$$= \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

Ex:-1 By transforming into Cylindrical Coordinates, evaluate the integral $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the region of space defined by $x^2+y^2 \le 1$ and $0 \le z \le 1$.

Soln: Here the region of space is enclosed by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$.

The radius of the cylinder is 1.

Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

we have

(i) $x^2+y^2=1$ becomes

$$r^2\cos^2\theta + r^2\sin^2\theta = 1$$

$$r = \pm 1$$

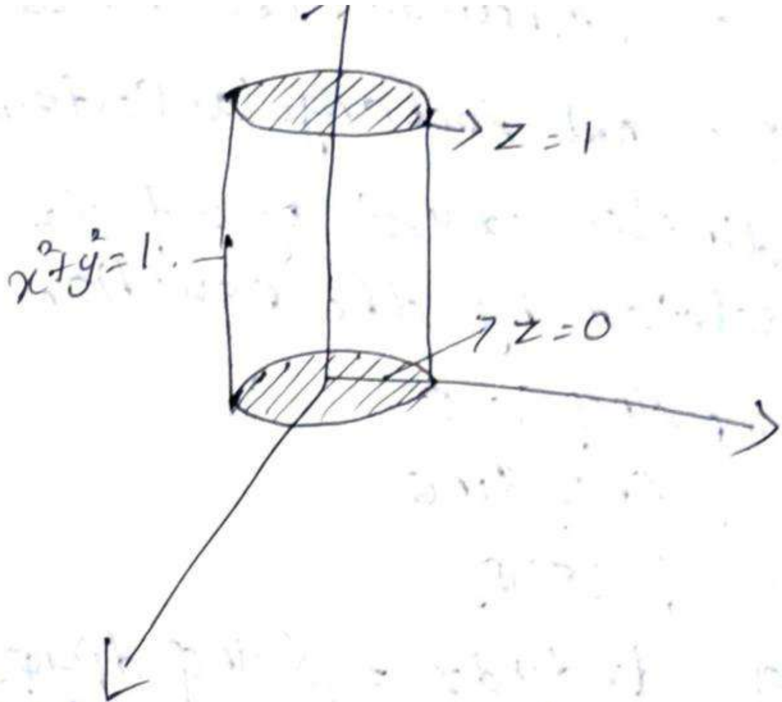
(ii) $dx\,dy\,dz = r\,dr\,d\theta\,dz$.

The limits are as follows

$r \Rightarrow \quad r=0$ to $r=1$

$\theta \Rightarrow \quad \theta=0$ to $\theta=2\pi$

$z \Rightarrow \quad z=0$ to $z=1$



$$\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_0^1 (r^2+z^2)\,r\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\int_0^1 (r^3+rz^2)\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\left(\frac{r^4}{4}+\frac{z^2r^2}{2}\right)_0^1 d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\left(\frac{1}{4}+\frac{z^2}{2}\right) d\theta\,dz$$

$$= \int_0^1\left(\frac{\theta}{4}+\frac{z^2\theta}{2}\right)_0^{2\pi} dz.$$

$$= \int_0^1\left(\frac{\cancel{2}\pi}{\cancel{4}_2}+\frac{\cancel{2}\pi z^2}{\cancel{2}}\right) dz$$

$$= \frac{\pi}{2}\int_0^1 (1+2z^2)\,dz$$

$$= \frac{\pi}{2}\left(z+\frac{2z^3}{3}\right)_0^1 = \frac{\pi}{2}\left(1+\frac{2}{3}\right) = \frac{\pi}{2}\left(\frac{5}{3}\right) = \frac{5\pi}{6}.$$

Note:-2 Spherical Coordinates (ρ, φ, θ) and rectangular Cartesian Coordinates (x, y, z) of a point P are related by the equation

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

and $dx\,dy\,dz = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

Suppose if we want to evaluate $\iiint_V f(x, y, z)\, dV$ where V is a bounded region then

$$\iiint_V f(x,y,z)\, dx\,dy\,dz = \iiint_V f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Here $dV = dx\,dy\,dz$

$= \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta.$

Example:-1 Evaluate the integration $\iiint xyz\, dx\,dy\,dz$ taken throughout the volume for which $x, y, z \geq 0$ and $x^2+y^2+z^2 \leq 9$.

Soln: We shall use the spherical polar coordinates to evaluate is

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\varphi$.

$$\iiint xyz\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 r^5\sin^3\theta\cos\theta\sin\varphi\cos\varphi\,dr\,d\theta\,d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{r^6}{6}\right)_0^3 \sin^3\theta\cos\theta\left(\frac{\sin 2\varphi}{2}\right)d\theta\,d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{729}{6}\right)\sin^3\theta\cos\theta\left(\frac{\sin 2\varphi}{2}\right)d\theta\,d\varphi$$

$$= \frac{729}{12}\left(\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\right)\left(\int_0^{\pi/2}\sin 2\varphi\,d\varphi\right)$$

$$= \frac{729}{12}\left(\int_0^1 t^3\,dt\right)\left(-\frac{\cos 2\varphi}{2}\right)_0^{\pi/2}$$

Put $t = \sin\theta$

$dt = \cos\theta\,d\theta$

$\theta = 0 \Rightarrow t = 0$

$\theta = \pi/2 \Rightarrow t = 1$

$$= \frac{729}{12}\left(\frac{t^4}{4}\right)_0^1\left(-\frac{1}{2}\right)\left(\cos\pi - \cos 0\right)$$

$= \frac{729}{12}\left(\frac{1}{4}\right)\left(\frac{-1}{2}\right)(-1-1)$

$= \frac{\cancel{729}^{243}}{\cancel{12}_{4} \times 4 \times \cancel{2}} \times \cancel{2}$

$= \frac{243}{16}$

Ex: 2 Find the Volume of $\iiint xyz\, dz\, dy\, dx$ through the positive spherical octant for which $x^2+y^2+z^2 \leq a^2$

Soln: Let us transform this integral in spherical polar coordinates by using

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\varphi$

$x^2+y^2+z^2 = a^2$

limits of r, $r=0$ to $r=a$

limits of $\theta \Rightarrow 0$ to $\pi/2$

limits of $\varphi \Rightarrow \varphi=0$ to $\varphi=\pi/2$

$$V = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^5 \sin^3\theta \cos\theta \sin\varphi \cos\varphi \, dr\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left(\frac{r^6}{6}\right)_0^a \sin^3\theta \cos\theta \left(\frac{\sin 2\varphi}{2}\right) d\varphi\, d\theta$$

$$= \frac{a^6}{2\times 6}\int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta \int_0^{\pi/2} \sin 2\varphi \, d\varphi.$$

$$= \frac{a^6}{12}\int_0^1 t^3\, dt \int_0^{\pi/2} \sin 2\varphi \, d\varphi$$

put $t = \sin\theta$
$dt = \cos\theta \, d\theta$

$\theta = 0 \Rightarrow t = 0$
$\theta = \pi/2 \Rightarrow t = 1$

$$= \frac{a^6}{12}\left(\frac{t^4}{4}\right)_0^1 \left(-\frac{\cos 2\varphi}{2}\right)_0^{\pi/2}$$

$$= \frac{a^6}{12}\left(\frac{1}{4}-0\right)\left(-\frac{1}{2}\right)\left(\cos\pi - \cos 0\right).$$

$$= \frac{a^6}{12}\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right)\left(-1-1\right)$$

$$= \frac{a^6}{12\times 4}\frac{(\cancel{2})}{\cancel{2}}$$

$$V = \frac{a^6}{48} = \frac{a^6}{48}.$$

Ex 1.3 Evaluate $\iiint_V dx\,dy\,dz$

where V is the finite region of space formed by the planes $x=0$, $y=0$, $z=0$ and $x/a + y/b + z/c = 1$

Soln:

z varies from

$z=0$ to $z = c\left(1 - x/a - y/b\right)$.

y varies from

$y=0$ to $y = b\left(1 - x/a\right)$

x varies from

$x=0$ to $x=a$.

$$\iiint_V dx\,dy\,dz = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\,dy\,dx$$

$$= c\int_0^a \int_0^{b(1-x/a)} \left(1 - \frac{x}{a} - y/b\right) dy\,dx$$

$$= c\int_0^a \left[y - \frac{xy}{a} - y^2/2b \right)_0^{b(1-x/a)} dx$$

$$= c\int_0^a \left(b\left(1-x/a\right) - \frac{xb\left(1-x/a\right)}{a} - \frac{b^2\left(1-x/a\right)^2}{2b} \right) dx$$

$$= c\int_0^a b - \frac{bx}{a} - \frac{xb}{a} + \frac{x^2b}{a^2} - \frac{b^2}{2b}\left(1 + \frac{x^2}{a^2} - \frac{2x}{a}\right)dx$$

$$= c\int_0^a \left(b - \cancel{\frac{bx}{a}} - \frac{xb}{a} + \frac{x^2b}{a^2} - \frac{b}{2} - \frac{x^2b}{2a^2} + \cancel{\frac{xb}{a}}\right)dx$$

$$= c\int_0^a \left(\frac{b}{2} - \frac{bx}{a} + \frac{x^2b}{2a^2}\right)dx$$

$$= c\left(\frac{bx}{2} - \frac{bx^2}{2a} + \frac{x^3b}{6a^2}\right)_0^a$$

$$= c\left(\frac{ab}{2} - \frac{ab}{2} + \frac{ab}{6}\right)$$

$$= c\left(\frac{3ab - 3ab + ab}{6}\right)$$

$$= \frac{abc}{6}$$

$$\iiint dx\,dy\,dz = \frac{abc}{6}$$

Applications of Multiple Integrals

Moments and Centre of mass.

The moment about x-axis is given by

$$M_x = \iint_D y\, \rho(x,y)\, dA.$$

Similarly, the moment about the y-axis is given by

$$M_y = \iint_D x\, \rho(x,y)\, dA$$

Where D is the corresponding region and $\rho(x,y)$ is the density function.

Centre of mass $(\bar{x}, \bar{y})$

Where $\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x.\rho(x,y)\, dA$

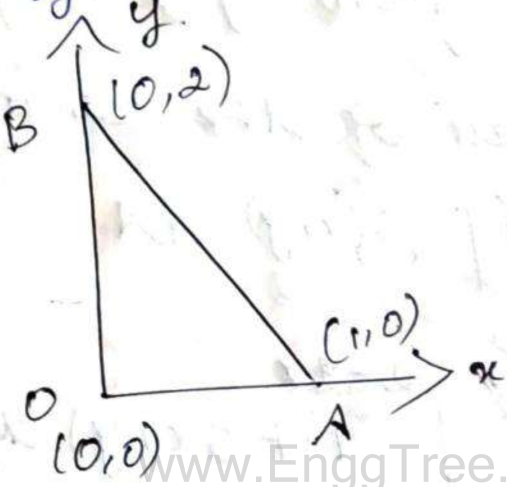
$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y.\, \rho(x,y)\, dA$$

The mass m is given by

$$m = \iint_D \rho(x,y)\, dA$$

Ex:1 Find the mass and centre of mass of a triangular lamina with vertices (0,0), (1,0), (0,2) if the density function is $\rho(x,y) = 1+3x+y$.

Soln: First let us find the mass of the lamina.



$$m = \iint_D \rho(x,y)\, dA$$

The triangle is bounded by the vertices (0,0), (1,0) and (0,2), from the above diagram, the equation of AB is obtained by using the formula

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

$$\Rightarrow \frac{x-1}{0-1} = \frac{y-0}{2-0}$$

$$\Rightarrow x-1 = -\left(\frac{y}{2}\right)$$

$-2(x-1)=y$

$\Rightarrow -2x+2=y$

$\Rightarrow y=2-2x.$

$\therefore$ In the region, x varies from from $x=0$ to $x=1$ and y varies from $y=0$ to $y=2-2x$.

The mass of the lamina is

$$m=\iint_D \rho(x,y)\,dA$$

$$=\int_0^1\int_0^{2-2x}(1+3x+y)\,dy\,dx$$

$$=\int_0^1\left(y+3xy+\frac{y^2}{2}\right)_0^{2-2x}dx$$

$$=\int_0^1\left((2-2x)+3x(2-2x)+\frac{(2-2x)^2}{2}\right)dx$$

$$=\int_0^1 2-2x+6x-6x^2+\frac{1}{2}(4+4x^2-8x)\,dx$$

$$=\int_0^1(2-\cancel{2x}+\cancel{6x}-6x^2+2+2x^2-\cancel{4x})\,dx$$

$$=\int_0^1(4-4x^2)\,dx$$

$$= \left[-\frac{4x^3}{3} + 4x\right]_0^1$$

$$= -\frac{4}{3} + 4$$

$$m = \frac{8}{3}$$

$$\bar{x} = \frac{1}{m} \iint_D x \cdot \rho(x,y)\, dA$$

$$= \frac{3}{8} \int_0^1 \int_0^{2-2x} x(1+3x+y)\, dy\, dx$$

$$= \frac{3}{8} \int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx.$$

$$= \frac{3}{8} \int_0^1 \left(xy + 3x^2 y + \frac{xy^2}{2}\right)_0^{2-2x} dx$$

$$= \frac{3}{8} \int_0^1 \left(x(2-2x) + 3x^2(2-2x) + \frac{x}{2}(2-2x)^2\right) dx$$

$$= \frac{3}{8} \int_0^1 \left(2x - 2x^2 + 6x^2 - 6x^3 + \frac{1}{2}(4x + 4x^3 - 8x^2)\right) dx$$

$$= \frac{3}{8} \int_0^1 \left(2x - \cancel{2x^2} + \cancel{6x^2} - 6x^3 + 2x + 2x^3 - \cancel{4x^2}\right) dx$$

$$= \frac{3}{8} \int_0^1 (4x - 4x^3)\, dx$$

$$= \frac{3}{8} \left(\frac{4x^2}{2} - \frac{4x^4}{4}\right)_0^1$$

$$= \frac{3}{8}(+2-1) = \frac{3}{8}$$

$$\bar{x} = \frac{3}{8}$$

We know that:

$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

$$\bar{y} = \frac{3}{8}\int_0^1\int_0^{2-2x} y(1+3x+y)\,dy\,dx$$

$$= \frac{3}{8}\int_0^1\int_0^{2-2x}(y+3xy+y^2)\,dy\,dx$$

$$= \frac{3}{8}\int_0^1\left(\frac{y^2}{2}+3x\frac{y^2}{2}+\frac{y^3}{3}\right)_0^{2-2x} dx$$

$$= \frac{3}{8}\int_0^1\left(\frac{(2-2x)^2}{2}+\frac{3x}{2}(2-2x)^2 + \frac{(2-2x)^3}{3}\right)dx$$

$$= \frac{3}{8\times 6}\int_0^1 3(4+4x^2-8x) + 9x(4+4x^2-8x) + 2(8-24x+24x^2-8x^3)\,dx$$

$$= \frac{3}{8\times 6}\int_0^1 (12+12x^2-24x+36x+36x^3-72x^2 + 2(8-24x+24x^2-8x^3))\,dx$$

$$= \frac{1}{16}\int_0^1 (12+12x^2-24x+36x+36x^3-72x^2 + 16-48x+48x^2-16x^3)\,dx$$

$$= \frac{1}{16}\int_0^1 \left(28-12x^2-36x+20x^3\right)dx$$

$$= \frac{1}{16}\int_0^1 (28 - 12x^2 - 36x - 20x^3)\,dx$$

$$= \frac{1}{16}\left(28x - \frac{12x^3}{3} - \frac{36x^2}{2} + \frac{20x^4}{4}\right)_0^1$$

$$= \frac{1}{16}\left(28 - \frac{12}{3} - \frac{36}{2} + \frac{20}{4}\right)$$

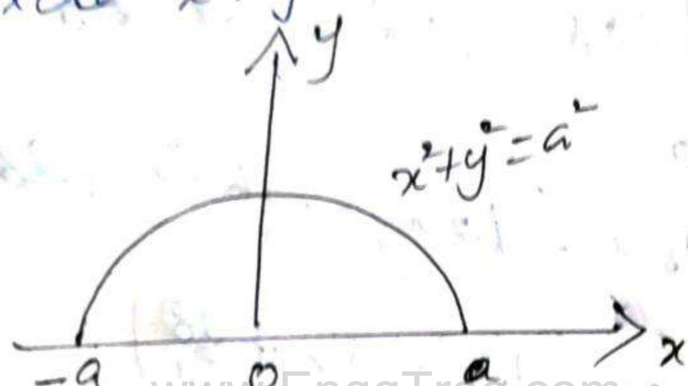
$$= \frac{1}{16}\left(28 - 4 - 18 + 5\right)$$

$$= \frac{1}{16}(10+1) = \frac{11}{16}.$$

The Centre of mass at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$

Ex:2 The density at any point on a semicircular lamina is proportional to the distance from the centre of the circle. Find the centre of mass of the lamina

Soln: Let us consider the lamina as the upper half of the circle $x^2+y^2=a^2$



Then the distance from a point (x,y) to the centre of the circle is $\sqrt{x^2+y^2}$.

Therefore, the density function is $\rho(x,y) = C\sqrt{x^2+y^2}$, where C is a constant.

Convert into polar coordinates.

The radius r varies from $r=0$ to $r=a$

and the angle θ varies from $\theta=0$ to $\theta=\pi$.

The mass of the lamina is.

$$m = \iint_D \rho(x,y)\, dA$$

$$= \iint_D c\sqrt{x^2+y^2}\, dA$$

$$= \int_0^{\pi}\int_0^{a} (cr)\, r\, dr\, d\theta$$

$$= c\int_0^{\pi} d\theta \int_0^{a} r^2\, dr.$$

$$= c(\theta)_0^{\pi} \left(\frac{r^3}{3}\right)_0^{a}$$

$$= \cancel{c(\theta)}\ c(\pi)\frac{a^3}{3} = \frac{c\pi a^3}{3}.$$

Here both the lamina and the density function are symmetric with respect to the y-axis, so the centre of mass must lie on the y-axis. ie) $\bar{x} = 0$.

$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\, dA$$

$$= \frac{3}{c\pi a^3}\int_0^{\pi}\int_0^{a} r\sin\theta\,(cr)\, r\, dr\, d\theta$$

$$= \frac{3}{\pi a^3}\int_0^{\pi}\sin\theta\, d\theta \int_0^{a} r^3\, dr$$

$$= \frac{3}{\pi a^3} \left(-\cos\theta\right)_0^{\pi} \left(\frac{r^4}{4}\right)_0^{a}$$

$$= \frac{3}{\pi a^3} \left(-\cos\pi - (\cos 0)\right) \left(\frac{1}{4} a^4\right)$$

$$= \frac{3}{\pi a^3} \left(-(-1) - (-1)\right) \left(\frac{1}{4} a^4\right)$$

$$= \frac{3}{\pi a^3} (2) \left(\frac{1}{4} a^4\right)$$

$$= \frac{3a}{\pi 2}.$$

$\therefore$ The Centre of the mass is located at the point $\left(0, \frac{3a}{2\pi}\right)$.

Moments of Inertia

The moment of inertia (or second moment) of a particle of mass m about an axis is defined as mr^2. Here r is the distance from the particle to the axis. Now we extend this concept to a lamina with density function $\rho(x,y)$ and occupying a region D. We subdivide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large

The moment of inertia of the lamina about the x-axis is given by $I_x = \iint_D y^2 \rho(x,y)\, dA$

The moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2 \rho(x,y)\, dA$$

The moment of inertia about the origin which is also called as polar moment of inertia given by $I_0 = \iint_D (x^2+y^2)\rho(x,y)\, dA$

Here $I_0 = I_x + I_y$

Ex:-1 Find the moments of inertia I_x, I_y, I_0 of a homogeneous disk D with the density $\rho(x,y) = C$, centre the origin and radius a.

Soln: The boundary of D is the circle $x^2+y^2=a^2$,

In polar coordinates the given region D is bounded by $0 \le \theta \le 2\pi$, $0 \le r \le a$.

We know that the moment of Inertia about the origin is

$$I_0 = \iint_D (x^2+y^2)\, \rho(x,y)\, dA$$

$$I_0 = \int_0^{2\pi} \int_0^a r^2 \, C \, r \, dr \, d\theta$$

$$= C \int_0^{2\pi} d\theta \int_0^a r^3 \, dr.$$

$$= C \left(\theta\right)_0^{2\pi} \left(\frac{r^4}{4}\right)_0^a$$

$$= C(2\pi)\left(\frac{a^4}{4}\right).$$

$$I_0 = \frac{C\pi a^4}{2}.$$

Instead of computing I_x and I_y directly, we use the relation

$I_x + I_y = I_o$.

and $I_x = I_y$

Because from the symmetry of the problem

$$I_x = I_y = \frac{I_o}{2} = \frac{C\pi a^4}{2/2} = \frac{C\pi a^4}{4}.$$

The mass of the disk is m

$= \text{density} \times \text{area}$

$= C \times \pi a^2$

$= C\pi a^2$

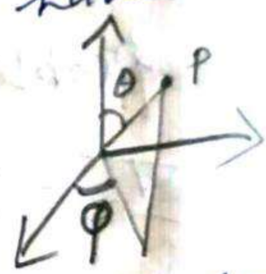
The moment of inertia of the disk about the origin can be written as

$$I_o = \frac{C\pi a^4}{2} = \frac{1}{2}(C\pi a^2)a^2$$

$$I_o = \frac{1}{2}ma^2.$$

Ex:- Find the Volume of the Sphere $x^2+y^2+z^2=a^2$ by transforming into spherical polar Coordinates

Soln: In spherical polar Coordinate system, we have



$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$dx\,dy\,dz = \cancel{\rho^2 \sin}\; r^2\sin\theta\, dr\, d\theta\, d\varphi$

Limits are as follows

r	$r=0$	$r=a$
θ	$\theta=0$	$\theta=\pi$
φ	$\varphi=0$	$\varphi=2\pi$

Required Volume

$$= \iiint_V dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi}\sin\theta\, d\theta \int_0^{a} r^2\, dr$$

$$= (\phi)_0^{2\pi} (-\cos\theta)_0^{\pi} \left(\frac{r^3}{3}\right)_0^a$$

$$= 2\pi (2) \left(\frac{a^3}{3}\right)$$

$$= \frac{4\pi a^3}{3}$$