unit I matrices.

Matrix was Introduced by James Joseph. Sylvester. in 1850.

A matrix is defined as a rectangular array arranged by number of rows. and Columns. If mn numbers or functions are arranged in the form of a rectangular array A. having m rows and n Columns A is Called an $m \times n$ matrix. Each of the mn numbers is Called an element of the matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix}_{m \times n}.$$

A be a square matrix of order n.

(a) If $A = A'$ then the matrix is said to be a symmetric matrix. Here $a_{ij} = a_{ji}\ \forall i,j$

(b) If $A = -A'$ then the matrix is said to be a skew symmetric matrix. Here $a_{ij} = -a_{ji}$ for all i,j.

(c) Any matrix of order $m \times 1$ is Called Column matrix eg: $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}_{3 \times 1}$

(d) Any matrix of order $1 \times n$ is Called row matrix. eg: $(1\ -1\ 2)_{1 \times 3}$

(e) In a square matrix $A = (a_{ij})$ the elements $a_{11}, a_{22}, \ldots a_{nn}$

are Called diagonal elements. It will form principal diagonal or leading diagonal (or) main diagonal of that matrix A.

f. If all elements except principal diagonal elements are zero, then the matrix is called diagonal matrix ex. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

g. In the diagonal matrix all the elements of principal diagonal is 1 then it is called unit matrix ex: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

h. In a square matrix if all the elements below the principal diagonal are zero, then the matrix is upper triangular matrix eg $\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 4 \end{pmatrix}$

i In a square matrix if all the elements above the principal diagonal are zero. then the matrix is called lower triangular matrix eg: $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$

j. A matrix is singular if $|A| = 0$. Otherwise A is called non singular ($|A| \neq 0$).

k. Inverse exist only for non singular matrices.

A row vector is of the form

$X = [x_1, x_2, x_3 \ldots x_n]_{1 \times n}$

Column vector is of the form.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix}_{n\times 1}.$$

Eigen values and Eigen vectors.

Let $A = (a_{ij})$ be a square matrix of order n. If there exists a non zero column vector X and a scalar λ. such that $AX = \lambda X$. Then λ is called an eigen value of the matrix A. and X is called eigen vector corresponding to the eigen value λ.

Let λ be an eigen value and X be the corresponding eigen vector.

Then $AX = \lambda I X$ where I is unit matrix of order n.

$$(A - \lambda I) X = 0$$

$$\left[\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

$$(a_{11} - \lambda) x_1 + a_{12} x_2 \cdots + a_{1n} x_n = 0.$$
$$a_{21} x_1 + (a_{22} - \lambda) x_2 + \cdots + a_{2n} x_n = 0 \quad \text{——} (1)$$
$$\cdots\cdots$$
$$a_{n1} x_1 + a_{n2} x_2 + \cdots + (a_{nn} - \lambda) x_n = 0$$

This is known as system of linear equation. with $x_1, x_2 \ldots x_n$ unknowns.

The equation $|A-\lambda I|=0$ is called the characteristic equation of A.

· Solving this equation we get n values for λ. These n roots are called characteristic roots (or) latent roots (or) eigen values of A. Corresponding to each λ, the equation (1) have a non zero solution X.

This X is Invariant vector (or) latent vector (or) Eigen vector of A. Corresponding to eigen value λ.

Find the eigen values and Eigen vectors of the matrix.

$$A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$$

The characteristic equation is

$$|A-\lambda I|=0$$

$$\left|\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0$$

$$\begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0$$

$$(11-\lambda)\left[(-2-\lambda)(-6-\lambda)-20\right]-(-4)\left[7(-6-\lambda)-(-50)\right]$$

$-7\left[-28-10(-2-\lambda)\right]$

$(11-\lambda)\left[(\lambda+2)(\lambda+6)-20\right]+4\left[-42-7\lambda+50\right]$

$-7\left[-28+20+10\lambda\right]$

$(11-\lambda)\left[\lambda^2+8\lambda+12-20\right]+4\left[8-7\lambda\right]$

$-7\left[10\lambda-8\right]=0$

$(11-\lambda)\left[\lambda^2+8\lambda-8\right]+32-28\lambda-70\lambda+56=0$

$11\lambda^2+88\lambda-88-\lambda^3-8\lambda^2+8\lambda-98\lambda+88=0$

$-\lambda^3+3\lambda^2-2\lambda=0$

$\lambda^3-3\lambda^2+2\lambda=0$

$\lambda(\lambda^2-3\lambda+2)=0$

$\lambda(\lambda-2)(\lambda-1)=0$

$\lambda=0,1,2.$

The eigen values (or) characteristic values are 0, 1, 2.

Find eigen vectors

The eigen vector X is given by.

$(A-\lambda I)X=0 \quad X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$$\begin{pmatrix}11-\lambda & -14 & -7\\ 7 & -2-\lambda & -5\\ 10 & -4 & -6-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$(11-\lambda)x_1-14x_2-7x_3=0.$

⑥

$$\left.\begin{aligned} 7x_1 + (-2-\lambda)x_2 - 5x_3 &= 0 \\ 10x_1 - 4x_2 + (-6-\lambda)x_3 &= 0 \end{aligned}\right\} \text{—} ①$$

$\lambda = 0$. $\quad 11x_1 - 4x_2 - 7x_3 = 0$

① becomes. $\quad 7x_1 - 2x_2 - 5x_3 = 0$

$10x_1 - 4x_2 - 6x_3 = 0.$

Taking last two equations.

$$\frac{x_1}{\begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 7 & -5 \\ 10 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 7 & -2 \\ 10 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-8} = -\frac{x_2}{8} = \frac{x_3}{-28+20}$$

$$\frac{x_1}{-8} = \frac{x_2}{-8} = \frac{x_3}{-8}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \text{Eigen vectors}$$

put $\lambda = 1$ eqn ① becomes.

$10x_1 - 4x_2 - 7x_3 = 0$

$7x_1 - 3x_2 - 5x_3 = 0$

$10x_1 - 4x_2 - 7x_3 = 0.$

using cross rule for first two equations.

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -4 & -7 & & 10 & & -4 \\ -3 & -5 & & 7 & & -3 \end{array}$$

$$\frac{x_1}{20-21} = \frac{x_2}{-49+50} = \frac{x_3}{-30+28}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-2}.$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{2}.$$

$$X_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

put $\lambda = 2$. The system ① becomes.

$$9x_1 - 4x_2 - 7x_3 = 0.$$
$$7x_1 - 4x_2 - 5x_3 = 0$$
$$10x_1 - 4x_2 - 8x_3 = 0.$$

Taking last two equations.

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -4 & -5 & & 7 & & -4 \\ -4 & -8 & & 10 & & -4 \end{array}$$

$$\frac{x_1}{32-20} = \frac{x_2}{-50+56} = \frac{x_3}{-28+40}.$$

$$\frac{x_1}{12} = \frac{x_2}{6} = \frac{x_3}{12}.$$

8

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$x_1 = 2, \quad x_2 = 1, \quad x_3 = 2.$

$X_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is Eigen vector. ②

Defn: A square matrix A is said to be orthogonal if $AA' = A'A = I$, where I is the Identity matrix. ③

Find the eigen value of an orthogonal matrix

Soln λ be an eigen value of an orthogonal matrix A. X be the corresponding eigen vector. ④

$AX = \lambda X$ — ①

$(Ax)' = (\lambda x)'$

$x' A' = \lambda x'$ — ② A is orthogonal

multiply ① & ② $\qquad AA' = I = A'A.$

$(x'A')(Ax) = (\lambda x')(\lambda x)$ ⑤

$x'(A'A)x = \lambda^2 x' x.$

$x' I x = \lambda^2 \cancel{x} x.$

$x' x = \lambda^2 x' x$

$\therefore \lambda^2 = 1 \Rightarrow \lambda = \pm 1.$ ⑥

Find the eigen values of $\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ Corresponding to the eigen vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Soln Let λ be the eigen value.

$$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \lambda = 2.$$

(10)

Find the eigen value of null matrix.

Soln Let A be a zero matrix of order 3, $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The characteristic eqn is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 0-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0.$$

$$(0-\lambda)(\lambda^2) = 0 \Rightarrow \lambda^3 = 0.$$

$\therefore$ The eigen values are 0, 0, 0.

If $A = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ The eigen values of A^T

Soln The eigen values of A are 2, 2, 3

$A, A^T (= A')$ have the same eigen valu

$\therefore$ A^T have the eigen value (2, 2, 3).

Find the eigen value of $2A^2$ if $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

Soln $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

The characteristic eqn is

$$|A - \lambda I| = 0 \Rightarrow \left| \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0.$$

$$\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(2-\lambda) - 3 = 0$$

$$(\lambda - 2)(\lambda - 4) - 3 = 0.$$

$$\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 1, 5.$$

Eigen value of A is 1, 5

Eigen value of $2A^2$ is $2(1)^2$ $2(5)^2$.

ie (2, 50)

(18)

If $X = (-1, 0, 1)^T$ is the eigen vector of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ find the Corresponding eigen value.

Soln $X = (-1\ 0\ 1)^T = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ Let λ be the eigen value.

$$(A - \lambda I)X = 0.$$

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

$$-(1-\lambda) + 3 = 0 \Rightarrow -1 + \lambda + 3 = 0 \Rightarrow \lambda = -2$$

If the sum of two eigen values and trace of 3×3 matrix A are equal find the value of $|A|$.

Soln given $\lambda_1 + \lambda_2 =$ trace of 3×3 matrix A

$$= \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_3 = 0.$$

$$|A| = \lambda_1 \lambda_2 \lambda_3 = 0.$$

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one of the eigen value of $\begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{pmatrix}$ is -9, find other two eigen value

Soln $A = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{pmatrix}$ Let $\lambda_1 = -9$, $\lambda_2 \lambda_3$ are other two.

Sum of eigen value = Trace

$$-9 + \lambda_2 + \lambda_3 = 7 - 8 - 8 = -9$$

$$\lambda_2 + \lambda_3 = 0. \Rightarrow \lambda_2 = -\lambda_3$$

Product of Eigen values $= |A|$.

$\lambda_1 \lambda_2 \lambda_3 = 441$

$(-9)\lambda_2\lambda_3 = 441$

$\lambda_2\lambda_3 = \frac{441}{-9} = -49.$

$(-\lambda_3)\lambda_3 = -49$

$-\lambda_3^2 = -49 \Rightarrow \lambda_3^2 = 49$

$\lambda_3 = 7, \lambda_2 = -7.$

$\therefore$ The other two eigen values are 7 and -7

Two of the eigen values of $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & -1 & 3 \end{pmatrix}$ are 3, 6 Find eigen value of A^{-1}

Soln The 3 eigen values are $3, 6, \lambda_3$

$3+6+\lambda_3 = \text{Trace of } A = 3+5+3$

$9+\lambda_3 = 11 \Rightarrow \lambda_3 = 2.$

The eigen values of A are 3, 6, 2.

$\therefore$ The eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

Results (1) If λ is an eigen value of A, Then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

(2) If λ is an eigen value of A, Then λ^2 is an eigen value of A^2

(3) If λ is an eigen value of A, Then $k\lambda$ is an eigen value of kA.

12

Note

1) The characteristic roots of a square matrix A and A^T are same

2) The eigen values of A and $P^{-1}AP$ are same.

3) If A and B are two non singular matrices prove that AB and BA have the same eigen value.

Find the characteristic roots of the orthogonal matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Soln

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$|A - \lambda I| = \left| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0.$$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$\lambda = \cos\theta \pm i\sin\theta$. are the characteristic roots of A.

5-4-21 (13)

Note: The above matrix have unit modula.

Result ① The eigen. values of a diagonal matrix are its leading diagonals.

② If λ is an eigen value of a non singular matrix A, then $\frac{|A|}{\lambda}$ is a characteristic root of adj A.

1.3. Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

ie A is a square matrix, then A satisfies the characteristic equation $|A - \lambda I| = 0$.

Verify Cayley Hamilton Theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ Hence find A^3 and A^{-1}

Soln $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a square matrix of order 2. It satisfies characteristic Equation $|A - \lambda I| = 0$.

$$\Rightarrow \left| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0.$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0. \Rightarrow (1-\lambda)(4-\lambda) - 6 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 5\lambda - 2 = 0.$$

14

Prove that $A^2 - 5A - 2I = 0$ for verify Cayley - Hamilton Theorem.

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$A^2 - 5A - 2I = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence Cayley Hamilton Theorem is verified.

Find A^3: $A^2 - 5A - 2I = 0$

$$\Rightarrow A^2 = 5A + 2I \quad \text{---} (1)$$

multiply by A on both sides.

$$(A^2)A = (5A)A + 2IA$$

$$A^3 = 5A^2 + 2A$$

$$= 5(5A + 2I) + 2A$$

$$= 25A + 10I + 2A$$

$$= 27A + 10I$$

$$= 27\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 27 & 54 \\ 81 & 108 \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

$$2I = -5A + A^2$$

Pre multiply by A^{-1}.

$$2A^{-1} = A^{-1}(A^2 - 5A)$$

$$2A^{-1} = A - 5I$$

$$2A^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$2A^{-1} = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

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verify Cayley-Hamilton theorem for $\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ and hence find the Inverse.

Soln $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0.$$

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

16

$$(1-\lambda)\left[(1-\lambda)(1-\lambda)-1\right]-0\left[2(1-\lambda)+1\right]+3\left[-2-(1-\lambda)\right]=0$$

$$[1-\lambda]\left[\lambda^2-2\lambda+\cancel{1}-\cancel{1}\right]+3[\lambda-3]=0$$

$$\lambda^2-2\lambda-\lambda^3+2\lambda^2+3\lambda-9=0$$

$$-\lambda^3+3\lambda^2+\lambda-9=0$$

$$\lambda^3-3\lambda^2-\lambda+9=0.$$

replace λ by A

$$A^3-3A^2-A+9I=0$$

$$A^2=A.A=\begin{pmatrix}1&0&3\\2&1&-1\\1&-1&1\end{pmatrix}\begin{pmatrix}1&0&3\\2&1&-1\\1&-1&1\end{pmatrix}$$

$$=\begin{pmatrix}1+0+3 & 0+0-3 & 3+0+3\\ 2+2-1 & 0+1+1 & 6-1-1\\ 1-2+1 & 0-1-1 & 3+1+1\end{pmatrix}$$

$$=\begin{pmatrix}4&-3&6\\3&2&4\\0&-2&5\end{pmatrix}$$

$$A^3=A^2.A=\begin{pmatrix}4&-3&6\\3&2&4\\0&-2&5\end{pmatrix}\begin{pmatrix}1&0&3\\2&1&-1\\1&-1&1\end{pmatrix}$$

$$=\begin{pmatrix}4&-9&21\\11&-2&11\\1&-7&7\end{pmatrix}$$

$$A^3-3A^2-A+9I=\begin{pmatrix}4&-9&21\\11&-2&11\\1&-7&7\end{pmatrix}-$$

$$3\begin{pmatrix}4 & -3 & 6\\ 3 & 2 & 4\\ 0 & -2 & 5\end{pmatrix} - \begin{pmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 1 & -1 & 1\end{pmatrix} + 9\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Hence Cayley-Hamilton Theorem is verified.

Find A^{-1}

Consider the equation.

$$A^3 - 3A^2 - A + 9 = 0$$

multiply by A^{-1}

$$A^2 - 3A - I + 9A^{-1} = 0.$$

$$9A^{-1} = -A^2 + 3A + I$$

$$9A^{-1} = -\begin{pmatrix}4 & -3 & 6\\ 3 & 2 & 4\\ 0 & -2 & 5\end{pmatrix} + 3\begin{pmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 1 & -1 & 1\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$9A^{-1} = \begin{pmatrix}0 & 3 & 3\\ 3 & 2 & -7\\ 3 & -1 & -1\end{pmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{pmatrix}0 & 3 & 3\\ 3 & 2 & -7\\ 3 & -1 & -1\end{pmatrix}$$

Note using Cayley Hamilton Theorem we can find the Inverse of the matrix by above method.

(18) 7-4-21. Diagonalization of a Matrix.

Defn: Let A be a square matrix of order n. Then A is said to be diagonalizable if there exist another non singular square matrix M of order n. such that $M^{-1}AM$ is a diagonal matrix. It is denoted by D.

Defn: Two matrices A and B are said to be similar, if there exists a non singular matrix P such that $B = P^{-1}AP$

Result: Two similar matrices $(B, P^{-1}AP)$ have the same eigen values.

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prob: Diagonalise the given matrix $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ by using similarity transformation.

Soln $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$ The ch. equation is $|A - \lambda I| = 0$.

$$\left| \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0.$$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)\left[(1-\lambda)(-3-\lambda) - 2\right] - 2\left[2(-3-\lambda) + 7\right] + 0[\,] = 0.$$

(19)

$(2-\lambda)[-3-\lambda+3\lambda+\lambda^2-2]-2[-6-2\lambda+7]=0$

$(2-\lambda)[\lambda^2+2\lambda-5]-2[1-2\lambda]=0.$

$2\lambda^2+4\lambda-10-\lambda^3-2\lambda^2+5\lambda-2+4\lambda=0.$

$-\lambda^3+13\lambda-12=0$

$\lambda^3-13\lambda+12=0.$

$\lambda=1,\quad 1-13+12=0 \Rightarrow \lambda=1$ is a root

To find remaining roots

$$\begin{array}{c|cccc} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$\lambda^2+\lambda-12=0$

$\lambda^2+4\lambda-3\lambda-12=0$

$(\lambda+4)(\lambda-3)=0. \Rightarrow \lambda=3,-4$

Hence $\lambda=1,3,-4$ are eigen Values

Find eigen vectors Corresponding to

$\lambda=-4.$ $\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

where x_1, x_2, x_3 is eigen vector Corresponding to $\lambda=-4.$

$$\begin{pmatrix} 2+4 & 2 & 0 \\ 2 & 1+4 & 1 \\ -7 & 2 & -3+4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$6x_1+2x_2+0x_3=0.$

$2x_1+5x_2+x_3=0$

$-7x_1+2x_2+x_3=0.$

20

Taking first two equations.

$$6x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 + 5x_2 + x_3 = 0.$$

$$\frac{x_1}{2-0} = \frac{-x_2}{6-0} = \frac{x_3}{30-4}.$$

$$\frac{x_1}{2} = \frac{x_2}{-6} = \frac{x_3}{26}.$$

$$\frac{x_1}{1} = \frac{x_2}{-3} = \frac{x_3}{13}.$$

$\therefore$ we get $X_1 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$ as eigen vector

put $\lambda = 1$ the system of Equation.

becomes. $\begin{pmatrix} 2-1 & 2 & 0 \\ 2 & 1-1 & 1 \\ -7 & 2 & -3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 + 0x_2 + x_3 = 0.$$
$$-7x_1 + 2x_2 - 4x_3 = 0.$$

First two eqns.

$$\frac{x_1}{1-0} = \frac{-x_2}{1-0} = \frac{x_3}{0-4}$$

$$\frac{x_1}{2} = -\frac{x_2}{1} = \frac{x_3}{-4}.$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-4}.$$

$\therefore$ $X_2 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ is an eigen vector for $\lambda = 1$

3

21

put $\lambda = 3$. The system of Equation becomes

$$\begin{pmatrix} 2-3 & 2 & 0 \\ 2 & 1-3 & 1 \\ -7 & 2 & -3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 - 2x_2 + x_3 = 0$$
$$-7x_1 + 2x_2 - 6x_3 = 0.$$

$$\frac{x_1}{2-0} = \frac{-x_2}{-1-0} = \frac{x_3}{2-4}.$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}.$$

we get $X_3 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ as eigenvector for $\lambda = 3$.

Consider $M = \begin{pmatrix} 1 & 2 & 2 \\ -3 & -1 & 1 \\ 13 & -4 & -2 \end{pmatrix}$

Diagonal matrix $D = M^{-1} A M$.

$$M^{-1} = \frac{\operatorname{adj}(M)}{|M|} \qquad |M| = 70.$$

$$D = \begin{pmatrix} 1 & 2 & 2 \\ -3 & -1 & 1 \\ 13 & -4 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ -3 & -1 & 1 \\ 13 & -4 & -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

(22)

Home work.

Diagonalize the matrix $A = \begin{pmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$ using similarity transformation.

8-4-21 Diagonalization by orthogonal Transformation

If A is a real symmetric matrix, then we can diagonalize the matrix by orthogonal transformations.

Procedure to diagonalize the matrix by orthogonal transformation.

① Normalise each eigen vector X_r i.e divide each element of the eigen vector by the square root of the sum of the square of all elements of X_r.

② Form the normalised modal matrix N by using the normalised eigen vectors of A.

③ Find $D = N^t A N$ as a diagonal matrix by orthogonal transformation.

Result ① For orthogonal matrix A, $A^{-1} = A^T$.

② For the diagonal matrix D, the diagonal elements are the eigen values of A, where $D = N^T A N$.

23

Propositions.

1. If A is an orthogonal matrix then A^T and A^{-1} are also an orthogonal matrix.

2. If A and B are orthogonal then AB is also orthogonal.

3. If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is an eigen value of A.

4. A square matrix A is orthogonal if and only if $A^{-1} = A^T$

Soln 1 A is orthogonal $AA^T = I = A^T A$.

$$(A^T)(A^T)^T = A^T A = I$$

similarly

$$(A^T)^T A^T = I \Rightarrow A^T \text{ is orthogonal.}$$

A is orthogonal A^{-1} exist and $A^{-1} = A^T$

$$\therefore (A^{-1})(A^{-1})^T = A^T (A^T)^T = A^T A = I$$

$$\text{llly } (A^{-1})^T A^{-1} = I$$

$$\text{Hence } (A^{-1})(A^{-1})^T = (A^{-1})^T A^{-1} = I$$

$$\Rightarrow A^{-1} \text{ is orthogonal.}$$

Soln 2 A is orthogonal $AA^T = I = A^T A$

B is orthogonal $BB^T = I = B^T B$

P.T. AB is orthogonal

$$(AB)(AB)^T = (AB)(B^T A^T)$$

$$= A(BB^T)A^T = AIA^T = AA^T = I$$

Similarly $(AB)^T(AB) = I$

Hence AB is orthogonal.

<u>Soln 3</u> A is an orthogonal matrix.

$$A^{-1} = A^T.$$

λ be an eigen value of A

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^T

Also A and A^T have the same eigen value $\therefore \frac{1}{\lambda}$ is an eigen value of A.

<u>Soln 4</u> Suppose A is orthogonal.

$$AA^T = A^TA = I$$

$$|AA^T| = |I| = 1$$

$$|A||A^T| = 1$$

$\Rightarrow |A||A| = 1 \quad (\because |A| = |A^T|$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| \neq 0 \Rightarrow A$ is non singular

$\Rightarrow A^{-1}$ exist.

$$A^{-1} = A^{-1}I = A^{-1}(AA^T)$$

$$= (A^{-1}A)A^T = IA^T = A^T$$

$$\therefore A^{-1} = A^T$$

also $AA^T = I = AA^{-1}$.

$$A^T A = A^{-1} A = I$$

Hence $AA^T = A^T A = I$

$\Rightarrow A$ is orthogonal.

Problem: prove that $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Soln A is orthogonal if $AA^T = A^T A = I$

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

similarly $A^T A = I$

$AA^T = A^T A = I \Rightarrow A$ is orthogonal

Exercise: prove that $\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ is an orthogonal matrix.

9-4-21

Problem: Diagonalise the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ by means of orthogonal transformation.

Soln $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ The characteristic equation is

$$|A - \lambda I| = 0.$$

26

$$\left|\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0.$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(3-\lambda)(3-\lambda)-1\right] - 1\left[(3-\lambda)+1\right]$$
$$+1\left[-1-(3-\lambda)\right] = 0.$$

$$(3-\lambda)^3 - 4 + \lambda + \lambda - 4 - 3 + \lambda = 0.$$

$$3^3 - \lambda^3 - 27\lambda + 9\lambda^2 + 3\lambda - 11 = 0.$$

$$-\lambda^3 + 9\lambda^2 - 24\lambda + 16 = 0.$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$$

$$\lambda = 1, 4, 4.$$

Find Eigen vector
$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(3-\lambda)x_1 + x_2 + x_3 = 0$$
$$x_1 + (3-\lambda)x_2 - x_3 = 0$$
$$x_1 - x_2 + (3-\lambda)x_3 = 0$$

$\lambda = 1$.
$$2x_1 + x_2 + x_3 = 0$$
$$x_1 + 2x_2 - x_3 = 0$$
$$x_1 - x_2 + 2x_3 = 0.$$

Solving First two equations.

$$\frac{x_1}{-3} = \frac{-x_2}{-3} = \frac{x_3}{3}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}.$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}.$$

The eigen vector Corresponding to $\lambda = 1$

is $X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 4$, $-x_1 + x_2 + x_3 = 0$

$x_1 - x_2 - x_3 = 0$

$x_1 - x_2 - x_3 = 0.$

all are same

$x_1 = x_2 + x_3.$

put. $x_2 = 1$, $x_3 = 0$, $x_1 = 1$

Hence $X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 4$, $X_3 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

Since X_3 is orthogonal to X_2.

we get $l + m + 0n = 0.$ —— (1)

X_3 satisfies $l - m - n = 0.$ —— (2)

Solving (1) & (2) $\frac{l}{-1} = \frac{-m}{-1} = \frac{n}{-2}$

$X_3 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

28

These 3 vectors X_1, X_2, X_3 are pairwise orthogonal.

Normalised matrix is $\begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$

$$D = N^T A N.$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Quadratic form - orthogonal Reduction to its Canonical form.

The most general Quadratic form for n variables is $a_{11}x_1^2 + a_{12}x_1x_2 + \dots a_{1n}x_1x_n + a_{21}x_2x_1 + \dots + a_{n1}x_nx_1 + \dots + a_{nn}x_n^2$.

Ex: write the matrix of Quadratic form. $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

The matrix corresponding to Quadratic form is

$$\begin{pmatrix} \text{Coef } x_1^2 & \frac{1}{2}\text{Coef of } x_1x_2 & \frac{1}{2}\text{Coef of } x_2x_3 \\ \frac{1}{2}\text{Coef } x_2x_1 & \text{Coef of } x_2^2 & \frac{1}{2}\text{Coef of } x_2x_3 \\ \frac{1}{2}\text{Coef } x_3x_1 & \frac{1}{2}\text{Coef of } x_3x_2 & \text{Coef of } x_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Ex: write the Quadratic form of following matrix. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{pmatrix}$

Soln

The quadratic form is

$$x^2 - 2y^2 - 3z^2 + 4xy + 6xz - 8yz.$$

 10-4-21

Nature of quadratic form.

Given a quadratic form $D = X^T AX$.

$P_1 = |a_{11}|$ $P_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $P_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

... $P_n = |A|$. be the principle sub determinants of A.

Then the quadratic form is

① positive definite if $P_1 P_2 .. P_n$ are all positive.

② positive Semidefinite if some of the determinants vanish and the others are positive.

③ Negative definite if $P_1 P_3 P_5 ....$ all are all negative and $P_2 P_4 P_6 ...$ are all positive.

④ negative Semidefinite if some of the determinants vanish and others are negative.

⑤ Indefinite in all other Cases.

Discuss the nature of the quadratic form. ① $11x^2 + 2x_2^2 + 2x_3^2 + 4x_1x_2 - 2x_2x_3 + 4x_1x_3$.

∴ The matrix form is

$$A = \begin{pmatrix} 11 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix} \quad P_1 = |11| > 0$$

$$P_2 = \begin{vmatrix} 11 & 2 \\ 2 & 2 \end{vmatrix} = 22 - 4 = 18 > 0.$$

$$P_3 = \begin{vmatrix} 11 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{vmatrix} = 9 > 0.$$

all are positive. ∴ The given Quadratic form is positive definite.

(2) $2x_1^2 + 2x_1x_2 + 3x_3^2$

matrix form is $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

$P_1 = |2| > 0$

$P_2 = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0.$ $P_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 0$

∴ P_1 is +ve P_2, P_3 vanishing. The quadratic form is positive semidefinite

(3) $6x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 + 8x_1x_3$

matrix form. $A = \begin{pmatrix} 6 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & 3 \end{pmatrix}$ $P_1 = |6| > 0$

$P_2 = \begin{vmatrix} 6 & -2 \\ -2 & 2 \end{vmatrix} = 8 > 0$ $P_3 = |A| = -8.$

Hence the quadratic form is Indefinite.

(32) 12-4-21.

Reduce the quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ to a Canonical form. Discuss its nature

Soln The given quadratic form is

$10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$

The matrix form is

$$A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$$

Find Eigen values of A

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 17\lambda^2 + 42\lambda = 0$$

$$\lambda(\lambda^2 - 17\lambda + 42) = 0.$$

Eigen values are 0, 3, 14.

Find Eigen vectors $(A - \lambda I)X = 0$.

$$\begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (10-\lambda)x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 + (2-\lambda)x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 + (5-\lambda)x_3 &= 0 \end{aligned}\right\} \quad (1)$$

$\lambda = 0$, ① becomes $10x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 + 2x_2 + 3x_3 = 0$

$-5x_1 + 3x_2 + 5x_3 = 0.$

solving first two eqns.

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16} \quad X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

similarly $\lambda = 3$. eqn ① becomes.

$7x_1 - 2x_2 - 5x_3 = 0$

$-2x_1 - x_2 + 3x_3 = 0$

$-5x_1 + 3x_2 + 2x_3 = 0.$

Solving first 2 Eqns.

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11} \quad X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

similarly $\lambda = 14$. we get

$$X_3 = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$N = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & 3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & -1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & -2/\sqrt{14} \end{pmatrix} \quad N^T = \begin{pmatrix} \frac{1}{\sqrt{42}} & -\frac{5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & -2/\sqrt{14} \end{pmatrix}$$

$$N^T A N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} = D.$$

Canonical form. is

$$(y_1\, y_2\, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0y_1^2 + 3y_2^2 + 14y_3^2$$

Nature of the quadratic form.

Rank = 2 Index = 2 Signature = 2

The quadratic form is positive semi definite

Exercise: Reduce the quadratic form. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$ to a Canonical form. Discuss the nature.

Application of matrices.

Stretching of an electric membrane

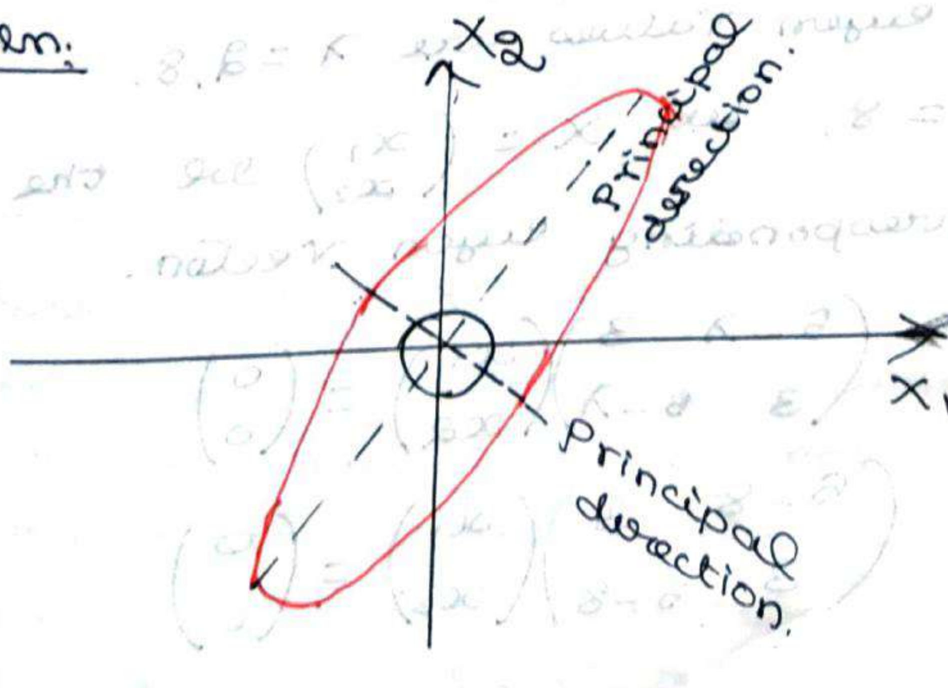
An elastic membrane in the $x_1 x_2$-plane with boundary circles $x_1^2 + x_2^2 = 1$ is stretched so that a point $P : (x_1, x_2)$ goes into a plane point $Q : (y_1, y_2)$ given by

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = AX = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

In components $y_1 = 5x_1 + 3x_2$

$y_2 = 3x_1 + 5x_2$. Find the principal directions. That means the direction of the position vector X of P for which the direction of position vector Y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Soln.



so let us consider for vector x such that $y = \lambda x$ since we know that $y = Ax$ then $Ax = \lambda x$.

$$\Rightarrow 5x_1 + 3x_2 = \lambda x_1$$

$$3x_1 + 5x_2 = \lambda x_2$$

$$\therefore (5-\lambda)x_1 + 3x_2 = 0.$$

$$3x_1 + (5-\lambda)x_2 = 0$$

The characteristic equation is

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(5-\lambda) - 9 = 0$$

$$\lambda^2 - 10\lambda + 25 - 9 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0.$$

$$(\lambda-2)(\lambda-8) = 0$$

eigen values are $\lambda = 2, 8$.

$\lambda = 8$, let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the corresponding eigen vector.

$$\begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5-8 & 3 \\ 3 & 5-8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 + 3x_2 = 0$$

$$3x_1 - 3x_2 = 0$$

$$x_1 - x_2 = 0 \quad x_1 = x_2$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigen vector.

$\lambda = 2$. The Corresponding eigen vector is $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 + 3x_2 = 0$$

$$3x_1 + 3x_2 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 = -x_2.$$

Thus $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the eigen vector.

These eigen vectors make $45°$ and $135°$ angles with the positive direction. They give the principal direction. These eigen values shows that the principal directions the membrane is stretched by factors 8 and 2

Choose principal directions as directions of new Cartesian Coordinate system u_1, u_2.

u_1 - positive semi axes in the first quadrant.

u_2 - semi axes in the second quadrant of $x_1 x_2$ system.

$$u_1 = r\cos\theta$$

$$u_2 = r\sin\theta.$$

Then a boundary point of the unstretched circular membrane the Coordinate $\cos\theta$ and $\sin\theta$.

$$z_1 = 8\cos\theta \quad z_2 = 2\sin\theta$$

$$z_1^2 = 64\cos^2\theta \quad z_2^2 = 4\sin^2\theta.$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{z_1^2}{64} + \frac{z_2^2}{4} = 1.$$

$$\frac{z_1^2}{8^2} + \frac{z_2^2}{2^2} = 1.$$

This is the deformed boundary is an ellipse as given in diagram.

Application 2

Given $A = \begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix}$ in $Y = AX$.

Find the principle direction and corresponding factor of extension or contraction.

Soln: Let us consider the vector X such that $Y = \lambda X$.

Since $Y = AX$, $AX = \lambda X$.

Find eigen values and Eigen vectors.

The characteristic eqn is $|A - \lambda I| = 0$.

$$\left| \begin{pmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$(7-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 9\lambda + 8 = 0 \quad \lambda = 1, 8.$$

Find Eigen vectors $\lambda = 1, 8$

$\lambda = 1$,

$$\begin{pmatrix} 7-1 & \sqrt{6} \\ \sqrt{6} & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$6x_1 + \sqrt{6}x_2 = 0$$

$$\sqrt{6}x_1 + x_2 = 0$$

$$6x_1 = -\sqrt{6}x_2$$

$$\frac{6}{\sqrt{6}} x_1 = -x_2.$$

$$x_1 = -\frac{1}{\sqrt{6}} x_2.$$

put $x_2 = 1$

$$x_1 = -\frac{1}{\sqrt{6}}$$

The eigen vector is $\begin{pmatrix} -\frac{1}{\sqrt{6}} \\ 1 \end{pmatrix}$

This vector makes angle with the positive direction (x_1)

$$\tan^{-1}\frac{x_2}{x_1} = \tan^{-1}\frac{1}{-\frac{1}{\sqrt{6}}}$$

$$= \tan^{-1}(-\sqrt{6})$$

$$= 180^\circ - \tan^{-1}\sqrt{6}$$

$$= 112.2^\circ$$

put $\lambda_2 = 8$

we get $-x_1 + \sqrt{6}x_2 = 0$

$$\sqrt{6}x_1 - 6x_2 = 0$$

$$-x_1 = -\sqrt{6}x_2$$

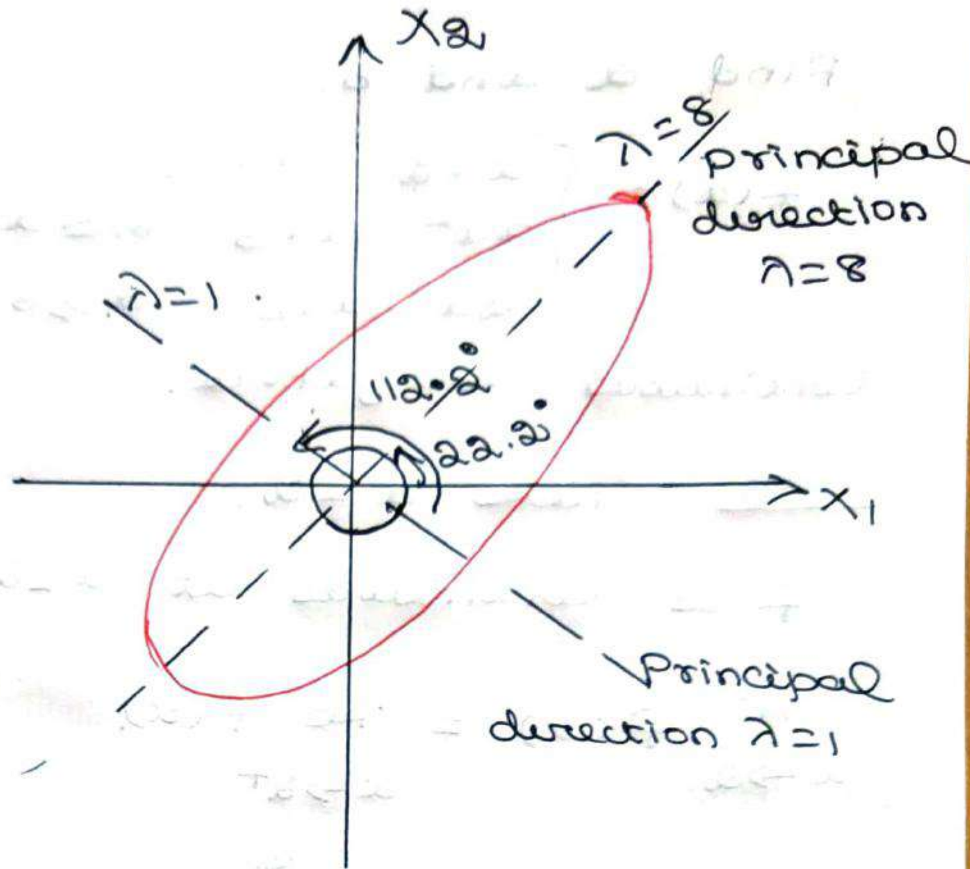
$$x_1 = \sqrt{6}x_2$$

$x_2 = 1, \quad x_1 = \sqrt{6}$

The eigen vector $\begin{pmatrix} \sqrt{6} \\ 1 \end{pmatrix}$

$$\tan^{-1}\frac{x_2}{x_1} = \tan^{-1}\frac{1}{\sqrt{6}} = 22.2^\circ$$

This vector makes 22.2° with +ve direction of x_1-axis



Section B.

1. Verify Cayley Hamilton Theorem for $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ Hence find A^4.

2. Diagonalise the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ by similarity transformation.

3. Diagonalise the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ by orthogonal transformation.

4. Reduce the quadratic form $x^2+y^2+z^2-2xy-2yz-2zx$ by orthogonal transformation. Also find the nature of quadratic form.

5 (a)

Differential Calculus

Function: A function is a rule that assigns to each element x in A to exactly one element called $f(x)$ in a set B.

$$A \xrightarrow{f} B, \quad x \dashrightarrow f(x).$$

Note: A and B are the set of real nos.
A - domain, B - Codomain. $f: A \to B$
range of $f = \{ f(x) / x \in \text{domain } A \}$

1.1 Representation of function.

There are 4 possible ways to represent the function.

① Represent the function algebrically. (by formula)

ex: $y = x^2$. Connect x & y by the eqn.

$A = \pi r^2$ (Area of Circle of radius r)

A as a function of r.

$y = \sin x$ (Trigonometric fn)

$y = e^{ax}$ (exponential fn)

$y = \log(ax+b)$ (logarithmic fn)

② Represent the function verbally

Cost of mailing an envelope depends on the weight w. For determining cost C when w is known.

③ Represent the function numerically.

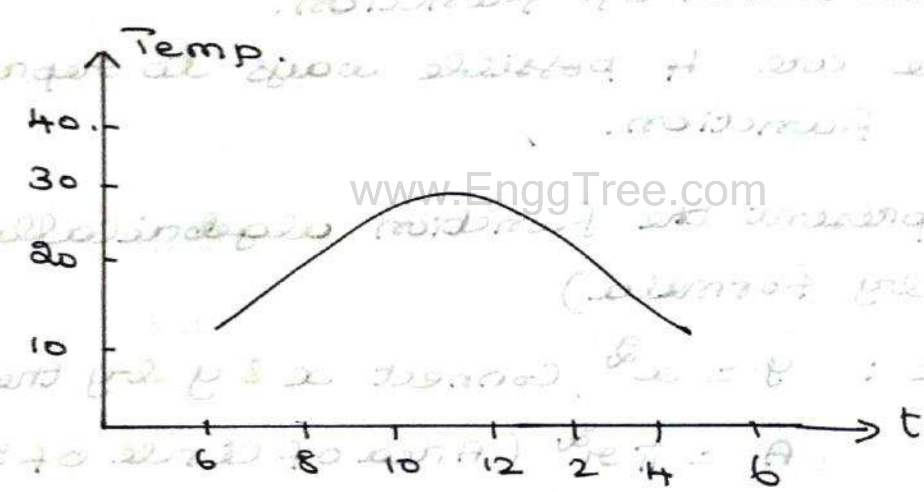
population depends on t.

The human population P depends on time t. mentioned in Crores.

t	1980	1990	2000	2010	2020
P	100	106	112	120	130

④ Represent the function visually.

In a certain place the temperature is recorded continuously from 6 AM to 6 P.M.



Examples.

Find the domain of the function

$f(x) = \sqrt{x+2}$

Soln: Find the domain of fn.

The square root of a number must be positive. (-ve value for square root can't be defined)

$\therefore x+2 \geq 0$

$\Rightarrow x \geq -2$

Hence domain is $[-2,\infty)$ ③

Ex: Find the domain of the function.

$f(x) = \sqrt{3-x} - \sqrt{2+x}$

Soln square root of negative no can't be defined.

$\therefore 3-x \geq 0, \quad 2+x \geq 0$

$3 \geq x, \quad x \geq -2$

ie $x \leq 3, \quad -2 \leq x$

Hence domain is $[-2,3]$

Ex: Find the domain of the function

$f(x) = \dfrac{1}{x^2-x}$

Soln $f(x) = \dfrac{1}{x^2-x} = \dfrac{1}{x(x-1)}$

The fn is not defined at $x=0$ and $x=1$.

Hence domain is $\{x / x \neq 0 \text{ and } x \neq 1\}$

Hence domain of the function is given $(-\infty,0) \cup (0,1) \cup (1,\infty)$

Find the domain of the function.

Soln. $f(x) = \dfrac{x+4}{x^2-9} = \dfrac{x+4}{(x+3)(x-3)}$

$(x+3)(x-3) \neq 0$

$x \neq -3$ and $x \neq 3$

domain is $(-\infty,-3) \cup (-3,3) \cup (3,\infty)$. (or)

Find the domain $\dfrac{2x^2-5}{x^2+x-6} = \dfrac{2x^2-5}{x^2+3x-2x-6}$

$x^2+x-6 = (x+3)(x-2)$

$x \neq -3, \; x \neq 2.$

domain is $(-\infty,-3)\cup(-3,2)\cup(2,\infty)$ (or) $\{x : x \neq -3, x \neq -2\}$

3. $f(x) = \dfrac{x+1}{1+\dfrac{1}{x+1}}$

$\dfrac{1}{x+1} \neq -1$ if $x+1 \neq -1$

$x \neq -2$.

domain is $(-\infty,-2)\cup(-2,\infty)$ (or) $\{x : x \neq -2\}$

4. $f(x) = \dfrac{1}{\sqrt{x^2-5x}}$

$x^2-5x \geq 0$

$x^2 \geq 5x$. $x \geq 5$.

domain is $[5,\infty)$

Piecewise function.

The function f defined by

Eg: $f(x) = \begin{cases} 1-x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Eg: $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$

odd and Even function

$f : A \to B$ be a function.

$f(x) = f(-x)$ for every $x \in$ domain A

Then f is called an even function

$f(-x) = -f(x)$ for every x in domain A.

Then f is called an odd function

eg $f(x) = x^3$ is odd

$f(-x) = (-x)^3 = -x^3 = -f(x)$.

eg $f(x) = \cos x,\ x^2$

$f(-x) = \cos(-x) = \cos x = f(x)$

$f(-x) = (-x)^2 = x^2 = f(x)$.

Determine $f(x) = x|x|$

$f(-x) = -x|-x| = -x|x| = -f(x)$

$\Rightarrow f$ is odd.

Increasing and decreasing function.

Increasing function: A function $f(x)$ is called increasing on an Interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

eg $f(x) = x^2$. in $(0, \infty)$ is an

Decreasing function. A function $f(x)$ is called decreasing on an Interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I

Check the following are odd or even.

$f(x) = \dfrac{x^2}{x^4+1}$ is even.

$f(x) = 1 + 3x^3 - x^5$ is odd.

$f(x) = 1 - 3x^3 + x^5$. $\Rightarrow f$ is neither odd nor even.

Find the domain and range of each function.

(a) $f(x) = 1+x^2 \geq 1$ since $x^2 \geq 0$ always.

domain is $(-\infty, \infty)$

range is $[1, \infty)$

(b) $f(x) = \sqrt{5x+10}$

$5x+10 \geq 0$

$5x \geq -10$

$x \geq -2$

domain is $[-2, \infty)$

range is $[0, \infty)$

(c) $f(x) = \dfrac{4}{3-x}$.

$3-x \neq 0$ ie $x \neq 3$

domain is $(-\infty, 3) \cup (3, \infty)$

range is $(-\infty, 0) \cup (0, \infty)$

(d) Find the domain and range of the function defined by the coordinates $\{(-4,1)(-2,2.5)(2,-1)(3,-2)\}$

domain is x values $\{-4, -2, 2, 3\}$

range is $\{1, 2.5, -1, -2\}$

$f(x)$ is a discontinuous function.

A.U.J. Jan. 2018 Sketch the graph of the function.

$$f(x) = \begin{cases} 1+x & x < -1 \\ x^2 & -1 \leq x \leq 1 \\ 2-x & x \geq 1 \end{cases}$$

1.2 Limit of a function.

<u>Defn:</u> Suppose $f(x)$ is defined when x is near the number a Then we write

$$\lim_{x \to a} f(x) = L$$

ie $f(x) \to L$ as $x \to a$.

<u>Defn:</u> Suppose $f(x)$ is defined when x is near the number from Left hand side of a, we write

$\lim_{x \to a^-} f(x) = L$. is called the left hand limit of $f(x)$. here x is less than a.

<u>Defn:</u> Suppose $f(x)$ is defined when x is near the number from Right hand side of a we write

$$\lim_{x \to a^+} f(x) = L.$$

<u>Defn:</u> Suppose $f(x)$ is defined when x is near the number a. Then.

$\lim_{x \to a} f(x) = L$. if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

<u>Defn:</u> Infinite limits.

$\lim_{x \to a} f(x) = \infty$ means the value of $f(x)$ is arbitrarily large.

Ex: what is wrong in the eqn $\frac{x^2+x-6}{x-2} = x+3$ and explain why the equation $\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2}(x+3)$ is Correct

Soln

$$\frac{x^2+x+6}{x-2} = \frac{(x-2)(x+3)}{(x-2)} \neq x+3.$$

because the factor $(x-2)$ is zero at $x=2$ we cannot cancel the zero factor.

But $$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2}$$

$$= \lim_{x\to 2}(x+3) \text{ is Correct}$$

because the limit of the function is not same as the value of the function

Example. Evaluate the limit if it exist $\lim_{x\to -3} \frac{x^2-9}{2x^2+7x+3}$.

Soln The given function $f(x) = \frac{x^2-9}{2x^2+7x+3}$

$$\lim_{x\to 3} \frac{x^2-9}{2x^2+7x+3} = \lim_{x\to -3} \frac{(x+3)(x-3)}{(2x+1)(x+3)}$$

$$= \lim_{x\to -3} \frac{x-3}{2x+1}$$

$$= \frac{-3-3}{-6+1} = \frac{6}{5}.$$

Example. Evaluate $\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$

$$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}.$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$

P.T. $\lim_{x \to 0} \frac{|x|}{x}$ does not exist

<u>Proof</u>: The absolute value function is defined as.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

First find LHL.

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

RHL is

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1.$$

$$\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x).$$

The limit does not exist

Evaluate the limit $\lim_{x \to -6} \frac{2x+12}{|x+6|}$

The given function is

$$f(x) = \frac{2x+12}{|x+6|}$$

$$= \frac{2(x+6)}{x+6} \quad x+6 \geq 0$$

$$= \frac{2(x+6)}{-(x+6)} \text{ if } x+6<0$$

$$= \begin{cases} 2, & x \geq -6 \\ -2 & x < -6 \end{cases}$$

$\therefore \lim\limits_{x \to -6^-} f(x) = -2$

$$\lim_{x \to -6^-} \frac{2x+12}{|x+6|} = -2$$

$$\lim_{x \to -6^+} \frac{2x+12}{|x+6|} = 2$$

$\therefore$ The LHL $\neq$ RHL.

$\therefore$ The limit does not exist.

Formulas using in limit of a functio

① $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

② $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ $\quad \lim\limits_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$

③ $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

④ $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \log a \quad a > 0$

⑥ $\lim\limits_{x \to 1} \frac{\log x}{x - 1} = 1$

(11) General Rules.

① $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

② $\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

③ $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$

④ $\lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$.

⑤ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$.

⑥ $\lim_{x \to a} f(x)^n = \left[\lim_{x \to a} f(x)\right]^n$, n is a +ve Integer

⑦ $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ n is a +ve Integer.

Ex: $\lim_{x \to -1} (x^4 - 3x)(x^2 + 5x + 3)$

$= \lim_{x \to -1} (x^4 - 3x) \lim_{x \to -1} (x^2 + 5x + 3)$

$= \left[(-1)^4 - 3(-1)\right]\left[(-1)^2 + 5(-1) + 3\right]$ $\left(\because \lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)\right)$

$= [1 + 3][1 - 5 + 3]$

$= 4(4 - 5)$

$= 4(-1) = -4$.

Evaluate

① $\underset{x\to 3}{Lt} \dfrac{x^4-81}{x^2-9} = \underset{x\to 3}{Lt} \dfrac{(x^2)^2-9^2}{x^2-9}$

$= \underset{x\to 3}{Lt} \dfrac{(x^2+9)(x^2-9)}{(x^2-9)}$ $(\because (a+b)(a-b) = a^2-b^2)$

$= \underset{x\to 3}{Lt}\; x^2+9$

$= 3^2+9 = 18$

② $\underset{x\to 0}{Lt} \dfrac{\sin mx}{\tan nx} = \underset{x\to 0}{Lt} \dfrac{\sin mx}{mx} \cdot \dfrac{mx}{nx} \dfrac{1}{\frac{\tan nx}{nx}}$

$= \dfrac{m}{n} \underset{x\to 0}{Lt} \dfrac{\sin mx}{mx} \dfrac{1}{\underset{x\to 0}{Lt} \frac{\tan nx}{nx}}$

$= \dfrac{m}{n}\, 1 \cdot \dfrac{1}{1} = \dfrac{m}{n}.$

③ $\underset{x\to 0}{Lt} \dfrac{(1+x)^4-1}{(1+x)^2-1} = \underset{x\to 0}{Lt} \dfrac{\frac{(1+x)^4-1}{(1+x)-1}}{\frac{(1+x)^2-1}{(1+x)-1}}.$

Let $1+x = y$

as $x\to 0,\ y\to 1$

$\therefore \underset{x\to 0}{Lt} \dfrac{(1+x)^4-1}{(1+x)^2-1} = \underset{y\to 1}{Lt} \dfrac{\frac{y^4-1}{y-1}}{\frac{y^2-1}{y-1}} = \dfrac{4.(1)^{4-1}}{2(1)^{2-1}}.$

$= \dfrac{4}{2} = 2$

$\left(\underset{x\to a}{Lt} \dfrac{x^n-a^n}{x-a} = na^{n-1}\right)$

(13)

Trigonometric Values

θ	0	$\frac{\pi}{6}=60$	$\frac{\pi}{4}=45$	$\frac{\pi}{3}=60$	$\frac{\pi}{2}=90$	$\pi=180$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0

① $\lim_{x \to \frac{\pi}{2}} \sec x - \tan x$

$$= \lim_{x \to \frac{\pi}{2}} \sec x - \lim_{x \to \frac{\pi}{2}} \tan x$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\cos x} - \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\cos x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \frac{1-1}{\infty} = \frac{0}{\infty}$$

It is an Indeterminate form.

$$\therefore \lim_{x \to \frac{\pi}{2}} \sec x - \tan x = \lim_{x \to \frac{\pi}{2}} \frac{(1-\sin x)(1+\sin x)}{\cos x(1+\sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1-\sin^2 x}{\cos x(1+\sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{\cos x(1+\sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{1+\sin x} = \frac{\cos \frac{\pi}{2}}{1+\sin \frac{\pi}{2}} = \frac{0}{1+1}$$

$$= \frac{0}{2} = 0$$

Formula: $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$

Ex: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+5} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^5$

$= e\,(1+0)^5$

$= e$

Ex: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^3$

$= e^3$

Ex: $\lim_{x\to 0}\frac{5^x-6^x}{x} = \lim_{x\to 0}\frac{5^x-1+1-6^x}{x}$

$= \lim_{x\to 0}\frac{5^x-1}{x} - \lim_{x\to 0}\frac{6^x-1}{x}$.

By Formula $\lim_{x\to 0}\frac{a^x-1}{x} = \log a$.

$\lim_{x\to 0}\frac{5^x-6^x}{x} = \log 5 - \log 6$

$= \log\frac{5}{6}$

Ex: $\lim_{x\to\infty}\frac{(3x+1)(2x+4)}{(x+1)(x-7)}$

$= \lim_{x\to\infty}\frac{x\left(3+\frac{1}{x}\right)x\left(2+\frac{4}{x}\right)}{x\left(1+\frac{1}{x}\right)x\left(1-\frac{7}{x}\right)}$

$= \frac{(3+0)(2+0)}{(1+0)(1-0)} = 6.$

(16)

Ex $\lim_{x\to 1} \frac{\log x}{x-1} = \lim_{x\to 1} \log (x-1+1)^{\frac{1}{x-1}}$

$= \lim_{x\to 1} \log [(x-1)+1]^{\frac{1}{x-1}}$

$= \lim_{x-1\to 0} \log [(x-1)+1]^{\frac{1}{x-1}}$

put $\frac{1}{x-1} = n$

as $x-1 \to 0$, $n \to \infty$

$\therefore \lim_{x\to 1} \frac{\log x}{x-1} = \lim_{\frac{1}{x-1}\to \infty} \log \left[1 + \frac{1}{\frac{1}{x-1}}\right]^{\frac{1}{x-1}}$

$= \lim_{n\to\infty} \log \left[1 + \frac{1}{n}\right]^{n}$

$= \log e = 1.$

Ex: $\lim_{x\to 0} \frac{e^{\tan x} - 1}{\tan x}$

put $t = \tan x$.

as $x \to 0$, $t \to 0$.

$\lim_{x\to 0} \frac{e^{\tan x} - 1}{\tan x} = \lim_{t\to 0} \frac{e^t - 1}{t} = 1.$ (by Formula)

Assignment problems

① $\lim_{x\to 1} \frac{x^2 + 2x + 1}{x+1 - 1}$.

(16)

(2) $\lim_{x \to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x}$.

(3) $\lim_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x}$

(4) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

(5) $\lim_{x \to 0} \frac{\sin 5x}{\sin 3x}$.

(6) $\lim_{\theta \to 0} \frac{1 - \cos m\theta}{1 - \cos n\theta}$ (Hint: $1-\cos\theta = 2\sin^2\frac{\theta}{2}$).

(7) $\lim_{x \to -6} \frac{2x+12}{|x+6|}$.

(17) 28-11-20

1.3. Continuity.

A function f is said to be Continuous at a point c where $a < c < b$. if $\lim_{x \to c} f(x) = f(c)$, f is well defined at $x = c$ ie $f(c)$ exist, $\lim_{x \to c} f(x)$ exist.

Continuity in an Interval.

A function f is said to be Continuous in an Interval $[a,b]$ if it is Continuous at each and every point of the Interval.

Discontinuous function.

A function f is said to be discontinuous at a point c if it is not Continuous at c.

Ex: ① Every Constant function $f(x) = c$ is Continuous.

② Every polynomial is Continuous.

③ The exponential function is Continuous. (e^x, e^{-2x} ...)

④ The logarithmic function $f(x) = \log x$ is Continuous.

⑤ Trigonometric function $\sin 2x$, $\cos x$ is Continuous.

(18)

Ex:1 $f(x) = \begin{cases} \frac{x^3-8}{x^2-4} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} \frac{(2-h)^3 - 8}{(2-h)^2 - 4}$$

$$= \lim_{h \to 0} \frac{8 - h^3 - 12h + 6h^2 - 8}{4 - 4h + h^2 - 4}$$

$$= \lim_{h \to 0} \frac{h(6h - h^2 - 12)}{h(h-4)}$$

$$= \frac{-12}{-4} = 3.$$

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} \frac{(2+h)^3 - 8}{(2+h)^2 - 4}$$

$$= \lim_{h \to 0} \frac{8 + h^3 + 12h + 6h^2 - 8}{4 + h^2 + 4h - 4}$$

$$= \lim_{h \to 0} \frac{h(h^2 + 6h + 12)}{h(h+4)}$$

$$= \frac{12}{4} = 3$$

$\therefore$ f is Continuous at $x = 2$.

Ex2 $f(x) = x - |x|$ at $x = 0$ verify the Continuity.

$x < 0, \quad |x| = -x$

$x > 0 \quad |x| = x.$

$$\lim_{x \to 0^-} f(x) = x - |x| = x - (-x) = x + x$$

$$= \lim_{x \to 0^-} 2x = 0$$

(19)

$\lim_{x \to 0^+} f(x) = x - |x| = x - x = 0$

$\therefore$ f is Continuous at $x = 0$.

Ex3. $f(x) = \begin{cases} 2x & \text{when } 0 \le x < 1 \\ 3 & \text{when } x = 1 \\ 4x & \text{when } 1 < x \le 2 \end{cases}$

Check f is Continuous at $x = 1$

$\lim_{x \to 1^-} f(x) = 2x = 2$

$\lim_{x \to 1^+} f(x) = 4x = 4$

$\therefore \lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$.

$\therefore$ f is discontinuous at $x = 1$.

Ex4 $f(x) = \begin{cases} -x^2 & x \le 0 \\ 5x - 4 & 0 < x \le 1 \\ 4x^2 - 3x & 1 < x \le 2 \\ 3x + 4 & x \ge 2 \end{cases}$

Check f is Continuous at $x = 0, 1, 2$.

Soln

$x = 0$. $\lim_{x \to 0^-} f(x) = -x^2 = 0$

$\lim_{x \to 0^+} f(x) = 5(0) - 4 = -4$

$\lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$.

$\therefore$ f is not Continuous at $x = 0$

$x = 1$. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 5x - 4 = 5(1) - 4 = 1$

(20)

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4x^2 - 3x = 4(1) - 3(1) = 1$$

$\therefore$ f is Continuous at $x = 1$.

$x = 2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 4x^2 - 3x$

$$= 4(2)^2 - 3(2)$$

$$= 16 - 6 = 10$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3x + 4$$

$$= 3(2) + 4 = 6$$

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$\therefore$ f is Continuous at $x = 2$.

Ex: 6. $f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

whether f is Continuous at $x = 0$

Soln:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x \sin\frac{1}{x}$$

$$= \lim_{h\to 0} (0-h) \sin\frac{1}{0-h} = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} (h+0) \sin\left(\frac{1}{h+0}\right)$$

$$= \lim_{h\to 0} h \sin\frac{1}{h} = 0$$

$$\therefore \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$

f is Continuous at $x=0$.

Ex: 6. $f(x) = \frac{x^2-9}{x^2-5x+6}$

$= \frac{x^2-9}{(x-2)(x-3)}$

f is discontinuous at $x=2, 3$.

Ex: 7 $f(x) = \sqrt{9-x^2}$ on the interval $[-3,3]$

$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{9-x^2}$

$= \sqrt{9-c^2} = f(c)$

f is Continuous at each pt $(-3,3)$

$\lim_{x\to 3^-} f(x) = \sqrt{9-3^2} = 0$

$\lim_{x\to 3^+} f(x) = \sqrt{9-3^2} = 0.$

f is Cont at $x=3$, Similarly

$x=-3$, $\therefore$ f is Continuous on $[-3,3]$

Ex: 7. $f(x) = \begin{cases} \frac{\sin 2x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} 2\frac{\sin 2x}{2x} = 2(1) = 2$

$f(0) = 1$

$\therefore \lim_{x\to 0} f(x) \neq f(0).$

f is not continuous at $x=0$.

$$f(t) = \begin{cases} \dfrac{\cos t}{\frac{\pi}{2}-t} & t \neq \frac{\pi}{2} \\ 1 & t = \frac{\pi}{2} \end{cases}$$

$$f(t) = \frac{\cos t}{\frac{\pi}{2}-t}$$

$$\lim_{t \to \frac{\pi}{2}^-} f(t) = \lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2}-h\right)}{\frac{\pi}{2}-\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\lim_{t \to \frac{\pi}{2}^+} f(t) = \lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)}{\frac{\pi}{2}-\left(\frac{\pi}{2}+h\right)}$$

$$= \lim_{h \to 0} \frac{-\sin h}{-h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

$$\therefore \lim_{t \to \frac{\pi}{2}^-} f(t) = \lim_{t \to \frac{\pi}{2}^+} f(t)$$

$\therefore$ f is continuous at $t = \frac{\pi}{2}$

Ex: $$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & x<0 \\ a & x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & x>0 \end{cases}$$

Determine a if f is continuous at $x=0$

(23)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\left(\sqrt{16+\sqrt{x}}+4\right)}{\left(\sqrt{16+\sqrt{x}}-4\right)\left(\sqrt{16+\sqrt{x}}+4\right)}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\left(\sqrt{16+\sqrt{x}}+4\right)}{16+\sqrt{x}-16}$$

$$= \sqrt{16}+4 = 8$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{x^2}$$

$$= \lim_{x\to 0^-} \frac{2\sin^2 2x}{x^2}$$

$$= \lim_{x\to 0^-} \frac{8\sin^2 2x}{(2x)^2}$$

$$= 8 \lim_{x\to 0^-} \left(\frac{\sin 2x}{2x}\right)^2$$

$$= 8$$

hence $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

$\therefore \boxed{8 = a}$

Find the value of a, b so that the function f given by

$$f(x) = \begin{cases} 1 & x \le 3 \\ ax+b & 3 < x < 5 \\ 7 & x \ge 5 \end{cases}$$

24

f is Continuous at $x=3$ & $x=5$

at $x=3$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$1 = ax+b.$$

$$x=3, \quad 3a+b=1 \quad \text{—①}$$

$$x=5. \quad \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$$

$$5a+b = 7 \quad \text{—②}$$

Solve ① & ②

$$\therefore \quad 5a+b=7$$
$$3a+b=1$$
$$2a = 6$$

$$\boxed{a=3}$$

$$3(3)+b=1$$
$$b = 1-9 = -8$$

$$\therefore \quad a=3, \; b=-8$$

A.U. 2017. Find a and b

$$f(x) = \begin{cases} x+2 & x<2 \\ ax^2-bx+3, & 2\le x<3 \\ 2x-a+b & x\geq 3 \end{cases}$$

Continuous everywhere.

$x=2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$. since

f is Continuous at $x=2$.

(26)

$$2+2 = a(2^2) - b(2) + 3$$

$$4 = 4a - 2b + 3$$

$$4a - 2b = 1 \quad \text{—} ①$$

f is Continuous at $x = 3$.

$$\therefore \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x).$$

$$a(3^2) - 3b + 3 = 2(3) - a + b$$

$$9a - 3b + 3 = 6 - a + b$$

$$10a - 4b = 3 \quad \text{—} ②$$

Solving ① and ②

$$4a - 2b = 1 \times 2$$
$$10a - 4b = 3$$

$$8a - 4b = 2$$
$$10a - 4b = 3$$
$$\underline{\;-\qquad +\qquad -\;}$$
$$-2a = -1$$

$$a = \frac{-1}{-2} = \frac{1}{2}$$

Sub in ①

$$4\left(\tfrac{1}{2}\right) - 2b = 1$$

$$2 - 2b = 1$$

$$-2b = 1 - 2$$

$$-2b = -1$$

$$b = \frac{1}{2}$$

$$\therefore a = \frac{1}{2},\ b = \frac{1}{2}.$$

30-11-20

1.4. Derivatives.

$y = f(x)$. The differentiation of y w.r.t. x. defined by $f'(x)$ or $\frac{dy}{dx}$.

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

<u>Ex: 1</u> $y = x^n$ find $\frac{dy}{dx}$.

$$y + \Delta y = (x+\Delta x)^n$$

$$\Delta y = (x+\Delta x)^n - y$$

$$\Delta y = \lim (x+\Delta x)^n - x^n.$$

$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{(x+\Delta x) - x}$$

$$\frac{dy}{dx} = n x^{n-1}$$

Ex 2 $y = e^x$

$$y + \Delta y = e^{x+\Delta x}$$

$$\Delta y = e^{x+\Delta x} - e^x$$

$$\frac{\Delta y}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x (e^{\Delta x} - 1)}{\Delta x}$$

$$= e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x}$$

$$\frac{dy}{dx} = e^x \cdot 1.$$

27

③ $y = \sin x$

$y + \Delta y = \sin(x + \Delta x)$

$\Delta y = \sin(x+\Delta x) - \sin x$

$= 2 \sin \frac{\Delta x}{2} \cos\left(x + \frac{\Delta x}{2}\right)$ $\quad \sin C - \sin D = 2 \sin \frac{C-D}{2} \cos \frac{C+D}{2}$

$$\frac{\Delta y}{\Delta x} = \frac{2 \sin \frac{\Delta x}{2} \cdot \cos\left(x + \frac{\Delta x}{2}\right)}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos\left(x + \frac{\Delta x}{2}\right)$$

$$= 1 \cdot \cos x$$

$\therefore \frac{dy}{dx} = \cos x.$

Assignment problems.

find $\frac{dy}{dx}$ for the functions

$\cos x$, $\tan x$, $\log x$.

Differentiate the following

① $y = x^5 + 4x^4 + 7x^3 + 6x^2 - 11x + 12$

$$\frac{dy}{dx} = 5x^{5-1} + 4(4x^{4-1}) + 7(3x^{3-1}) + 6(2x^{2-1}) - 11 + 0$$

$$= 5x^4 + 16x^3 + 21x^2 + 12x - 11$$

(2) $y = \dfrac{x^2 + x + 1}{\sqrt{x}}$

$= x^{\frac{3}{2}} + \sqrt{x} + x^{-\frac{1}{2}}$

$y = x^{\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = \frac{3}{2} x^{\frac{3}{2}-1} + \frac{1}{2} x^{\frac{1}{2}-1} - \frac{1}{2} x^{-\frac{1}{2}-1}.$

$= \frac{3}{2} x^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}}$

$= \frac{3}{2}\sqrt{x} + \frac{1}{2\sqrt{x}} - \frac{1}{2} \frac{1}{x\sqrt{x}}.$

(3) $y = \left(\frac{1}{x} - x\right)^3$

$= \frac{1}{x^3} - x^3 - 3\left(\frac{1}{x} - x\right)$

$y = \frac{1}{x^3} - x^3 - \frac{3}{x} + 3x$

$\dfrac{dy}{dx} = -3x^{-3-1} - 3x^2 - 3\left(-\frac{1}{x^2}\right) + 3.$

$= -\frac{3}{x^4} - 3x^2 + \frac{3}{x^2} + 3.$

(4) $y = \sqrt{2}\sin x + 4x^5 - \frac{3}{x^4}$

$\dfrac{dy}{dx} = \sqrt{2}\cos x + 20x^4 - 3(-4)x^{-4-1}.$

$= \sqrt{2}\cos x + 20x^4 + \frac{12}{x^5}$

(5) $y = \log x + e^x$

$\dfrac{dy}{dx} = \frac{1}{x} + e^x$

(29)

Differentiation Rules.

① Product rule.

$y = uv$ where u, v are functions of x alone.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\cdot\frac{du}{dx}.$$

If $y = uvw$, then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + u\frac{dv}{dx}w + \frac{du}{dx}vw.$$

② Quotient Rule.

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\cdot\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

ex:

$$y = e^x \sin x \log x$$

$$\frac{dy}{dx} = e^x \sin x \log x + e^x \log x \cos x + \frac{e^x \sin x}{x}$$

ex

$$y = x^2 e^x \sin x$$

$$\frac{dy}{dx} = 2x e^x \sin x + x^2 e^x \sin x + x^2 e^x \cos x.$$

ex

$$y = (x^2-1)(x^2+2)$$

$$\frac{dy}{dx} = (x^2-1)(2x) + (x^2+2)(2x)$$

$$= 2x[x^2-1+x^2+2]$$

$$= 2x(2x^2+1)$$

ex:

$$y = \frac{x^2 + e^x \sin x}{\cos x + \log x}$$

$$\frac{dy}{dx} = \frac{(\cos x + \log x)(2x + e^x \sin x + e^x \cos x) - (x^2 + e^x \sin x)(-\sin x + \frac{1}{x})}{(\cos x + \log x)^2}$$

ex:

$$y = \frac{\tan x + 1}{\tan x - 1}$$

$$\frac{dy}{dx} = \frac{(\tan x - 1)(\sec^2 x) - (\tan x + 1)(\sec^2 x)}{(\tan x - 1)^2}$$

$$= \frac{\tan x \sec^2 x - \sec^2 x - \tan x \sec^2 x - \sec^2 x}{(\tan x - 1)^2}$$

$$= \frac{-2\sec^2 x}{(\tan x - 1)^2}$$

ex

$$y = \frac{\log x^2}{e^x}$$

$$\frac{dy}{dx} = \frac{e^x \frac{1}{x^2}(2x) - (\log x^2) e^x}{(e^x)^2}$$

$$= \frac{\frac{2e^x}{x} - e^x \log x^2}{e^{2x}}$$

$$= \frac{e^x\left[\frac{2}{x} - \log x^2\right]}{e^{2x}}$$

$$= \frac{\frac{2}{x} - \log x^2}{e^x}.$$

30/11/20

31

Ex: If $f(x) = \left(\frac{1}{x^2} - \frac{3}{x^4}\right)(x + 5x^3)$

$f'(x) = \left(\frac{1}{x^2} - \frac{3}{x^4}\right)(1 + 15x^2) +$

$(x + 5x^3)\left(-\frac{2}{x^3} + \frac{12}{x^5}\right)$

$= \frac{1}{x^2} + 15x^2\left(\frac{1}{x^2}\right) - \frac{3}{x^4} - \frac{45x^2}{x^4} - 2x\left(\frac{1}{x^3}\right)$

$+ \frac{12x}{x^5} - 10x^3\frac{1}{x^3} + 60x^3\left(\frac{1}{x^5}\right)$

$= \frac{1}{x^2} + 15 - \frac{3}{x^4} - \frac{45}{x^2} - \frac{2}{x^2} + \frac{12}{x^4}$

$-10 + \frac{60}{x^2}$

$= \frac{9}{x^4} + \frac{14}{x^2} + 5.$

Ex: $f(x) = \left(x^2 + \frac{1}{x^2}\right)\tan x$

$f'(x) = \left(x^2 + \frac{1}{x^2}\right)\frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}\left(x^2 + \frac{1}{x^2}\right)$

$= \left(x^2 + \frac{1}{x^2}\right)\sec^2 x + \tan x\left(2x - \frac{2}{x^3}\right)$

Ex: $f(x) = \frac{1 - xe^x}{x + e^x}$ find $f'(x)$

$f'(x) = \frac{(x+e^x)(0 - xe^x - e^x) - (1 - xe^x)(1 + e^x)}{(x+e^x)^2}$

$= \frac{-x^2e^x - xe^x - xe^{2x} - e^{2x} - (1 - xe^x + e^x - xe^{2x})}{(x+e^x)^2}$

$$-\frac{(x^2 e^x + e^{2x} + e^x + 1)}{(x + e^x)^2}$$

Chain Rule.

Suppose $y = f(u)$, $u = f(v)$, $v = \phi(x)$.

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$.

ex: $y = \sin(2x+3)$

Soln: put $u = 2x+3$. $\frac{du}{dx} = 2$.

$\therefore y = \sin u$

$\frac{dy}{du} = \cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$= \cos u \cdot 2$

$= \cos(2x+3)2$

$= 2\cos(2x+3)$

ex: $y = \sqrt[4]{x^3+2x+1} = (x^3+2x+1)^{1/4}$

Soln: $u = x^3+2x+1$

$\frac{du}{dx} = 3x^2+2$.

$y = u^{1/4}$

$\frac{dy}{du} = \frac{1}{4} u^{1/4 - 1} = \frac{1}{4} u^{-\frac{3}{4}}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$= \frac{1}{4} \cdot \frac{1}{u^{3/4}} (3x^2+2)$

33

$$f'(x) = \frac{3x^2+2}{4(x^3+2x+1)^{\frac{3}{4}}}$$

$$y = \tan(\sin x)$$

$$\frac{dy}{dx} = \sec^2(\sin x).\cos x.$$

$$y = \log(2+\sin x).$$

$$\frac{dy}{dx} = \frac{1}{2+\sin x}(\cos x)$$

$$= \frac{\cos x}{2+\sin x}.$$

$$y = \sin(\cos\tan x)$$

$$\frac{dy}{dx} = \cos(\cos\tan x)(-\sin(\tan x)\sec^2 x$$

$$= -\cos(\cos\tan x)\sin(\tan x)\sec^2 x$$

$$y = \log(x+\sqrt{x^2-1})$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2-1}}\left[1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-1}}\right]$$

$$= \frac{1}{(x+\sqrt{x^2-1})}\frac{(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}.$$

$$y = \sin^{-1}\sqrt{\sin x}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{\sin x})^2}} \cdot \frac{1}{2\sqrt{\sin x}} \cdot \cos x$$

$$= \frac{\cos x}{2\sqrt{\sin x}\sqrt{1-\sin x}}.$$

Assignment Problems.

① $y = \sin(\cos\tan x)$ find $\frac{dy}{dx}$.

② $y = \cos^{-1} e^{2x}$ find $\frac{dy}{dx}$.

Implicit Differentiation.

If x, y are connected by a relation $f(x,y) = C$. Then it is called as Implicit function. Differentiate the equation w.r.t. x. and solve the equation for $\frac{dy}{dx}$.

Ex: If $\sqrt{x} + \sqrt{y} = 1$ then find $\frac{dy}{dx}$.

Soln: given $\sqrt{x} + \sqrt{y} = 1$

Differentiate w.r.t. x.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0.$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Ex: $x^3 + y^3 = 3axy$.

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3a\left[x \cdot \frac{dy}{dx} + y\right]$$

$$x^2 + y^2 \frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Ex: $\sin(x+y) = y^2 \cos x$

Differentiate w.r.t. x.

$$\cos(x+y)\left[1 + \frac{dy}{dx}\right] = y^2(-\sin x) + \cos x(2y)\frac{dy}{dx}$$

$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = -y^2 \sin x + 2y\cos x \frac{dy}{dx}$$

$$\left[\cos(x+y) - 2y\cos x\right]\frac{dy}{dx} = -y^2 \sin x - \cos(x+y)$$

$$\frac{dy}{dx} = -\frac{\left[y^2 \sin x + \cos(x+y)\right]}{\cos(x+y) - 2y\cos x}$$

$$\frac{dy}{dx} = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Exercise ① 98 $x\cos y + y\cos x = 1$ find $\frac{dy}{dx}$

② $x^4 + y^4 = 4a^2 xy$ find $\frac{dy}{dx}$.

③ $xy = \sin(x^2 + y^2)$ find $\frac{dy}{dx}$

(35)

A.U. J. 2018 January. Find y'' if $x^4+y^4=16$ 8-12-20 (36)

$$4x^3+4y^3\cdot\frac{dy}{dx}=0$$

$\div$ by 4

$$x^3+y^3\frac{dy}{dx}=0$$

$$y^3\frac{dy}{dx}=-x^3$$

$$\frac{dy}{dx}=-\frac{x^3}{y^3}$$

$$\therefore \frac{d^2y}{dx^2}=-\left[\frac{y^3(3x^2)-x^3(3y^2)\cdot\frac{dy}{dx}}{(y^3)^2}\right]$$

$$=-\frac{1}{y^6}\left[3x^2y^3-3x^3y^2\left(-\frac{x^3}{y^3}\right)\right]$$

$$=-\frac{1}{y^6}\left[3x^2y^3+3\frac{x^6}{y}\right]$$

$$=-\frac{1}{y^7}\left[3x^2y^4+3x^6\right]$$

$$=-\frac{3x^2}{y^7}\left[y^4+x^4\right]$$

$$=-\frac{3x^2}{y^7}(16)$$

$$=-\frac{48x^2}{y^7}$$

Derivative of Trigonometric function.

$y=\sin^{-1}x$ find $\frac{dy}{dx}$.

$$y=\sin^{-1}x$$

$$x=\sin y.$$

Differentiating w.r.t. x.

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

hence $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Exercise find derivative of $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$.

Find the derivative of $\sec^{-1}x$.

$$y = \sec^{-1}x.$$

$$\sec y = x$$

differentiating.

$$\sec y \tan y \cdot \frac{dy}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \frac{1}{\sec y\sqrt{\sec^2 y - 1}} \quad (\because 1+\tan^2 y = \sec^2 y$$

$$\frac{dy}{dx}. = \frac{1}{x\sqrt{x^2-1}}.$$

Find the derivative of $\operatorname{cosec}^{-1}x$.

98 $y = e^{\tan^{-1}\frac{x}{\sqrt{1-x^2}}}$, find $\frac{dy}{dx}$.

Soln First take $\frac{x}{\sqrt{1-x^2}}$

put $x = \sin\theta$. —(1)

$$\frac{x}{\sqrt{1-x^2}} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\therefore \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \tan^{-1}(\tan\theta)$$

$$= \theta \cdot = \sin^{-1}x \text{ by } (1).$$

$$\therefore y = e^{\sin^{-1}x}.$$

$$\frac{dy}{dx} = e^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

Q7 $f(x) = \tan^{-1}\frac{a-x}{1+ax}$ find the derivative of $f(x)$.

Soln Let $\frac{a-x}{1+ax} = u$

$$\therefore f(x) = \tan^{-1}u$$

$$f'(x) = \frac{1}{1+u^2}\cdot\frac{du}{dx}.$$

$$= \frac{1}{1+\left(\frac{a-x}{1+ax}\right)^2}\cdot\frac{(1+ax)(-1)-(a-x)(a)}{(1+ax)^2}.$$

$$= \frac{1}{\frac{(1+ax)^2+(a-x)^2}{(1+ax)^2}}\frac{[-1-ax-a^2+ax]}{(1+ax)^2}.$$

$$= \frac{(1+ax)^2[-(1+a^2)]}{1+a^2x^2+2ax+a^2+x^2-2ax}\cdot\frac{1}{(1+ax)^2}$$

$$= \frac{-(1+a^2)}{(1+a^2)+x^2(1+a^2)}$$

$$= -\frac{(1+a^2)}{(1+a^2)(1+x^2)}$$

$$= -\frac{1}{1+x^2}.$$

Assignment.

① $f(x) = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$ find $f^1(x)$.

② $f(x) = \frac{1+\cos x}{1-\cos x}, \frac{1+\sin x}{1-\sin x}$ find $f^1(x)$

Derivative of logarithmic function.

Basic rules in logarithms

① $\log a^b = b\log a,$

② $\log ab = \log a + \log b$

③ $\log\frac{a}{b} = \log a - \log b.$

$y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$ find $\frac{dy}{dx}$.

Taking logarithm on both sides.

$$\log y = \log \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}.$$

$$\log y = \log x^{\frac{3}{4}} + \log(x^2+1)^{\frac{1}{2}} - \log(3x+2)^5$$

$$= \frac{3}{4}\log x + \frac{1}{2}\log(x^2+1) - 5\log(3x+2)$$

Differentiate

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\frac{(2x)}{x^2+1} - \frac{5(3)}{3x+2}$$

$$\frac{dy}{dx} = y\left[\frac{3}{4x} + \frac{1}{x^2+1} - \frac{15}{3x+2}\right]$$

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

$y = \log(x+\sqrt{x^2-1})$ find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2-1}}\left[1 + \frac{(2x)}{2\sqrt{x^2-1}}\right]$$

$$= \frac{1}{x+\sqrt{x^2-1}}\,\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}.$$

$$= \frac{1}{\sqrt{x^2-1}}.$$

Find $\frac{dy}{dx}$ where ① $y = \log(e^{-x} + xe^{-x})$

② $y = \log x\sqrt{x^2-1}$

---

③ If $f(x) = \frac{\cos^m x \cdot \sin^n x}{\cosh^2 x}$. find the derivative of $f(x)$.

④ If $y = e^{ax}\cos^3 x \sin^2 x$ find $\frac{dy}{dx}$.

(H1) Hyperbolic Functions. 3-12-20

① $\sinh x = \frac{e^{x} - e^{-x}}{2}$

② $\cosh x = \frac{e^{x} + e^{-x}}{2}$

③ $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{x}}$

④ $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sin x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$.

⑤ $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$

⑥ $\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$.

Formulae in hyperbolic functions.

① $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

② $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

③ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

④ $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

⑤ $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

⑥ $\cosh^2 x - \sinh^2 x = 1$

⑦ $\operatorname{sech}^2 x + \tanh^2 x = 1$

⑧ $\coth^2 x - \operatorname{cosech}^2 x = 1$

⑨ $\cosh^2 x + \sinh^2 x = \cosh 2x$

⑩ $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$

Inverse Hyperbolic function.

① $\sinh^{-1} x = \log(x + \sqrt{x^2 - 1})$

<u>Soln:</u>

$$y = \sinh^{-1} x$$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - \frac{1}{e^y}$$

$$2x = \frac{e^{2y} - 1}{e^y}$$

$$2x e^y = e^{2y} - 1$$

$$e^{2x} - 2x e^y - 1 = 0$$

$$e^{2x} - 2x e^y + x^2 = x^2 + 1$$

$$(e^y - x)^2 = x^2 + 1$$

$$e^y - x = \pm \sqrt{x^2 + 1}$$

$$e^y = x + \sqrt{x^2 + 1}$$ e^y is always positive.

$$\therefore y = \log(x + \sqrt{x^2 + 1})$$

② $y = \cosh^{-1} x = \log(x + \sqrt{x^2 - 1})$

Soln:

$$y = \cosh^{-1} x$$

$$x = \cosh y$$

$$x = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y}$$

$$2x = \frac{e^{2y} + 1}{e^y}$$

(43)

$$2xe^y = e^{2y}+1$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^{2y} - 2xe^y + x^2 = x^2 - 1$$

$$(e^y)^2 - 2xe^y + x^2 = x^2 - 1$$

$$(e^y - x)^2 = x^2 - 1$$

$$e^y - x = \pm\sqrt{x^2-1}$$

$$e^y = x + \sqrt{x^2-1}$$ (e^y is always positive)

$$y = \log(x + \sqrt{x^2-1})$$

② Find the value of $\tanh^{-1} x$.

<u>Soln</u>

$$y = \tanh^{-1} x$$

$$\tanh y = x$$

$$x = \frac{\sinh y}{\cosh y}$$

$$= \frac{e^y - e^{-y}}{2} \cdot \frac{2}{e^y + e^{-y}}$$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$x(e^y + e^{-y}) = e^y - e^{-y}$$

$$xe^y + xe^{-y} = e^y - e^{-y}$$

$$xe^{-y} + e^{-y} = e^y - xe^y$$

$$e^{-y}(1+x) = e^y(1-x)$$

$$e^y(1-x) = \frac{1+x}{e^y}$$

$$e^{2y} = \frac{1+x}{1-x}$$

$$2y = \log \frac{1+x}{1-x}$$

$$y = \frac{1}{2} \log \frac{1+x}{1-x}.$$

$$\therefore \tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}.$$

Differentiation of Hyperbolic and Inverse hyperbolic functions.

(1) $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\frac{dy}{dx} = \frac{1}{2}\left[e^x - e^{-x}(-1)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[e^x + e^{-x}\right]$$

$$= \cosh x.$$

Exercise: Find the derivative of $\cosh x$, $\tanh x$, $\coth x$.

Find the derivative of $y = \operatorname{sech} x$

$$y = \operatorname{sech} x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\cosh x}\right)$$

$$= -\frac{1}{\cosh^2 x}(\sinh x)$$

$$= -\frac{\sinh x}{\cosh x} \cdot \frac{1}{\cosh x}$$

$$= -\operatorname{sech} x \tanh x.$$

(45) derive the derivative of $\operatorname{Cosech} x$.

$$\frac{d}{dx}(\operatorname{Cosech} x) = -\operatorname{Cosech} x \operatorname{Coth} x.$$

① $y = \sinh^{-1} x$

$x = \sinh y$

$$\frac{dx}{dy} = \cosh y = \sqrt{1+\sinh^2 y} \quad (\because \cosh^2 x - \sinh^2 x)$$

$$= \sqrt{1+x^2}.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}.$$

② $y = \tanh^{-1} x$ find $\frac{dy}{dx}$

$x = \tanh y$

$$\frac{dx}{dy} = \operatorname{sech}^2 y = 1-\tanh^2 y = 1-x^2.$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}.$$

③ $y = \operatorname{sech}^{-1} x$. find $\frac{dy}{dx}$.

$\operatorname{sech} y = x$

$$\frac{dx}{dy} = -\operatorname{sech} y \tanh y$$

$$= -\operatorname{sech} y \sqrt{1-\operatorname{sech}^2 y}$$

$$\frac{dx}{dy} = -x\sqrt{1-x^2}$$

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}.$$

Exercise: Find the derivative of.

$\cosh^{-1}x$, $\coth^{-1}x$, $\operatorname{cosech}^{-1}x$.

Q. $f(x) = \frac{1-\cosh x}{1+\cosh x}$ find $f'(x)$.

$$f'(x) = \frac{(1+\cosh x)(-\sinh x)-(1-\cosh x)(\sinh x)}{(1+\cosh x)^2}$$

$$= \frac{-\sinh x - \sinh x\cosh x - \sinh x + \sinh x\cosh x}{(1+\cosh x)^2}$$

$$= -\frac{2\sinh x}{(1+\cosh x)^2}.$$

Q. $f(x) = x\sinh^{-1}\frac{x}{3} - \sqrt{x^2+9}$.

$$f'(x) = x.\frac{1}{\sqrt{1+\left(\frac{x}{3}\right)^2}}.\frac{1}{3} + \sinh^{-1}\frac{x}{3} (-) \frac{1(2x)}{2\sqrt{x^2+9}}$$

$$= \sinh^{-1}\frac{x}{3} + \frac{x}{3\sqrt{1+\frac{x^2}{9}}} - \frac{x}{\sqrt{x^2+9}}.$$

$$= \sinh^{-1}\frac{x}{3} + \frac{x(3)}{3\sqrt{9+x^2}} - \frac{x}{\sqrt{x^2+9}}.$$

$$= \sinh^{-1}\frac{x}{3} + \frac{x}{\sqrt{x^2+9}} - \frac{x}{\sqrt{x^2+9}}.$$

$$= \sinh^{-1}\frac{x}{3}.$$

Assignment. Differentiate the following.

① $x\coth(x^2+1)$ ② $\operatorname{sech}^{-1}\sqrt{1-x^2}$.

③ $\tanh^{-1}x + \log\sqrt{1-x^2}$

(47) 5-12-2

16 Maxima and minima of functions of one variable.

Defn: 1 Let c be a number in the Domain D of a function f. Then $f(c)$ is the

<u>absolute maximum</u> value of f on D if $f(c) \geq f(x)$ for all x in D.

<u>absolute minimum</u> value of f on D if $f(c) \leq f(x)$ for all x in D.

Defn: 2. The number $f(c)$ is a

<u>local maximum</u> value of f if $f(c) \geq f(x)$ when x is near c

<u>local minimum</u> value of f if $f(c) \leq f(x)$ when x is near c.

<u>Critical number</u> of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

<u>Defn</u>: If f has a local maximum or minimum at c, then c is a <u>critical number</u> of f.

① Ex: Find the critical value of the function $f(x) = 5x^2 + 4x$.

<u>Soln</u> Given $f(x) = 5x^2 + 4x$.

critical numbers of f occur at $f'(x)=0$ — ①

$f'(x) = 10x+4$

By ① $\therefore$ $10x+4 = 0 \Rightarrow x = -\frac{4}{10} = -\frac{2}{5}$.

$\therefore$ The critical number is $-\frac{2}{5}$.

② Find the critical value of the function $f(x) = x^4 + x^3 + x^2 + 1$

The critical numbers of f occur at $f'(x)=0$ — ①

$f'(x) = 4x^3 + 3x^2 + 2x$

$\Rightarrow 4x^3 + 3x^2 + 2x = 0$ by ①

$\Rightarrow x(4x^2 + 3x + 2) = 0$

$\Rightarrow x = 0, \; 4x^2 + 3x + 2 = 0$

$\Rightarrow x = 0, \; x = \frac{-3 \pm \sqrt{9-4(8)}}{2(4)}$

Formula. $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-3 \pm \sqrt{9-32}}{8}$

$= \frac{-3 \pm \sqrt{-23}}{8} = \frac{-3+\sqrt{23}i}{8}, \frac{-3-\sqrt{23}i}{8}$

$x = 0, \frac{-3+\sqrt{23}i}{8}, \frac{-3-\sqrt{23}i}{8}$

Critical numbers are real numbers only $\therefore$ 0 is the critical number

③ Find the critical values of $x^{3/4} - 2x^{1/4}$

Soln: Critical numbers of f occur at $f'(x) = 0$

(49)

$$f'(x) = \frac{3}{4}x^{\frac{3}{4}-1} - 2\left(\frac{1}{4}\right)x^{\frac{1}{4}-1}$$

$$= \frac{3}{4}x^{-\frac{1}{4}} - \frac{1}{2}x^{-\frac{3}{4}}$$

$$f'(x) = 0 \Rightarrow \frac{3}{4}x^{-\frac{1}{4}} = \frac{1}{2}x^{-\frac{3}{4}}$$

$$x^{-\frac{1}{4}} \cdot x^{\frac{3}{4}} = \frac{4}{6} = \frac{2}{3}$$

$$x^{-\frac{1}{4}+\frac{3}{4}} = \frac{2}{3}$$

$$x^{\frac{1}{2}} = \frac{2}{3}$$

$$x = \frac{4}{9}.$$

$f'(x)$ does not exist when $x = 0$

$\therefore$ critical values are $0, \frac{4}{9}$.

---

(4) Find the critical value of $x^{3/5}(4-x)$.

Soln $f(x) = x^{3/5}(4-x)$

$$= 4x^{3/5} - x^{8/5}$$

$$f'(x) = 4\left(\frac{3}{5}x^{\frac{3}{5}-1}\right) - \frac{8}{5}x^{\frac{8}{5}-1}.$$

$$= \frac{12}{5}x^{-\frac{2}{5}} - \frac{8}{5}x^{3/5}$$

Critical values occur at $f'(x) = 0$

$$\frac{12}{5}x^{-\frac{2}{5}} = \frac{8}{5}x^{3/5},$$

$$\frac{12}{8} = \frac{x^{3/5}}{x^{-2/5}} = x^{\frac{3}{5}} \cdot x^{\frac{2}{5}} = x$$

$$\therefore x = \frac{3}{2}$$

$f'(x)$ does not exist $x = 0$.

The critical values are $0, \frac{3}{2}$.

Absolute maximum and absolute minimum value of a continuous function on closed interval $[a,b]$

1. Find the derivative of f in (a,b)
2. Find the critical points of f in (a,b)
3. Find the values of f at the critical points of f in (a,b).
4. Find the values of f at the end points of the interval $[a,b]$
5. The largest value is absolute maximum the smallest value is absolute minimum.

Ex: Find the absolute maximum and absolute minimum value of the function $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1,4]$

Soln: The given function $f(x) = 3x^4 - 16x^3 + 18x^2$ is continuous on $[-1,4]$

$$f'(x) = 12x^3 - 48x^2 + 36x$$

The critical pts of $f(x)$ are

$$f'(x) = 0$$

$$12x^3 - 48x^2 + 36x = 0$$

$$12x(x^2 - 4x + 3) = 0.$$

$x(x-1)(x-3) = 0$

$x = 0, 1, 3$ critical values.

The values of $f(x)$ at these critical points are:

$f(0) = 3(0)^4 - 16(0)^3 + 18(0)^2 = 0$

$f(1) = 3(1)^4 - 16(1)^3 + 18(1)^2 = 5$

$f(3) = 3(3)^4 - 16(3)^3 + 18(3)^2$

$= -27$

The value of $f(x)$ at end pts of $[-1, 4]$

$f(-1) = 3(-1)^4 - 16(-1)^3 + 18(-1)^2$

$= 3 + 16 + 18 = 37$

$f(4) = 3(4)^4 - 16(4)^3 + 18(4)^2$

$= 768 - 1024 + 288$

$= 32$

The absolute maximum value is

$f(-1) = 37$

The absolute minimum value is $f(3) = -27$.

A.U. AP 2019 Find critical pts of $y = 5x^3 - 6x$

(52)

Find the absolute maximum and absolute minimum value of $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

Soln The given function $f(x) = 2x^3 - 3x^2 - 12x + 1$ is continuous on $[-2, 3]$

The critical points of $f(x)$ are

$$f'(x) = 0$$

$$6x^2 - 6x - 12 = 0$$

$$6(x^2 - x - 2) = 0$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1, 2.$$

The values of $f(x)$ at critical points

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$$

$$= -2 - 3 + 12 + 1 = 8$$

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$$

$$= 16 - 12 - 24 + 1$$

$$= -19$$

The values of $f(x)$ at end points are. $f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1$

$$= 2(-8) - 12 + 24 + 1$$

$$= -16 - 12 + 25 = 3$$

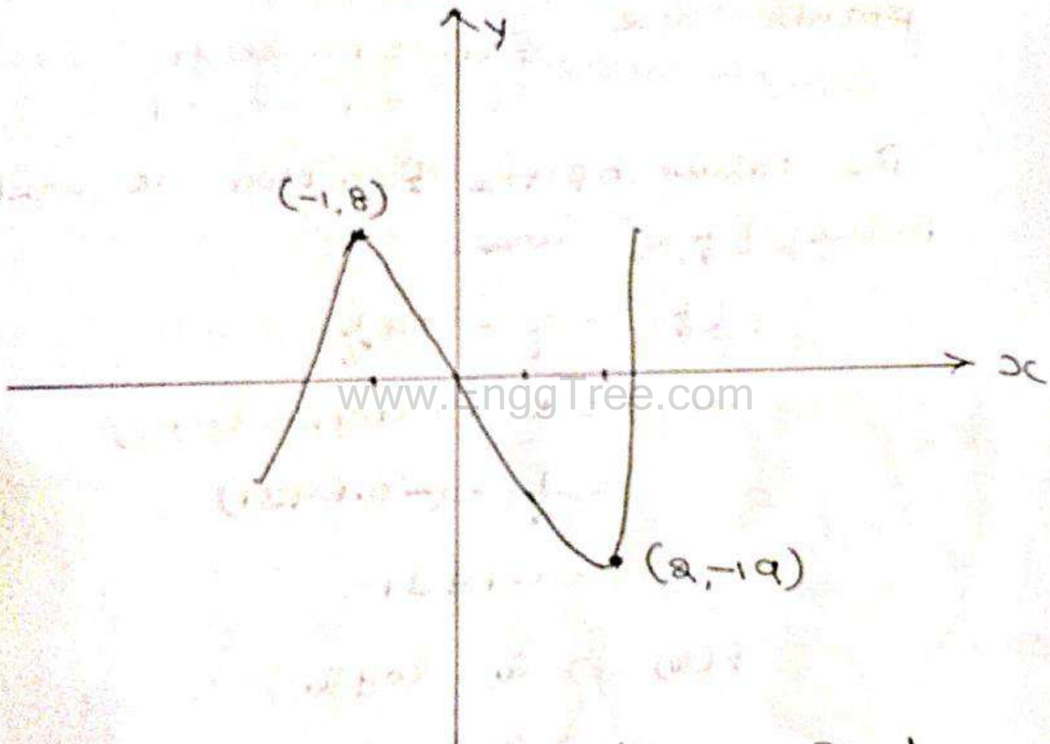
$$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 1$$
$$= 54 - 27 - 36 + 1$$
$$= -8$$

The absolute minimum value is

$f(2) = -19$.

The absolute maximum value is

$f(-1) = 8$



Find the absolute maximum and absolute minimum value of $f(x) = x - \log x$ on $[\frac{1}{2}, 2]$

<u>Soln:</u> $f(x) = x - \log x$ is Continuous on $[\frac{1}{2}, 2]$ and

$f'(x) = 1 - \frac{1}{x}$.

The critical values of $f(x)$ are $f'(x) = 0$.

$$1 - \frac{1}{x} = 0$$

$$\frac{x-1}{x} = 0$$

$$\therefore x = 1$$

The values of $f(x)$ at these critical points are $f(1) = 1 - \log 1$

$$= 1 - 0 = 1$$

The values of the function at end points $[\frac{1}{2}, 2]$ are

$$f(\tfrac{1}{2}) = \frac{1}{2} - \log \frac{1}{2}$$

$$= \frac{1}{2} - (\log 1 - \log 2)$$

$$= \frac{1}{2} - (-0.6931)$$

$$= 1.1931$$

$$f(2) = 2 - \log 2$$

$$= 2 - 0.6931$$

$$= 1.3068$$

The absolute maximum value is $f(2) = 1.3068$

The absolute minimum value is $f(1) = 1$.

(53) 7-12-20.

Increasing / Decreasing Test.

(a) If $f'(x) > 0$ on an Interval, then f is Increasing on that Interval.

(b) If $f'(x) < 0$ on an Interval, then f is decreasing on that Interval.

Ex: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and when it is decreasing.

Soln: The given function is

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0.$$

$$12x(x^2 - x - 2) = 0$$

$$12x(x+1)(x-2) = 0$$

The critical pts are the values of at $f'(x) = 0$

$\therefore x = 0, -1, 2$ are critical points.

we divide the real line into Intervals whose end points are the critical points $x = -1, 0, 2$. we list as a table.

Interval	$12x$	$x+1$	$x-2$	$f'(x)$	$f(x)$
$x<-1$	$-$	$-$	$-$	$-$	decreasing
$-1<x<0$	$-$	$+$	$-$	$+$	Increasing
$0<x<2$	$+$	$+$	$-$	$-$	decreasing
$x>2$	$+$	$+$	$+$	$+$	Increasing

First Derivative Test

Suppose c is a critical number of a continuous function f.

(i) If f' changes from positive to negative at c, then f has a local maximum at c

(ii) If f' changes from negative to positive at c, then f has a local minimum at c.

(iii) If f' does not change sign at c, then f has no local maximum or minimum at c.

Second derivative Test

Suppose f'' is continuous near c.

(i) If $f'(c)=0$ and $f''(c)>0$, then f has local minimum at c.

(ii) If $f'(c)=0$ and $f''(c)<0$ then f has local maximum at c.

A.U. 2019.

If $f(x) = 2x^3 + 3x^2 - 36x$. Find the intervals on which is increasing or decreasing, the local maximum and local minimum of the function $f(x)$.

Soln: The given function is

$$f(x) = 2x^3 + 3x^2 - 36x$$

Diff. w.r.t. x

$$f'(x) = 2(3x^2) + 3(2x) - 36$$

$$= 6x^2 + 6x - 36$$

$$f'(x) = 0 \Rightarrow 6x^2 + 6x - 36 = 0$$

$$6(x^2 + x - 6) = 0$$

$$6(x+3)(x-2) = 0$$

$\therefore x = -3, 2$ are the critical points of $f(x)$.

Divide the real line into intervals whose end points are the critical points $x = -3, 2$

Interval	$6(x+3)$	$(x-2)$	$f'(x)$	$f(x)$
$x < -3$	$-$	$-$	$+$	Increasing
$-3 < x < 2$	$+$	$-$	$-$	decreasing
$x > 2$	$+$	$+$	$+$	Increasing.

$f(x)$ changes from increasing to decreasing at $x=-3$, then the function has a local maximum at $x=-3$, the local maximum value is

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3)$$
$$= 2(-27) + 3(9) + 108$$

local max. value $= -54 + 27 + 108 = 81$

$f(x)$ changes from decreasing to increasing at $x=2$. Thus the function has a local minimum at $x=2$ and the local minimum value is

$$f(2) = 2(2)^3 + 3(2)^2 - 36(2)$$
$$= 2(8) + 3(4) - 72$$
$$= 16 + 12 - 72$$

local min. value $= -44$.

Concavity Test

(i) If $f''(x) > 0$ for all in I f is Concave upward on I

(ii) If $f''(x) < 0$ for all in I then f is Concave downward on I.

For the above problem $f(x) = 2x^3 + 3x^2 - 36x$ Find the Intervals of Concavity and the Inflexion points.

Soln For Concavity.

$$f''(x) = 0.$$

$$f(x) = 2x^3 + 3x^2 - 36x$$

$$f'(x) = 6x^2 + 6x - 36$$

$$f''(x) = 12x + 6.$$

$$f''(x) = 0 \Rightarrow 12x + 6 = 0$$

$$x = -\frac{6}{12} = -\frac{1}{2}.$$

Divide the real line into Intervals whose end points are the critical points $x = -\frac{1}{2}$. Form a Table.

Interval	$f''(x)$	Concavity
$x < -\frac{1}{2}$	$-$	downward
$x > -\frac{1}{2}$	$+$	upward

The curve changes concave downward to upward at $x = -\frac{1}{2}$.

The point of Inflexion is

$$\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right)$$

$$f\left(-\tfrac{1}{2}\right) = 2\left(-\tfrac{1}{2}\right)^3 + 3\left(-\tfrac{1}{2}\right)^2 - 36\left(-\tfrac{1}{2}\right)$$

$$= 2\left(-\tfrac{1}{8}\right) + 3\left(\tfrac{1}{4}\right) + \frac{36}{2}$$

$$= -\frac{1}{4} + \frac{3}{4} + 18$$

$$= \frac{1}{2} + 18$$

$$= \frac{37}{2}$$

points of Inflection is $\left(-\frac{1}{2}, \frac{37}{2}\right)$

Find the local maximum and local minimum values of f using first and second derivative test. $f(x) = x^4 - 4x^3$

Soln: $f(x) = x^4 - 4x^3$

$$f'(x) = 4x^3 - 12x^2$$

$$= 4x^2(x-3)$$

$$\Rightarrow x = 0, 3$$

The critical numbers are 0, 3.

Interval	Sign of f'	Behaviour of f
$x<0$ $(-\infty, 0)$	$-$	decreasing
$0<x<3$	$-$	decreasing
$x>3$	$+$	Increasing.

First derivative test tells us that f does not have a local maximum or local minimum at 0

$$f''(x) = 12x^2 - 24x$$

$$f''(x) = 0$$

$$12x^2 - 24x = 0$$

$$12x(x-2) = 0$$

$$x = 0, 2.$$

	Interval	$f''(x)$	Behaviour of f
$(-\infty, 0)$	$x < 0$	+	Concave upward
$(0, 2)$	$0 < x < 2$	−	Concave downward.
$(2, \infty)$	$x > 2$	+	Concave upward.

The curve changes from concave upward to concave downward at $x = 0$.

The pt $(0, f(0)) = (0,0)$ is Inflection point.

The curve changes from concave down-ward to concave upward at $x = 2$.

The pt $(2, f(2)) = (2, -16)$ is an Inflection point.

$f'(3) = 0$, $f''(3) > 0 \Rightarrow f$ is local minimum at $x = 3$, $f(3) = -27$ is a local minimum.

62

The second derivative test gives no Information about the critical number 0.

Find the local maximum and minimum values of f using first and second derivative test for $f(x) = x^5 - 5x + 3$.

Soln: $f(x) = x^5 - 5x + 3$.

$f'(x) = 5x^4 - 5$

$f'(x) = 0 \Rightarrow 5x^4 - 5 = 0$.

$\Rightarrow x^4 - 1 = 0$

$x^4 = 1 \Rightarrow (x^2)^2 = 1$

$x^2 = \pm 1 \Rightarrow x = +1, -1$.

Critical numbers are 1, -1.

Interval	Sign of f'	Behaviour of f
$-\infty < x < -1$	+	Increasing
$-1 < x < 1$	−	decreasing
$x > 1$	+	Increasing.

The first derivative tells us that local maximum at $x = -1$

$f(-1) = (-1)^5 - 5(-1) + 3$

$= -1 + 5 + 3 = 7$

The second derivative $f''(x) = 20x^3$

$f''(1) = 20(1)^3 > 0$

$\Rightarrow f$ has a local minimum at $x = 1$.

$f(1) = 1 - 5 + 3 = -1$.

$f''(x) = 0 \Rightarrow x = 0$.

	Interval	$f''(x)$	Behaviour of f
$(-\infty, 0)$	$x < 0$	$-$	Concave downward
$(0, \infty)$	$x > 0$	$+$	Concave upward.

$x = -1$, $f'(-1) = 0$, $f''(-1) = -20$

$\Rightarrow f(-1) = 7$ is local maximum.

$x = 1$, $f'(1) = 0$ $f''(1) = 20$

$\Rightarrow f(1) = -1$ is local minimum.

$(0, 3)$ is the point of Inflection

AU 2019 Find the local maximum and local minimum. $\sqrt{x} - \sqrt[4]{x}$ using first and second derivative Test.

Soln

$$f(x) = \sqrt{x} - \sqrt[4]{x} = \sqrt{x} - x^{1/4}$$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{4} x^{\frac{1}{4}-1}$$

$$= \frac{1}{2x^{1/2}} - \frac{1}{4x^{\frac{3}{4}}}$$

$$f'(x) = 0 \Rightarrow \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{4x^{\frac{3}{4}}} = 0.$$

$$\frac{1}{2x^{\frac{1}{2}}} = \frac{1}{4x^{\frac{3}{4}}}$$

$$\frac{x^{\frac{3}{4}}}{x^{\frac{1}{2}}} = \frac{2}{4} = \frac{1}{2}$$

$$x^{\frac{1}{4}} = \frac{1}{2}$$

$$x = \left(\frac{1}{2}\right)^4 = \frac{1}{2^4} = \frac{1}{16}.$$

The critical no is $\frac{1}{16}$.

$\therefore$ The critical nos $\frac{1}{16}$

Interval	Sign of f'	$f(x)$
$(0, \frac{1}{16})$	$-$	decreasing
$x > \frac{1}{16}$	$+$	Increasing.

In $(-\infty, 0)$ the $-$ve values $f'(x)$ does not give real values.

at $x = \frac{1}{16}$ f' changes from $-$ve to $+$ve. Then f has local minimum at $x = \frac{1}{16}$ is $f(\frac{1}{16}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

$$f''(x) = 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} - \frac{1}{4}\left(-\frac{3}{4}x^{-\frac{3}{4}-1}\right)$$

$$= -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{16}x^{-\frac{7}{4}}$$

$f''(0)$ does not exist

$$f''\left(\tfrac{1}{16}\right) = -\frac{1}{4}\left(\frac{1}{2^4}\right)^{-\frac{3}{2}} + \frac{3}{16}\left(\frac{1}{2^4}\right)^{-\frac{7}{4}}$$

$$= -\frac{1}{2^2}\left(\frac{1}{2^{-6}}\right) + \frac{3}{2^4}\left(\frac{1}{2^{-7}}\right)$$

$$= -\frac{2^6}{2^2} + \frac{3}{2^4}\,2^7$$

$$= -2^4 + 3(2^3)$$

$$= -16 + 24$$

$$= 8 > 0$$

$\therefore$ f has local minimum at $x = \frac{1}{16}$.

Local minimum value is

$$f\left(\tfrac{1}{16}\right) = \sqrt{\frac{1}{2^4}} - \left(\frac{1}{2^4}\right)^{\frac{1}{4}}$$

$$= \frac{1}{2^2} - \frac{1}{2}$$

$$= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

9-12-20.

Find the equation of tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$

$y = x^2 - 8x + 9 = f(x)$

$f'(x) = 2x - 8$

Slope of tangent $= f'(x)]_{(3,-6)}$

(or) $\left.\frac{dy}{dx}\right)_{(3,-6)}$

$= 2x - 8)_{(3,-6)}$

$= 2(3) - 8 = -2$

Equation of tangent at $(3, -6)$

$y - y_1 = \text{Slope}\,(x - x_1)$

$y - (-6) = -2(x - 3)$

$y + 6 = -2x + 6$

$2x + y = 0.$

H.W. Find Equation of tangent to the curve ① $y = 4x - 3x^2$ at $(2, -4)$

Ans $y = -8x + 12$

② $y = 3x^2 - x^3$ at $(1, 2)$

Ans $y = 3x - 1$.

Find the Equation of normal to the Curve $y = f(x)$. at (x_1, y_1).

Soln: $y = f(x)$

$$\frac{dy}{dx} = f'(x)$$

$$m = \left.\frac{dy}{dx}\right)_{(x_1, y_1)} = \left[f'(x)\right]_{(x_1, y_1)}$$ is

Called slope of tangent.

Equation of normal is

$$y - y_1 = -\frac{1}{m}(x - x_1).$$

Horizontal tangents.

Horizontal tangents occur where the derivative $\frac{dy}{dx}$ is zero ie $\frac{dy}{dx} = 0$.

Ex: Find the points on the curve $y = x^4 - 6x^2 + 8$ where the tangent line is horizontal.

$$\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 12x = 0.$$

$$4x(x^2 - 3) = 0$$

$$x = 0, \; x = \pm\sqrt{3}.$$

x :	0	$\sqrt{3}$	$-\sqrt{3}$
y :	8	-1	-1

The Corresponding points are $(0, 8)$ $(\sqrt{3}, -1)$ $(-\sqrt{3}, -1)$

Find the equation of tangent and normal to the curve $y = 3x^2 - x^3$ at $(1, 2)$

$$y = 3x^2 - x^3$$

$$\frac{dy}{dx} = 6x - 3x^2$$

$$\left.\frac{dy}{dx}\right)_{(x_1, y_1)} = m = (6x - 3x^2)_{(1,2)}$$

$$= 6(1) - 3(1)^2$$

$$m = 6 - 3 = 3.$$

Equation of tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 3(x - 1)$$

$$y - 2 = 3x - 3$$

$$y - 3x - 2 + 3 = 0$$

$$y - 3x + 1 = 0.$$

Equation of normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - 2 = -\frac{1}{3}(x - 1)$$

$$3(y - 2) = -x + 1$$

$$3y - 6 + x - 1 = 0$$

$$x + 3y - 7 = 0.$$

Find a and b.

$$f(x) = \begin{cases} x+2 & x<2 \\ ax^2-bx+3 & 2\le x<3 \\ 2x-a+b & x\geq 3 \end{cases}$$

Continuous everywhere.

Soln Take $x=2$.

f is continuous at $x=2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$2+2 = a(2)^2 - b(2)+3$$

$$4 = 4a-2b+3.$$

$$4a-2b=1 \quad \text{—} \ (1)$$

f is continuous at $x=3$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$a(3)^2 - b(3)+3 = 2(3)-a+b.$$

$$9a-3b+3 = 6-a+b$$

$$10a-4b=3 \quad \text{—} \ (2)$$

Solving (1) & (2)

$$4a-2b=1 \times 2$$

$$\Rightarrow 8a-4b=2$$

$$\underline{\underset{-}{10a}\ \underset{+}{-4b} = \underset{-}{3}}$$

$$-2a = -1$$

$a = \frac{1}{2}$.

sub in ① $4\left(\frac{1}{2}\right) - 2b = 1$.

$$2 - 2b = 1$$
$$2 - 1 = 2b$$
$$b = \frac{1}{2}.$$

For what value of the constant c is the function f continuous on $(-\infty, \infty)$

$$f(x) = \begin{cases} cx^2 + 2x & x < 2. \\ x^3 - cx & x \geq 2. \end{cases}$$

<u>Soln</u> f is continuous at $x = 2$

$$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x).$$

$$c(2)^2 + 2(2) = 2^3 - c(2).$$
$$4c + 4 = 8 - 2c.$$
$$6c = 8 - 4 = 4$$
$$c = \frac{4}{6} = \frac{2}{3}.$$
$$\therefore c = \frac{2}{3}.$$

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unit III

Functions of several variables.

partial Derivatives.

$z = f(x,y)$ be a function of two variables x and y.

The derivative of z w.r.t. x treating y as constant is called the partial derivative of z w.r.t x. denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x.$$

successive partial Differentiation.

$z = f(x,y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ (or) } \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ (or) } \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ (or) } \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ (or) } \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

Ex: If $u = (x-y)(y-z)(z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Soln $\quad u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)1\right]$$

$$= -(x-y)(y-z) + (y-z)(z-x)$$

$$\frac{\partial u}{\partial y} = (z-x)[(x-y)(1) + (y-z)(-1)]$$
$$= (x-y)(z-x) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial z} = (x-y)[(y-z)(1) + (z-x)(-1)]$$
$$= (x-y)(y-z) - (x-y)(z-x)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Ex: If $f(x,y) = \log\sqrt{x^2+y^2}$

Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$

Soln: $f = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\,\frac{1(2x)}{x^2+y^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$
$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$
$$= \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Similarly $\frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$= 0.$$

Ex: If $u = x^y$ then show that

(i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. (ii) $u_{xxy} = u_{xyx}$

Soln

$$u = x^y = e^{\log x^y}$$

$$u = e^{y \log x}$$

$$\frac{\partial u}{\partial x} = e^{y \log x} \frac{y}{x}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = e^{y \log x} \cdot \frac{1}{x}.$$

$$+ \frac{y}{x} e^{y \log x} (\log x).$$

$$= \frac{e^{y \log x}}{x}\left[1 + y \log x\right] \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = e^{y \log x} \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{e^{y \log x}}{x} + \log x \, e^{y \log x} \cdot \frac{y}{x}$$

$$= \frac{e^{y \log x}}{x}\left[1 + y \log x\right] \quad \text{--- (2)}$$

From ① & ②

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Diff. partially w.r.t. x on both sides.

$$u_{xxy} = u_{xyx}.$$

<u>Ex:</u> If $x = r\cos\theta$, $y = r\sin\theta$ find (i) $\frac{\partial x}{\partial r}$ (ii) $\frac{\partial y}{\partial \theta}$ (iii) $\frac{\partial r}{\partial x}$ (iv) $\frac{\partial \theta}{\partial y}$

(i) $\frac{\partial x}{\partial r} = \cos\theta$

(ii) $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$x^2 + y^2 = r^2$$

$$r = \sqrt{x^2+y^2}$$

(iii) $\frac{\partial r}{\partial x} = \frac{1(2x)}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$

(iv) $\frac{y}{x} = \tan\theta$, $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2}.$$

2018

If $u = f(2x-3y, 3y-4z, 4z-2x)$.
find $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$

Soln: Let $p = 2x-3y$, $q = 3y-4z$

$r = 4z-2x$

$u = f(p,q,r)$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}.$$

$$= \frac{\partial f}{\partial p}(2) + \frac{\partial f}{\partial q}\cdot 0 + \frac{\partial f}{\partial r}(-2).$$

$$\frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial p} - \frac{\partial f}{\partial r} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= \frac{\partial f}{\partial p}(-3) + \frac{\partial f}{\partial q}(3) + \frac{\partial f}{\partial r}\cdot 0$$

$$\frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \frac{\partial f}{\partial p}\cdot 0 + \frac{\partial f}{\partial q}(-4) + \frac{\partial f}{\partial r}(4)$$

$$\frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial q} + \frac{\partial f}{\partial r} \quad \text{--- (3)}$$

adding ① ② and ③

$$\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=0.$$

A.U. 2018 If $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$ then find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$

Soln $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\frac{\partial u}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$$

$$\frac{\partial u}{\partial x}=-x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 u}{\partial x^2}=-x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{3}{2}-1}(2x)+(x^2+y^2+z^2)^{-\frac{3}{2}}(-1)$$

$$\frac{\partial^2 u}{\partial x^2}=3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}-(x^2+y^2+z^2)^{-\frac{3}{2}}$$

Similarly $\frac{\partial^2 u}{\partial y^2}=3y^2(x^2+y^2+z^2)^{-\frac{5}{2}}-(x^2+y^2+z^2)^{-\frac{3}{2}}$

$$\frac{\partial^2 u}{\partial z^2}=3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}-(x^2+y^2+z^2)^{-\frac{3}{2}}.$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=3(x^2+y^2+z^2)^{-\frac{5}{2}}(x^2+y^2+z^2)-3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$=3(x^2+y^2+z^2)^{-\frac{3}{2}}-3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$=0.$$

A.U. Dec. 2019. For the given function $z = \tan^{-1}\frac{x}{y} - xy$ whether the statement $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ is correct or not.

Soln

$$z = \tan^{-1}\frac{x}{y} - xy$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y} - y$$

$$= \frac{y^2}{y^2+x^2}\left(\frac{1}{y}\right) - y$$

$$\frac{\partial z}{\partial x} = \frac{y}{x^2+y^2} - y$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} - 1$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} - 1 \quad \text{—} (1)$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}x\left(-\frac{1}{y^2}\right) - x$$

$$= \frac{y^2}{y^2+x^2}\left(-\frac{x}{y^2}\right) - x$$

$$\frac{\partial z}{\partial y} = -\frac{x}{x^2+y^2} - x$$

76

$$\frac{\partial^2 z}{\partial x \partial y} = -\left[\frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}\right] - 1$$

$$= -\left[\frac{x^2 + y^2 - 2x^2}{(x^2+y^2)^2}\right] - 1$$

$$= -\frac{(y^2 - x^2)}{(x^2+y^2)^2} - 1$$

$$= \frac{x^2 - y^2}{(x^2+y^2)^2} - 1 \quad \text{--- (2)}$$

From (1) & (2) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial y}$.

Eulers Theorem for Homogeneous Function.

If u is a homogeneous function of degree n in x and y then.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Ex: If $\log u = \dfrac{x^3+y^3}{3x+4y}$ find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$

Soln

$$z = \log u$$

$$z = \frac{x^3+y^3}{3x+4y}$$

$$= \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(3+4\frac{y}{x}\right)} = \frac{x^2\left(1+\frac{y^3}{x^3}\right)}{3+4\frac{y}{x}}.$$

$\therefore$ z is a homogeneous function of deg 2 in x and y.

By Eulers Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad \text{—①}$$

$$z = \log u$$

$$\frac{\partial z}{\partial x} = \frac{1}{u}\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{u}\frac{\partial u}{\partial y}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y}$$

From ① $\quad 2z = \frac{x}{u}\frac{\partial u}{\partial x} + \frac{y}{u}\frac{\partial u}{\partial y}$

$$u\,2z = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u.$$

Prob. ① If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Soln Let $z = \tan u = \frac{x^3+y^3}{x-y}$

$$z = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = \frac{x^2\left(1+\frac{y^3}{x^3}\right)}{\left(1-\frac{y}{x}\right)}$$

z is a homogeneous function of degree 2 in x and y.

By Eulers Theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

$$z = \tan u$$

$$\frac{\partial z}{\partial x}=\sec^2 u.\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y}=\sec^2 u.\frac{\partial u}{\partial y}$$

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$$

From ①

$$\sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=2\tan u$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\sin u}{\cos u}.\cos^2 u$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cos u=\sin 2u$$

2019. J. Verify Eulers Theorem for the function

$$u=x^2+y^2+2xy$$

Soln $u=x^2+y^2+2xy$

$$u(tx,ty)=t^2x^2+t^2y^2+2txty$$
$$=t^2(x^2+y^2+2xy)$$
$$n=2$$

$\therefore$ u is a homogeneous function of degree 2.

By Eulers Theorem.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

$$\frac{\partial u}{\partial x} = 2x + 2y$$

$$x \frac{\partial u}{\partial x} = 2x^2 + 2xy$$

$$\frac{\partial u}{\partial y} = 2y + 2x$$

$$y \frac{\partial u}{\partial y} = 2y^2 + 2xy.$$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2x^2 + 2xy + 2y^2 + 2xy$$

$$= 2\left[x^2 + y^2 + 2xy\right]$$

$$= 2u.$$

Hence Eulers Theorem is verified.

Prob. 92 $u = \sin^{-1} \frac{x^3 - y^3}{x+y}$ prove that

$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2\tan u$ by using Eulers Theo

Soln $\quad u = \sin^{-1} \frac{x^3 + y^3}{x+y}$

Let $z = \sin u = \frac{x^3 + y^3}{x+y}$

$$z = \frac{x^3 - y^3}{x+y} = \sin u$$

$$f(x,y) = \sin u = \frac{x^3 - y^3}{x+y}$$

$$f(tx,ty) = \frac{t^3x^3 - t^3y^3}{tx+ty} = \frac{t^3(x^3-y^3)}{t(x+y)}$$

$$\therefore z = t^2\left(\frac{x^3-y^3}{x+y}\right)$$

$\therefore$ z is a homogeneous function of degree 2 in x, y. By Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = 2z$$

$$x \cdot \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = 2z \quad \text{—(1)}$$

$$z = \sin u \qquad \frac{\partial z}{\partial u} = \cos u$$

Sub in (1)

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 2(\sin u)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u} = 2\tan u$$

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pro. If $z = \log(x^2+xy+y^2)$ show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$ using Euler's Theorem.

Soln $z = \log(x^2+xy+y^2)$

$$e^z = e^{\log(x^2+xy+y^2)}$$

$$e^z = x^2+xy+y^2$$

let $f = e^z = x^2+xy+y^2$

f is a homogeneous function of degree 2. in x and y. By Eulers Theorem

$$x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f$$

$$x \cdot \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 2f$$

$$x \cdot e^z \frac{\partial z}{\partial x} + y e^z \cdot \frac{\partial z}{\partial y} = 2e^z \quad (\because f = e^z \quad \frac{\partial f}{\partial z} = e^z.$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2.$$

If $u = \sin^{-1} \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3\tan u = 0$ using Euler's Theorem.

Soln $u = \sin^{-1} \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$

$$\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} = f \text{ (say)}$$

f is a homogeneous function of degree -3 in x and y.

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = -3f$$

$$x \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + y \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + z \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial z} = -3f$$

$$f = \sin u \quad \frac{\partial f}{\partial u} = \cos u$$

$$x \cdot \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z}$$
$$= -3 \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -3 \frac{\sin u}{\cos u} = -3 \tan u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -3 \tan u$$

If $u = f(x-y, y-z, z-x)$ then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln Let $x-y = p$ $y-z = q$ $z-x = r$.

$$u = f(p, q, r)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$p = x - y \qquad q = y - z \qquad r = z - x$$
$$\frac{\partial p}{\partial x} = 1 \qquad \frac{\partial q}{\partial x} = 0 \qquad \frac{\partial r}{\partial x} = -1.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$= -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}\cdot 0 + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1) \quad \text{--- (3)}$$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ (by adding (1) & (2) & (3))

If $u = \log(\tan x + \tan y + \tan z)$

S.T. $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$

Soln Given $u = \log(\tan x + \tan y + \tan z)$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \sec^2 x}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \sin x \cos x \frac{1}{\cos^2 x}}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \tan x}{\tan x + \tan y + \tan z}.$$

Similarly

$$\sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z}.$$

$$\sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z}.$$

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2.$$

Prob:

$u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$ then prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}.$$

Soln.

$$u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

z is a homogeneous function in x and y of degree $\frac{1}{2}$.

By Eulers Theorem.

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{1}{2} z$$

$$z = \sin u$$

$$\frac{\partial z}{\partial u} = \cos u$$

$$\therefore x \cdot \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + y \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{1}{2} z$$

$$\therefore x \cdot \cos u \frac{\partial u}{\partial x} + y \cdot \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \frac{\sin u}{\cos u} = \frac{1}{2} \tan u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

Differentiate partially w.r.t. x.

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u \cdot \frac{\partial u}{\partial x}.$$

multiply by x we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u \cdot x\frac{\partial u}{\partial x}. \quad \text{---} (1)$$

Similary we get

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + xy\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{2}\sec^2 u \cdot y\frac{\partial u}{\partial y}. \quad \text{---} (2)$$

adding (1) & (2) we get

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= \frac{1}{2}\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] \quad \left(\because \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\tan u$$

$$= \frac{1}{2}\sec^2 u\left(\frac{1}{2}\tan u\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\tan u\left[\frac{1}{2}\sec^2 u - 1\right]$$

$$= \frac{1}{2}\tan u\left[\frac{1}{2}\frac{1}{\cos^2 u} - 1\right]$$

$$= \frac{1}{2}\frac{\sin u}{\cos u}\left[\frac{1-2\cos^2 u}{2\cos^2 u}\right]$$

$$= -\frac{1}{4}\frac{\sin u}{\cos^3 u}\left[2\cos^2 u - 1\right]$$

$$= -\frac{1}{4}\frac{\sin u \cos 2u}{\cos^3 u}.$$

Total Differential Coefficient.

If $u = f(x, y)$ is a function of x and y. $x = f(t)$, $y = g(t)$

$\frac{du}{dt}$ is called the total derivative of u.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Ex: Find $\frac{du}{dt}$ if $u = x^2y^3$, $x = \log t$, $y = e^t$.

Solution given $u = x^2y^3$

$x = \log t$, $y = e^t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= 2xy^3\cdot\frac{1}{t} + 3x^2y^2\cdot e^t$$

$$= 2(\log t)\frac{e^{3t}}{t} + 3(\log t)^2 e^{2t}\cdot e^t$$

$$= 2(\log t)\frac{e^{3t}}{t} + 3(\log t)^2 e^{3t}$$

$$= e^{3t}\log t\left[\frac{2}{t} + 3(\log t)\right]$$

J.2022. $u = x^3 + y^3$, $x = a\cos t$ $y = b\sin t$ find $\frac{du}{dt}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= 3x^2(-a\sin t) + 3y^2(b\cos t)$$

$$= 3a^2\cos^2 t(-a\sin t) + 3(b^2\sin^2 t)(b\cos t)$$

$$= -3a^3\cos^2 t\sin t + 3b^3\sin^2 t\cos t.$$

(8.1)

$u = x \log xy$ where $x^3+y^3+3xy=1$

find $\frac{du}{dx}$.

soln $\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} \quad -① $$

$$\frac{\partial u}{\partial x} = \log xy + x\frac{1}{xy}y$$

$$= \log xy + 1$$

$$\frac{\partial u}{\partial y} = x\frac{1}{xy}\cdot x = \frac{x}{y}$$

Sub in ①

$$\frac{du}{dx} = \log xy + 1 + \frac{x}{y}\cdot\frac{dy}{dx}.$$

given $x^3+y^3+3xy=0$

$$3x^2+3y^2\cdot\frac{dy}{dx} + 3\left[x\cdot\frac{dy}{dx}+y\right]=0.$$

$$3x^2+3y = -(3y^2+3x)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{3(x^2+y)}{3(y^2+x)} = -\frac{x^2+y}{y^2+x}.$$

$$\therefore \frac{du}{dx} = \log xy + 1 + \frac{x}{y}\left[-\frac{(x^2+y)}{y^2+x}\right]$$

$$= 1+\log xy - \frac{x(x^2+y)}{y(y^2+x)}.$$

(88)

Jacobians.

If u_1, u_2 are functions of x_1 and x_2 Then

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$u_1\, u_2\, u_3$ are functions of x_1, x_2, x_3 then

$$\frac{\partial(u_1,u_2 u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

Similarly we can extend these variables.

Ex: If $x = r\cos\theta$, $y = r\sin\theta$ then find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

<u>Soln</u> given $x = r\cos\theta$ $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial r} = r\cos\theta.$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

Ex: If $x = u(1+v)$, $y = v(1+u)$

find $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$\frac{\partial x}{\partial u} = 1+v$ $\quad \frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = v$ $\quad \frac{\partial y}{\partial v} = 1+u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1+u+v.$$

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(90)

If $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$. find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Soln:

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} \quad \text{—(1)}$$

$u = 2xy \qquad v = x^2 - y^2$

$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 2x \qquad \frac{\partial v}{\partial y} = -2y.$

$x = r\cos\theta \qquad y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4y^2 - 4x^2$$

$$= -4(x^2+y^2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r.$$

Sub in ①.

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4(x^2+y^2).r$$

$x = r\cos\theta \quad y = r\sin\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2.$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = -4r^2.r = -4r^3.$$

---

97 $u = \log r$ where $r^2 = (x-a)^2 + (y-b)^2$.

Then the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =$

ⓐ r ⓑ $-r$ ⓒ 0 ⓓ none.

Soln: $r^2 = (x-a)^2 + (y-b)^2$

$$u = \log r$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} = \frac{du}{dr}.\frac{\partial r}{\partial x} \text{ — ①}$$ u is a fn of r alone.

$\frac{du}{dr} = \frac{1}{r}$. $\quad r^2 = (x-a)^2 + (y-b)^2$

$$2r.\frac{\partial r}{\partial x} = 2(x-a).$$

(92)

$$\therefore \frac{\partial r}{\partial x} = \frac{x-a}{r}.$$

Sub in ①

$$\frac{\partial u}{\partial x} = \frac{1}{r} \cdot \frac{(x-a)}{r} = \frac{(x-a)}{r^2} = (x-a)r^{-2}$$

Diff. w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = (x-a)(-2r^{-3})\frac{\partial r}{\partial x} + r^{-2}(1)$$

$$= (x-a) - 2r^{-3}\frac{(x-a)}{r} + \frac{1}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2(x-a)^2}{r^4} + \frac{1}{r^2}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{2(y-b)^2}{r^4} + \frac{1}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{2}{r^4}\left[(x-a)^2 + (y-b)^2\right] + \frac{2}{r^2}$$

$$= -\frac{2}{r^4}\left[r^2\right] + \frac{2}{r^2}$$

$$= -\frac{2}{r^2} + \frac{2}{r^2} = 0.$$

Ans c.

Note: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as Laplace equation in two dimensions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ is known as Laplace Equation in three dimensions.

If u is a homogeneous function of degree n in x and y show that.

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\left(\frac{\partial^2 u}{\partial x \partial y}\right) + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

<u>Soln</u> Euler Equation is

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \quad \text{--- (1)}$$

Diff (1) partially w.r.t. x.

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = n \frac{\partial u}{\partial x}. \quad \text{--- (2)}$$

Diff (1) partially w.r.t. y.

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n \frac{\partial u}{\partial y} \quad \text{--- (3)}.$$

(2) $\times x$ + (3) $\times y$

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = nx \frac{\partial u}{\partial x}$$

$$xy \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = ny \frac{\partial u}{\partial y}$$

adding

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$
$$= n\left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+nu=n(nu)$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=n^2u-nu$$

$$=n(n-1)u.$$

If $u=\frac{x+y}{x-y}$ $v=\tan^{-1}x+\tan^{-1}y$

find Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$

Soln $u=\frac{x+y}{x-y}$ $\frac{\partial u}{\partial x}=\frac{(x-y)(1)-(x+y)\cdot 1}{(x-y)^2}$

$$=-\frac{2y}{(x-y)^2}.$$

$$\frac{\partial u}{\partial y}=\frac{(x-y)(1)-(x+y)(-1)}{(x-y)^2}=\frac{2x}{(x-y)^2}$$

$$v=\tan^{-1}x+\tan^{-1}y$$

$$\frac{\partial v}{\partial x}=\frac{1}{1+x^2}\quad \frac{\partial v}{\partial y}=\frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}-\frac{2y}{(x-y)^2} & \frac{2x}{(x-y)^2}\\ \frac{1}{1+x^2} & \frac{1}{1+y^2}\end{vmatrix}$$

$$=\frac{-2y}{(x-y)^2(1+y^2)}-\frac{2x}{(x-y)^2(1+x^2)}.$$

96

If $x = \frac{u^2}{v}$ $y = \frac{v^2}{w}$ $z = \frac{w^2}{u}$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

Soln $\frac{\partial x}{\partial u} = \frac{2u}{v}$ $\frac{\partial x}{\partial v} = -\frac{u^2}{v^2}$ $\frac{\partial x}{\partial w} = 0$

$\frac{\partial y}{\partial u} = 0$, $\frac{\partial y}{\partial v} = \frac{2v}{w}$ $\frac{\partial y}{\partial w} = -\frac{v^2}{w^2}$

$\frac{\partial z}{\partial u} = -\frac{w^2}{u^2}$ $\frac{\partial z}{\partial v} = 0$ $\frac{\partial z}{\partial w} = \frac{2w}{u}$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} & 0 \\ 0 & \frac{2v}{w} & -\frac{v^2}{w^2} \\ -\frac{w^2}{u^2} & 0 & \frac{2w}{u} \end{vmatrix} = \frac{2u}{v}\left(\frac{4vw}{uw}\right) - \frac{u^2}{v^2}\left(\frac{w^2 v^2}{u^2 w^2}\right)$$

$= 8 - 1 = 7$

If $u = x - y$, $v = y - z$, $w = z - x$

find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Soln given $u = x - y$, $v = y - z$, $w = z - x$.

96

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} \quad 1-1=0$$

97 $x = uv$, $y = \frac{u}{v}$ find $\frac{\partial(x,y)}{\partial(u,v)}$

Soln: $x = uv \quad y = \frac{u}{v}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{uv}{v^2} - \frac{u}{v}$$

$$= -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}$$

Jan 2019 97 $x = uv$. $y = \frac{u}{v}$ then find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln: $x = uv \quad y = u/v$

$xy = u^2$, $\quad \frac{x}{y} = \frac{uv}{u/v} = v^2$

$u = \sqrt{xy}$ $\quad v = \sqrt{x/y}$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \qquad \frac{\partial u}{\partial x} = \frac{y}{2\sqrt{xy}} = \frac{1}{2}\sqrt{\frac{y}{x}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{xy}}x = \frac{1}{2}\sqrt{\frac{x}{y}} \qquad \frac{\partial v}{\partial x} = \frac{1}{2\sqrt{x/y}} \cdot \frac{1}{y} = \frac{1}{2\sqrt{xy}}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2\sqrt{\frac{x}{y}}} \cdot \frac{x}{y^2} = \frac{\sqrt{x}}{2\,y^{\frac{3}{2}}} = \frac{\sqrt{x}}{2y\sqrt{y}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{y}{x}} & \frac{1}{2}\sqrt{\frac{x}{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{1}{2}\frac{\sqrt{x}}{y\sqrt{y}} \end{vmatrix}$$

$$= -\frac{1}{4y} - \frac{1}{4y} = -\frac{1}{2y}$$

If $x = uv$, $y = \frac{u}{v}$ P.T. $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$.

Soln From above two problems

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = -2y\left(-\frac{1}{2y}\right) = 1.$$

If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$ then prove that
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$$

Soln: $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$ is a homogeneous function in x, y of deg 2

By Eulers Theorem.
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u$$

If $u = \cos^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x}$ then $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

Soln
u is homo. in x, y of deg 0. hence
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 0.$$

$u = 3x + 2y - z$, $v = x - 2y + z$, $w = x(x + 2y - z)$

Are u, v, w are functionally related.

Taylors Series for functions of two variables.

Let $f(x,y)$ be a function of two variables x and y. Then $f(x+h, y+k)$ can be expanded in series of powers of h and k.

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) + \cdots$$

$x = a, \; y = b.$

$$f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \left[\frac{h^2}{2!} f_{xx}(a,b) + hk f_{xy}(a,b) + \frac{k^2}{2!} f_{yy}(a,b)\right] + \cdots$$

put $a+h = x$, $b+k = y$

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right] + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$

This is the formula for Taylors Series Expansion.

A.U.D. 2019.

Obtain the Taylor's series Expansion of $e^x \sin y$ in terms of powers of x and y upto third degree terms.

Soln. $f(x,y) = e^x \sin y \qquad f(0,0) = 0.$

$f_x(x,y) = e^x \sin y \qquad f_x(0,0) = 0.$

$f_y(x,y) = e^x \cos y \qquad f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \sin y \qquad f_{xx}(0,0) = 0$

$f_{xy}(x,y) = e^x \cos y \qquad f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -e^x \sin y \qquad f_{yy}(0,0) = 0$

$f_{xxx}(x,y) = e^x \sin y \qquad f_{xxx}(0,0) = 0$

$f_{xxy}(x,y) = e^x \cos y \qquad f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = -e^x \sin y \qquad f_{xyy}(0,0) = 0$

$f_{yyy}(x,y) = -e^x \cos y \qquad f_{yyy}(0,0) = -1.$

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)\right]$$

$$+ \frac{1}{3!}\Big[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b)$$

$$+ 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\Big]$$

put $a = 0,\ b = 0.$

$$e^x \sin y = f(x,y) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2}{2!} f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0) + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$e^x \sin y = 0 + y.1 + xy + \frac{1}{2}x^2 y - \frac{1}{6}y^3 + \cdots$$

$$e^x \sin y = y + xy + \frac{1}{2}x^2 y - \frac{y^3}{6} + \cdots$$

Expand $f(x,y) = e^{xy}$ in Taylor series at $(1,1)$ upto second degree terms.

<u>Soln.</u> $a = 1, \; b = 1$

$f(x,y) = e^{xy}$ $\quad f(1,1) = e$

$f_x = e^{xy} y$ $\quad f_x(1,1) = e$

$f_y = e^{xy} x$ $\quad f_y(1,1) = e$

$f_{xy} = e^{xy}.1 + y e^{xy} x$ $\quad f_{xy}(1,1) = 2e$

$f_{xx} = y e^{xy} y$ $\quad f_{xx}(1,1) = e$

$f_{yy} = x e^{xy} x$ $\quad f_{yy}(1,1) = e$

Taylors series expansion is

$$f(x,y) = f(1,1) + (x-1) f_x(1,1) + (y-1) f_y(1,1)$$

(10

$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

$$= e + (x-1)e + (y-1)e + \frac{1}{2}\left[(x-1)^2 e + 2(x-1)(y-1)(2e) + (y-1)^2 e\right] + \cdots$$

$$= e\left\{1 + (x-1) + (y-1) + \frac{1}{2}\left[(x-1)^2 + 4(x-1)(y-1) + (y-1)^2\right] + \cdots\right.$$

17/12/20
1 1/2 units covered

Maxima and minima for the functions of two variables.

Procedure to find maxima and minima of $f(x,y)$.

① find $\frac{\partial f}{\partial x}(f_x)$ and $\frac{\partial f}{\partial y}(f_y)$.

② $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$. find Solution (c,d) (stationary points).

③ Find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$ at (c,d).

④ Let $\frac{\partial^2 f}{\partial x^2}=A$, $\frac{\partial^2 f}{\partial x\partial y}=B$ and $\frac{\partial^2 f}{\partial y^2}=C$.

⑤ Find $AC-B^2$.

⑥ (i) If $AC-B^2$ is +ve (>0) and A is −ve (<0) Then $f(x,y)$ has a maximum at (c,d).

(ii) If $AC-B^2$ is +ve and A is +ve (>0) then $f(x,y)$ has a minimum at (c,d).

(iii) If $AC-B^2$ is −ve then $f(x,y)$ has neither a maximum nor a minimum at (c,d). such a point is called <u>Saddle point</u>.

(iv) If $AC-B^2=0$ then nothing is known

(v) If $AC-B^2<0$ $f(c,d)$ is not an Extremum.

Defn: A function $f(x,y)$ is said to be stationary at (c,d) of $f(c,d)$ is said to be a stationary value of $f(x,y)$ if $f_x(c,d) = f_y(c,d) = 0$.

Note: ① If $AC - B^2 > 0$ then $A \neq 0$, $C \neq 0$

② Every extreme value is a Stationary value but a Stationary value need not be an extremum value.

Ex: Find the maxima and minima of $f(x,y) = x^2 - xy + y^2 - 2x + y$

Solution: $f(x,y) = x^2 - xy + y^2 - 2x + y$

Find stationary points.

$$\frac{\partial f}{\partial x} = 2x - y - 2 \qquad \frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial f}{\partial y} = -x + 2y + 1 \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x - y - 2 = 0 \Rightarrow 2x - y = 2$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow -x + 2y + 1 = 0 \Rightarrow -x + 2y = -1$$

Solve
$$2x - y = 2 \quad \text{— ①}$$
$$-x + 2y = -1 \quad \text{— ②}$$

Solve ① & ②

$$2x - y = 2$$
$$-2x + 4y = -2$$

$3y = 0 \Rightarrow y = 0$, sub in ②

$-2x = -2 \Rightarrow x = -\frac{2}{-2} = 1.$

The Stationary points (or) Extreme points are $(1,0)$

$A = \frac{\partial^2 f}{\partial x^2} = 2$, $B = \frac{\partial^2 f}{\partial x \partial y} = -1$, $C = \frac{\partial^2 f}{\partial y^2} = 2$

$AC - B^2 = 4 - 1 = 3 > 0$, A is +ve.

$\therefore f(x,y)$ has a minimum value at $(1,0)$ and its minimum value is

$$f(1,0) = 1^2 - 1(0) + 0^2 - 2(1) + 0$$

$$= 1 - 2 = -1.$$

A.U Dec 2019. Find the maximum or minimum values of the function $x^2 + y^2 + 6x + 12$

Soln: given $f(x,y) = x^2 + y^2 + 6x + 12$.

$\frac{\partial f}{\partial x} = 2x + 6$, $\frac{\partial^2 f}{\partial x^2} = 2$ $\frac{\partial^2 f}{\partial x \partial y} = 0$

$\frac{\partial f}{\partial y} = 2y$, $\frac{\partial^2 f}{\partial y^2} = 2$

At the maximum point or minimum point

$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x + 6 = 0 \Rightarrow x = -3$

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$.

105

At the pt $(-3,0)$

$$AC - B^2 = 2(2) - 0 = 4 > 0$$

$$A = 2 > 0$$

$\therefore$ The point $(-3,0)$ is a minimum point

The minimum value is $f(x,y) = 9+0-18+12$

ie $f(-3,0) = 3$

Find the maxima and minima of

$f(x,y) = x^3+y^3-3x-12y+20$.

<u>Soln:</u> $f(x,y) = x^3+y^3-3x-12y+20$

find stationary points.

$$\frac{\partial f}{\partial x} = 3x^2-3, \quad \frac{\partial f}{\partial y} = 3y^2-12$$

$$A = \frac{\partial^2 f}{\partial x^2} = 6x, \quad B = \frac{\partial^2 f}{\partial x \partial y} = 0, \quad C = \frac{\partial^2 f}{\partial y^2} = 6y$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2-3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2-12 = 0 \Rightarrow y^2 = 4, \Rightarrow y = \pm 2$$

The stationary points are $(1,2)$ $(1,-2)$ $(-1,2)$ $(-1,-2)$.

check the maximum or minimum for the 4 points.

stationary points	$A=6x$	$B=0$	$C=6y$	$AC-B^2$
$(1,2)$	6	0	12	$72>0$ minimum $A>0$
$(1,-2)$	6	0	-12	$-72<0$ saddle point
$(-1,2)$	-6	0	12	$-72<0$ saddle point
$(-1,-2)$	-6	0	-12	$72>0$ maximum $A<0$

Maximum value at $(-1,-2)$ is

$$f(-1,-2) = -1-8+3+24+20 = 38$$

minimum value at $(1,2)$ is

$$f(1,2) = 1+8-3-12(2)+20$$

$$= 1+8-3-24+20 = 2.$$

Find the three positive numbers such that their sum is a constant a and their product is maximum.

<u>Soln</u> Let 3 numbers be x, y, z

then $x+y+z=a \Rightarrow z = a-x-y$

Product $= xyz = xy(a-x-y)$

Let $f(x,y) = xy(a-x-y)$

$$= axy - x^2y - xy^2$$

$$\frac{\partial f}{\partial x} = ay - 2xy - y^2$$

$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow ay - 2xy - y^2 = 0$$
$$\Rightarrow y(a - 2x - y) = 0$$
$$\Rightarrow 2x + y = a \quad — ①$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow ax - x^2 - 2xy = 0$$
$$\Rightarrow x(a - x - 2y) = 0$$
$$\Rightarrow 2y + x = a \quad — ②$$

Solving ① & ②

$$\begin{aligned} \cancel{2x} + y &= a \\ \cancel{2x} + 4y &= 2a \\ \hline -3y &= -a \\ y &= \frac{a}{3} \end{aligned}$$

Sub in ①

$$2x + \frac{a}{3} = a$$
$$\therefore 2x = a - \frac{a}{3} = \frac{2a}{3}$$
$$x = \frac{a}{3}$$

At the point $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$\therefore \frac{\partial^2 f}{\partial x^2} = -2y \quad \left(\frac{\partial^2 f}{\partial x^2}\right)_{\frac{a}{3}, \frac{a}{3}} = -\frac{2a}{3}$$

$$A = -\frac{2a}{3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y = a - 2\left(\frac{a}{3}\right) - 2\left(\frac{a}{3}\right)$$

$$B = -\frac{a}{3} \qquad = \frac{3a - 2a - 2a}{3}$$

$$= -\frac{a}{3}$$

$$\frac{\partial^2 f}{\partial y^2} = -2x = -\frac{2a}{3} \qquad C = -\frac{2a}{3}$$

$$AC - B^2 = \left(-\frac{2a}{3}\right)\left(-\frac{2a}{3}\right) - \left(-\frac{a}{3}\right)^2$$

$$= \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$$

$$A = -2\left(\frac{a}{3}\right) < 0$$

$\therefore f(x,y)$ is maximum at $\left(\frac{a}{3}, \frac{a}{3}\right)$.

also $z = a - x - y = a - \left(\frac{a}{3}\right) - \left(\frac{a}{3}\right) = \frac{a}{3}$

$\therefore$ The 3 numbers are $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}$.

Method of Lagrangian multiplier.

To find the maximum and minimum values of $f(x,y,z)$ where x,y,z are subject to constraint equation $g(x,y,z)=0$.

define $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$.

λ is Lagrangian multiplier independent of x,y,z

The necessary condition for maximum and minimum are

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0 \text{ and } \frac{\partial F}{\partial z} = 0$$

solving these equations for four unknowns λ, x, y, z we get x, y, z

Ex: Find the minimum value of $x^2+y^2+z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Soln: Let the Auxillary Equation

b)

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$F(x,y,z) = x^2+y^2+z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

λ is Lagranges multiplier.

$$\frac{\partial F}{\partial x} = 2x + \lambda\left(-\frac{1}{x^2}\right) = 2x - \frac{\lambda}{x^2}$$

$$\frac{\partial F}{\partial y} = 2y + \lambda\left(-\frac{1}{y^2}\right) = 2y - \frac{\lambda}{y^2}$$

$$\frac{\partial F}{\partial z} = 2z + \lambda\left(-\frac{1}{z^2}\right) = 2z - \frac{\lambda}{z^2}$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x - \frac{\lambda}{x^2} = 0 \Rightarrow 2x = \frac{\lambda}{x^2}$$

$$2x^3 = \lambda \quad x^3 = \frac{\lambda}{2} \quad x = \left(\frac{\lambda}{2}\right)^{1/3}$$

Similarly

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \quad x = \left(\frac{\lambda}{2}\right)^{1/3}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \quad z = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}}.$$

Thus we get $x = y = z$

given $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1$

$$\therefore \frac{1}{x}+\frac{1}{x}+\frac{1}{x} = 1 \Rightarrow \frac{3}{x} = 1$$

$$\Rightarrow x = 3$$

Hence $y = 3, \ z = 3$

Sub these values in $x^2+y^2+z^2$

The minimum value is $x^2+y^2+z^2 =$

$3^2+3^2+3^2 = 27.$

Find the minimum value of x^2+y^2 subject to the Condition $ax+by=c.$

Soln $F(x,y) = f(x,y) + \lambda g(x,y)$

$F(x,y) = x^2+y^2 + \lambda(ax+by-c)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x^2 + \lambda a = 0$

$\Rightarrow 2x = -\lambda a$

$\Rightarrow x = -\frac{\lambda a}{2}$

Similarly $\frac{\partial F}{\partial y} = 0 \Rightarrow y = -\frac{\lambda b}{2}$

Sub x, y in $ax+by=c$

$a\left(-\frac{\lambda a}{2}\right) + b\left(-\frac{\lambda b}{2}\right) = c$

$-\lambda \frac{a^2}{2} - \lambda \frac{b^2}{2} = c$

$-\lambda(a^2+b^2) = 2c$

$\lambda = \frac{-2c}{(a^2+b^2)}$ sub in x, y

$x = -\left(-\frac{2c}{a^2+b^2}\right)\frac{a}{2} = \frac{2ac}{(a^2+b^2)2} = \frac{ac}{a^2+b^2}$

$y = -\left(-\frac{2c}{a^2+b^2}\right)\frac{b}{2} = \frac{bc}{a^2+b^2}$

The minimum value is attained at $\left(\frac{ac}{a^2+b^2}, \frac{bc}{a^2+b^2}\right)$

The minimum value of x^2+y^2 is

$$\frac{a^2c^2}{(a^2+b^2)^2}+\frac{b^2c^2}{(a^2+b^2)^2}=\frac{a^2c^2+b^2c^2}{(a^2+b^2)^2}$$

A thin closed rectangular box is to have one edge equal to twice the other and constant volume $72\ m^3$. Find the least surface Area of the box.

Soln Let $x, y, 2y$ be length, bredth and height of rectangular box.

$$\text{Surface Area}=2xy+2y(2y)+2x(2y)$$

$$f(x,y,z)=2xy+4y^2+4xy$$

$$f(x,y,z)=6xy+4y^2$$

$$\text{Volume } g(x,y,z)=xyz=72$$

$$g(x,y,z)=xy(2y)=72$$

$$2xy^2=72$$

$$\Rightarrow xy^2=36 \Rightarrow xy^2-36=0$$

$$F(x,y,z)=f(x,y,z)+\lambda\, g(x,y,z)$$

$$=6xy+4y^2+\lambda(xy^2-36)$$

$$\frac{\partial F}{\partial x}=0 \Rightarrow 6y+\lambda y^2=0.$$

$$\frac{\partial F}{\partial y} = 6x + 8y + 2\lambda xy$$

$$\frac{\partial F}{\partial z} = 0.$$

F is extremum. $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$.

$$6y + \lambda y^2 = 0 \Rightarrow \lambda y^2 = -6y$$

$$\Rightarrow \lambda y = -6$$

$$\Rightarrow \lambda = -\frac{6}{y} \quad \text{—} (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 6x + 8y + 2\lambda xy = 0$$

$$2\lambda xy = -6x - 8y$$

$$\lambda xy = -3x - 4y.$$

$$\lambda = -\frac{(3x+4y)}{xy} \quad \text{—} (2)$$

From (1) & (2)

$$-\frac{6}{y} = -\frac{(3x+4y)}{xy}$$

$$6x = +(3x+4y)$$

$$6x - 3x = 4y$$

$$3x = 4y$$

$$y = \frac{3}{4}x.$$

$$xy^2 = 36; \quad x\left(\frac{3}{4}x\right)^2 = 36.$$

$$\frac{9x^3}{16} = 36$$

$$x^3 = \frac{36 \times 16}{9} = 4 \times 16$$

$x = 4, \quad y = \frac{3}{4} x$

$\Rightarrow y = \frac{3}{4}(4) = 3$

$\therefore$ The point is $(4,3)$

f is minimum at $(4,3)$

minimum surface Area

$= 6(4)(3) + 4(3)^2$

$= 72 + 36$

$= 108$

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obtain Taylors series Expansion of $e^x \log(1+y)$ in powers of x and y upto terms of third degree

Soln Here $a = 0, b = 0$.

$f(x,y) = e^x \log(1+y) \qquad f(0,0) = 0$

$f_x(x,y) = e^x \log(1+y) \qquad f_x(0,0) = 0$

$f_y(x,y) = e^x \frac{1}{1+y} \qquad f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \log(1+y) \qquad f_{xx}(0,0) = 0$

$f_{xy}(x,y) = \frac{e^x}{1+y} \qquad f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -e^x (1+y)^{-2} \qquad f_{yy} = -1$

$f_{xxx}(x,y) = e^x \log(1+y) \qquad f_{xxx} = 0$

$f_{xyy}(x,y) = -e^x (1+y)^{-2} \qquad f_{xyy} = -1$

$f_{xxy} = e^x \log(1+y) \qquad f_{xxy} = 1$

$f_{yyy}(xy) = 2e^{x}(1+y)^{-3} \quad f_{yyy}(0,0) = 2.$

$f(xy) = f(0,0) + x f_x(0,0) + y f_y(0,0)$

$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0)\right.$

$\left. + y^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0)\right.$

$\left. + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$

$= 0 + x(0) + y(1) + \frac{1}{2!}\left[x^2(0) + 2xy(1)\right.$

$\left. + y^2(-1)\right] + \frac{1}{3!}\left[x^3(0) + 3x^2 y(1) + 3xy^2(-1)\right.$

$\left. + y^3(2)\right] + \cdots$

$= y + xy - \frac{y^2}{2} + \frac{x^2 y - xy^2}{2} + \frac{y^3}{3} + \cdots$

unit IV Integral Calculus.

Indefinite Integrals.

① $\int c\, f(x)\, dx = c\int f(x)\, dx$, c is const.

② $\int k\, dx = k\int dx = kx + c$, k - Constant

③ $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c.$ $(n \neq -1)$

④ $\int \frac{dx}{x} = \log x + c.$

⑤ $\int e^x\, dx = e^x + c$

⑥ $\int a^x\, dx = \frac{a^x}{\log a} + c.$

⑦ $\int \sin x\, dx = -\cos x + c.$

⑧ $\int \cos x\, dx = \sin x + c.$

⑨ $\int \sec^2 x\, dx = \tan x + c.$

⑩ $\int \operatorname{cosec}^2 x\, dx = -\cot x + c.$

⑪ $\int \sec x \tan x\, dx = \sec x + c$

⑫ $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c.$

⑬ $\int \frac{1}{x^2+1}\, dx = \tan^{-1} x + c$ (or) $-\cot^{-1}(x) + c.$

⑭ $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$ (or) $-\cos^{-1} x + c.$

15. $\int \sinh x \, dx = \cosh x + C$

(16) $\int \cosh x \, dx = \sinh x + C$.

(17) $\int \operatorname{sech}^2 x \, dx = \tanh x + C$

(18) $\int \operatorname{cosech}^2 x \, dx = -\coth x + C$.

(19) $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$

(20) $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + C$.

(21) $\int \frac{dx}{\sqrt{x(x^2-1)}} = \sec^{-1} x + C$ (or) $-\operatorname{cosec}^{-1} x + C$.

(22) $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$.

(23) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$.

(24) $\int \frac{dx}{x^2-1} = \tanh^{-1} x + C$ (or) $\coth^{-1} x + C$.

The Fundamental Theorem of Calculus.

Theorem (part I): If f is continuous on $[a,b]$ then the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad a \le x \le b.$$ is continuous on $[a,b]$ and differentiable on (a,b)

and $g'(x) = f(x)$

Integration of a function is same as the joining many small quantities to create a large entity.

Fundamental Theorem of Calculus - part 2

If f is continuous on $[a,b]$ then:

$\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of 'f'

i.e a function such that $F' = f$.

Differentiation and Integration as Inverse Processes.

The fundamental Theorem of Calculus.

Suppose f is continuous on $[a,b]$

(i) If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$.

(ii) $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F' = f$

Ex: Find the derivative of the function

$g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Soln $f(t) = \sqrt{1+t^2}$ is continuous

by part (1) of fundamental Theorem of Calculus $g'(x) = \sqrt{1+x^2}$

Note The definite Integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

117

Definite Integral.

If f is defined for $a \le x \le b$. we divide the Interval $[a,b]$ into n sub Intervals.

$\Delta x = \frac{b-a}{n}$. $x_0 = a, x_1, x_2 \ldots x_n = b$.

be the end points.

Then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$ where

x_i^* lies in $[x_{i-1}, x_i]$ $x_i^* = x_i = a + i\Delta x$

The sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is Called a Riemann sum.

Find the general Indefinite Integral

$\int (10x^4 - 2\sec^2 x)\,dx$

$= \int 10x^4\,dx - 2\int \sec^2 x\,dx$

$= 10\frac{x^5}{5} - 2\tan x + C.$

$= 2x^5 - 2\tan x + C.$

Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$= \int \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}\,d\theta = \int \cot\theta \operatorname{cosec}\theta\,d\theta$

$= -\operatorname{cosec}\theta + C.$

Evaluate $\int_0^3 (x^3 - 6x)\,dx$

Soln $\int_0^3 x^3 - 6x\,dx = \int_0^3 x^3\,dx - 6\int_0^3 x\,dx$

$= \left[\frac{x^4}{4}\right]_0^3 - 6\left[\frac{x^2}{2}\right]_0^3$

$= \frac{3^4}{4} - \frac{0^4}{4} - 6\left[\frac{3^2}{2} - \frac{0^2}{2}\right]$

$= \frac{81}{4} - 6\,\frac{9}{2}$

$= \frac{81}{4} - 27$

$= \frac{81-108}{4} = \frac{-27}{4}$

Evaluate $\int_0^2 \left[2x^3 - 6x + \frac{3}{x^2+1}\right] dx$

Soln $\int_0^2 \left[2x^3 - 6x + \frac{3}{x^2+1}\right] dx = 2\left(\frac{x^4}{4}\right)_0^2$

$- 6\left(\frac{x^2}{2}\right)_0^2 + 3\left(\tan^{-1} x\right)_0^2$

$= 2\left(\frac{2^4}{4}\right) - 6\left(\frac{2^2}{2}\right) + 3\tan^{-1} 2$

$= 8 - 12 + 3\tan^{-1} 2$

$= -4 + 3\tan^{-1} 2.$

$$\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx = \int_1^2 x^{-2} - 4\int_1^2 x^{-3}$$

$$= \left[\frac{x^{-2+1}}{-2+1}\right]_1^2 - 4\left[\frac{x^{-3+1}}{-3+1}\right]_1^2$$

$$= \left[\frac{x^{-1}}{-1}\right]_1^2 + \frac{4}{2}\left[x^{-2}\right]_1^2$$

$$= -\left[\frac{1}{x}\right]_1^2 + 2\left[\frac{1}{x^2}\right]_1^2$$

$$= -\left[\frac{1}{2} - 1\right] + 2\left[\frac{1}{4} - 1\right]$$

$$= \frac{1}{2} + 2\left(-\frac{3}{4}\right)$$

$$= \frac{1}{2} - \frac{3}{2} = -1.$$

$$\int_1^4 \frac{\sqrt{y} - y}{y^2}\, dy = \int_1^4 \frac{\sqrt{y}}{y^2} - \frac{y}{y^2}\, dy$$

$$= \int_1^4 y^{-\frac{3}{2}} dy - \int_1^4 \frac{1}{y}\, dy$$

$$= \left[\frac{y^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right]_1^4 - (\log y)_1^4$$

$$= -2\left[y^{-\frac{1}{2}}\right]_1^4 - \left[\log 4 - \log 1\right]$$

$$= -2\left[4^{-\frac{1}{2}} - 1\right] - \log 4$$

$$= -2\left[\frac{1}{2} - 1\right] - \log 4$$

$$= -2\left(-\frac{1}{2}\right) - \log 4 = 1 - \log 4$$

P11
120

What is wrong with the equation

$$\int_{-2}^{1} x^{-4}\, dx = -\frac{3}{8}$$

Soln Here $f(x) = \frac{1}{x^4} \geq 0$

it is not Continuous on $[-2,1]$

f is discontinuous at $x=0$

we can't apply fundamental Theorem of Calculus.

Since $f \geq 0$ $\int_a^b f(x)dx \geq 0$ but the answer is negative. The Calculation is wrong.

What is wrong with the equation

$$\int_{-1}^{2} \frac{4}{x^3}\, dx = \left[-\frac{2}{x^2}\right]_{-1}^{2} = \frac{3}{2}$$

Soln The function $\frac{4}{x^3}$ is not Continuous on $[-1,2]$. It is discontinuous at $x=0$

$\therefore \int_{-1}^{2} \frac{4}{x^3}\, dx$ does not exist.

what is wrong the equation

$$\int_{\frac{\pi}{3}}^{\pi} \sec\theta \tan\theta\, d\theta = \sec\theta\Big]_{\frac{\pi}{3}}^{\pi} = -3$$

Soln $\sec\theta\tan\theta$ is not Continuous on $[\frac{\pi}{3}, \pi]$

Hence fundamental Theorem cannot be applied

12.1 2.5

9) $\int \sin x + \sinh x \, dx = \int \sin x \, dx + \int \sinh x \, dx$

$= -\cos x + \cosh x + C.$

$\int \frac{x^3 - 2\sqrt{x}}{x} \, dx = \int x^2 - 2\int \frac{dx}{\sqrt{x}}$

$= \frac{x^3}{3} - 2 \cdot \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$= \frac{x^3}{3} + 4 x^{\frac{1}{2}} + C$

$= \frac{x^3}{3} + 4\sqrt{x} + C.$

$\int (x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2) \, dx$

$= \int x^4 dx - \frac{1}{2}\int x^3 dx + \frac{1}{4}\int x \, dx - 2\int dx$

$= \frac{x^5}{5} - \frac{1}{2}\frac{x^4}{4} + \frac{1}{4}\frac{x^2}{2} - 2x + C.$

$= \frac{x^5}{5} - \frac{1}{8}x^4 + \frac{1}{8}x^2 - 2x + C.$

$\int_0^{\frac{\pi}{4}} \frac{1+\cos^2\theta}{\cos^2\theta} \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2\theta} d\theta + \int_0^{\frac{\pi}{4}} \frac{\cos^2\theta}{\cos^2\theta} d\theta$

$= \int_0^{\frac{\pi}{4}} \sec^2\theta \, d\theta + \int_0^{\frac{\pi}{4}} d\theta$

$= \tan\theta \Big]_0^{\frac{\pi}{4}} + \Big[\theta\Big]_0^{\frac{\pi}{4}}$

$= \tan\frac{\pi}{4} - \tan 0 + \frac{\pi}{4} - 0 = 1 + \frac{\pi}{4}$

The following formulae are needed to evaluate definite Integrals.

ⓐ $\sum_{i=1}^{n} i = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

ⓑ $\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$

ⓒ $\sum_{i=1}^{n} i^3 = 1^3+2^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2.$

Ex :

ⓐ Evaluate the Riemann sum for $f(x)=x^3-6x$ taking the sample points to be right end points and $a=0, b=3, n=6.$

ⓑ Evaluate $\int_0^3 (x^3-6x)\,dx$ using right end point

A.U. Dec 2019.

Soln: ⓐ

$n=6$

[0, .5, 1, 1.5, 2, 2.5, 3]

width $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{3}{6} = 0.5$

$x_i = a + i\Delta x$

Right End points are $x_1 = 0.5$ $x_2 = 1.0$

$x_3 = 1.5$ $x_4 = 2$ $x_5 = 2.5$ $x_6 = 3$

The Riemann Sum is

$$\int_0^3 (x^3-6x)\,dx = \sum_{i=1}^{6} f(x_i)\Delta x$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$= f(0.5)\tfrac{1}{2} + f(1)\tfrac{1}{2} + f(1.5)\tfrac{1}{2} + f(2)\tfrac{1}{2} + f(2.5)\tfrac{1}{2} + f(3)\tfrac{1}{2}$$

$= \frac{1}{2}\left[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)\right]$

$= \frac{1}{2}\left[-2.875 - 5 - 5.625 - 4 + 0.625 + 9\right]$

$= -3.9375.$

(b) with n sub intervals we have

$x_i = a + i\,\Delta x$

$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}.$

$\Rightarrow x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = \frac{6}{n},\ x_3 = \frac{9}{n}$

and in general $x_i = \frac{3i}{n}$

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{3^3 i^3}{n^3} - 18\frac{i}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\,\frac{27}{n^3} \sum_{i=1}^{n} i^3 - \frac{54}{n^2}\Sigma i$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2}\Sigma i$$

$$= \lim_{n\to\infty} \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{81}{n^4}\,\frac{n^2(n+1)^2}{4} - \frac{54}{n^2}\,\frac{n^2(1+\frac{1}{n})}{2}$$

$$= \lim_{n\to\infty} \frac{81}{4n^4} n^4 \left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75.$$

Integration use mid point rule:

$n=5$ find approximate value of

$$\int_1^2 \frac{dx}{x}$$

Soln.

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

$$\bar{x}_i = \frac{x_i + x_{i-1}}{2} \qquad \Delta x = \frac{b-a}{n}.$$

$n=5$, End points are $1, 1.2, 1.4, 1.6, 1.8, 2$.

mid points are $\frac{1+1.2}{2} = 1.1 = \bar{x}_1$

Similarly $\bar{x}_2 = \frac{1.2+1.4}{2} = 1.3$

$\bar{x}_3 = 1.5$ $\qquad \Delta x = \frac{2-1}{5}$

$\bar{x}_4 = 1.7$ $\qquad = \frac{1}{5}$

$\bar{x}_5 = 1.9$

$$\int_1^2 \frac{dx}{x} = \sum_{i=1}^{5} f(\bar{x}_i)\,\Delta x$$

$$= f(1.1)\frac{1}{5} + f(1.3)\frac{1}{5} + f(1.5)\frac{1}{5}$$

$$+ f(1.7)\frac{1}{5} + f(1.9)\frac{1}{5}$$

$$= \frac{1}{5}\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$$

$$= \frac{1}{5}\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right]$$

$$= 0.691908.$$

Integration By Substitution.

Formulae.

1. $\int f(ax+b)\,dx = \frac{1}{a}\int f(t)\,dt$, $t = ax+b$
2. $\int [f(x)]^n f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} + C$. $n \neq -1$.
3. $\int \frac{f'(x)\,dx}{f(x)} = \log f(x) + C$.
4. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{n+1} + C$.
5. $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + C$.
6. $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$
7. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$.
8. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$.
9. $\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + C$.
10. $\int \sec(ax+b)\tan(ax+b) = \frac{1}{a}\sec(ax+b) + C$

11. $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)$

Evaluate $\int \frac{x^3\,dx}{\sqrt{1-x^8}}$

Soln put $t = x^4$, $dt = 4x^3\,dx$

$x^3\,dx = \frac{dt}{4}$

$$\therefore \int \frac{x^3\,dx}{\sqrt{1-x^8}} = \int \frac{dt}{4\sqrt{1-t^2}} = \frac{1}{4}\sin^{-1}(t) + C$$

$$= \frac{1}{4}\sin^{-1}x^4 + C.$$

Evaluate $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$.

Soln. $t = \sin^{-1}x$ $dt = \frac{dx}{\sqrt{1-x^2}}$

$$\therefore \int \frac{\sin^{-1}x\,dx}{\sqrt{1-x^2}} = \int t\,dt = \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1}x)^2}{2} + C.$$

Evaluate $\int x^3\cos(x^4+2)\,dx$.

Soln $t = x^4+2$ $dt = 4x^3\,dx$

$x^3\,dx = \frac{dt}{4}$

$$\int x^3\cos(x^4+2)\,dx = \int \cos t\,\frac{dt}{4}$$

$$= \frac{1}{4}\sin t + C$$

$$= \frac{1}{4}\sin(x^4+2) + C.$$

Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$

Soln $t = 3-5x$ $x=1,\ t = 3-5=-2$

$dt = -5dx.$ $x=2\ t = 3-10=-7$

$dx = \frac{dt}{-5}$

$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{t^2} \frac{dt}{-5} = -\frac{1}{5}\int_{-2}^{-7} \frac{dt}{t^2} = -\frac{1}{5}\int_{-2}^{-7} t^{-2} dt$

$= -\frac{1}{5}\left[\frac{t^{-1}}{-1}\right]_{-2}^{-7}$

$= \frac{1}{5}\left[\frac{1}{t}\right]_{-2}^{-7}$

$= \frac{1}{5}\left[-\frac{1}{7}+\frac{1}{2}\right] = \frac{1}{5}\frac{5}{14} = \frac{1}{14}$

$\int_1^e \frac{\log x}{x} dx.$

Soln $u = \log x$ $du = \frac{1}{x}dx$

$\int \frac{\log x}{x} dx = \int u\, du$

$= \frac{u^2}{2} + c.$

$= \left[\frac{(\log x)^2}{2}\right]_1^e + c$

$= \frac{(\log e)^2 - (\log 1)^2}{2}$

$= \frac{1}{2} - 0 = \frac{1}{2}.$

Evaluate $\int \frac{dx}{1+\tan x}$

Soln: $\int \frac{dx}{1+\tan x} = \int \frac{dx}{1+\frac{\sin x}{\cos x}}$

$$= \int \frac{\cos x \, dx}{\sin x + \cos x}.$$

$$= \frac{1}{2}\int \frac{2\cos x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x} dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x} dx.$$

$$= \frac{x}{2} + \frac{1}{2}\log(\sin x + \cos x) + C.$$

Result f is continuous on $[-a, a]$

① f is even ie $f(-x) = f(x)$ then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

② f is odd $f(-x) = -f(x)$ then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$.

Soln: $f(x) = \tan^2 x \sec^2 x$.

$f(-x) = \tan^2(-x) \sec^2(-x)$

$= \tan^2 x \sec^2 x = f(x)$

$\Rightarrow f$ is even.

$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx = 2\int_{0}^{\frac{\pi}{4}} \tan^2 x \sec^2 x$

put $t = \tan x$, $dt = \sec^2 x \, dx$

$x = 0$, $t = 0$

$x = \frac{\pi}{4}$ $t = 1$.

$\therefore I = 2\int_0^1 t^2 . dt = 2\left(\frac{t^3}{3}\right)_0^1 = \frac{2}{3}$

Evaluate $\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4} dx$

Soln $f(x) = \frac{\tan x}{1+x^2+x^4}$ $f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4}$

$= -\frac{\tan x}{1+x^2+x^4} = -f$

f is odd

$\int_{-1}^{1} \frac{\tan x}{1+x^2+x^4} dx = 0$

Evaluate $\int \frac{x+2}{\sqrt{x^2+4x}}\,dx$.

Soln $t = x^2+4x$ $dt = (2x+4)\,dx$

$dt = 2(x+2)\,dx$

$(x+2)\,dx = \frac{dt}{2}$

$$\int \frac{(x+2)\,dx}{\sqrt{x^2+4x}} = \int \frac{\frac{dt}{2}}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{2}\,\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C. = \sqrt{t} + C$$

$$= \sqrt{x^2+4x} + C.$$

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{t^4 \tan t}{2+\cos t}\,dt$

$$f(t) = \frac{t^4 \tan t}{2+\cos t}$$

$$f(-t) = \frac{(-t)^4 \tan(-t)}{2+\cos(-t)} = -\frac{t^4 \tan t}{2+\cos t} = -f(t)$$

$\Rightarrow$ f is an odd fn.

$$\therefore \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{t^4 \tan t}{2+\cos t}\,dt = 0.$$

Integration by parts

$\int u\,dv = uv - \int v\,du$ where u, v are functions of x

Example. ① $\int x \sin x \, dx$

$u = x \quad dv = \sin x \, dx$

$v = -\cos x$

$\int u \, dv = uv - \int v \, du$

$\int x \sin x \, dx = -x\cos x - \int(-\cos x) dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + C.$

$\int \log x \, dx$

$u = \log x \quad dv = dx \quad v = x$

$du = \frac{1}{x} dx.$

$\int u \, dv = uv - \int v \, du$

$= x\log x - \int x \cdot \frac{1}{x} dx$

$= x \log x - \int dx$

$= x\log x - x + C.$

$= x(\log x - 1) + C.$

A.U. Dec. 2019 Eval. $\int e^x \sin x \, dx$ using integration by parts.

Soln let $I = \int e^x \sin x \, dx$

$u = e^x \quad dv = \sin x \, dx$

$du = e^x dx \quad v = -\cos x.$

$\therefore I = \int u\,dv = uv - \int v\,du$

$\therefore I = -e^{x}\cos x - \int -\cos x\, e^{x}\,dx$

$= -e^{x}\cos x + \int e^{x}\cos x\,dx$ — ①

again apply Integration by parts

put $u = e^{x}$, $dv = \cos x\,dx$

$du = e^{x}dx$ $v = \sin x$

$\therefore I = -e^{x}\cos x + \int e^{x}\cos x\,dx$

$\int e^{x}\cos x\,dx = e^{x}\sin x - \int e^{x}\sin x\,dx.$

$\int e^{x}\cos x\,dx = e^{x}\sin x - I$

Sub in ① $I = -e^{x}\cos x + e^{x}\sin x - I$

$2I = e^{x}(\sin x - \cos x)$

$I = \frac{1}{2}e^{x}(\sin x - \cos x)$

$\therefore \int e^{x}\sin x\,dx = \frac{1}{2}e^{x}(\sin x - \cos x)$

Evaluate $\int_0^1 \tan^{-1}x\,dx.$

Solution $u = \tan^{-1}x$ $dv = dx$

$du = \frac{dx}{1+x^2}$ $v = x$

$\int_0^1 \tan^{-1}x\,dx = x\tan^{-1}x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

134

$$= \tan^{-1} 1 - 0 \cdot \tan^{-1} 0 - \int_0^1 \frac{x}{1+x^2}\, dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\, dx$$

put $t = 1 + x^2$ $\quad dt = 2x\, dx$

$x\, dx = \frac{dt}{2}$

when $x = 0$, $t = 1$

$x = 1$, $t = 2$

$$\int \tan^{-1} x\, dx = \frac{\pi}{4} - \int_1^2 \frac{dt}{2t}$$

$$= \frac{\pi}{4} - \frac{1}{2} \int_1^2 \frac{dt}{t}$$

$$= \frac{\pi}{4} - \frac{1}{2} \left[\log t\right]_1^2$$

$$= \frac{\pi}{4} - \frac{1}{2} \left[\log 2 - \log 1\right]$$

$$= \frac{\pi}{4} - \frac{1}{2} \log 2.$$

$\int x^5 e^{x^3} dx$

<u>soln</u> $\int x^5 e^{x^3} dx$

$t = x^3$, $\quad dt = 3x^2\, dx$ $\quad x^2 dx = \frac{dt}{3}$

$$\therefore \int x^5 e^{x^3} dx = \int x^3 e^{x^3} x^2 dx$$

$$= \int t e^t \frac{dt}{3}$$

$I = \frac{1}{3}\int t e^t \, dt$

$u = t, \quad e^t dt = dv$

$du = dt \quad e^t = v$

$\therefore \int x^5 e^{x^3} dx = \frac{1}{3}\left[t e^t - \int e^t dt \right]$

$= \frac{1}{3} t e^t - \frac{1}{3} e^t + c.$

$\int x^5 e^{x^3} dx = \frac{1}{3} x^3 e^{x^3} - \frac{1}{3} e^{x^3} + c.$

A.U. J.2018 Evaluate $\int e^{ax} \cos bx \, dx$ by using Integration by parts.

<u>Soln</u> Let $I = \int e^{ax} \cos bx \, dx.$

$u = e^{ax} \quad dv = \cos bx \, dx$

$du = a e^{ax} dx \quad v = \frac{\sin bx}{b}.$

$I = \frac{1}{b} e^{ax} \sin bx - \frac{1}{b}\int \sin bx \, a e^{ax} dx$

$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b}\int e^{ax} \sin bx \, dx$

$u = e^{ax} \quad dv = \sin bx \, dx$

$du = a e^{ax} dx \quad v = -\frac{\cos bx}{b}$

$I = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b}\left[-e^{ax}\frac{\cos bx}{b} - \int -\frac{\cos bx}{b} a e^{ax} dx \right.$

$= \frac{1}{b} e^{ax} \sin bx + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2}\int e^{ax} \cos bx \, dx$

13-1-21

$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$$

$$I + \frac{a^2}{b^2}I = \frac{e^{ax}}{b}\left(\sin bx + \frac{a}{b}\cos bx\right)$$

$$\left(1+\frac{a^2}{b^2}\right)I = e^{ax}\left(\frac{b\sin bx + a\cos bx}{b^2}\right)$$

$$\frac{(a^2+b^2)I}{b^2} = \frac{1}{b^2}e^{ax}(a\cos bx + b\sin bx)$$

$$\therefore I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$$

---

3. Evaluate $\int \cos^n x\,dx$ by using Integration by parts.

Soln $I_n = \int \cos^n x\,dx = \int \cos^{n-1}x.\cos x\,dx$

$u = \cos^{n-1}x$ $\qquad dv = \cos x\,dx$

$du = n-1\cos^{n-2}x(-\sin x)dx$ $\qquad v = \sin x$

$$I_n = \cos^{n-1}x\sin x - \int \sin x(n-1)\cos^{n-2}x(-\sin x)dx$$

$$I_n = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x.\sin^2 x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$I_n = \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n.$$

$$I_n + (n-1)I_n = \cos^{n-1}x\sin x + (n-1)I_{n-2}$$

$$I_n[1+(n-1)] = \cos^{n-1}x\sin x + (n-1)I_{n-2}$$

$$nI_n = \cos^{n-1}x\sin x + (n-1)I_{n-2}$$

$$I_n = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}I_{n-2}.$$

Evaluate $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

Soln.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^{n-1}x.\sin x\,dx.$$

$$u = \sin^{n-1}x \qquad dv = \sin x\,dx$$

$$du = n-1\sin^{n-2}x\cos x\,dx \qquad v = -\cos x$$

$$I_n = -\sin^{n-1}x\cos x\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,(n-1)\sin^{n-2}x\cos x\,dx$$

$$I_n = 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\,dx.$$

$$= (n-1)I_{n-2} - (n-1)I_n.$$

$$I_n[1+(n-1)] = n-1\,I_{n-2}$$

$$nI_n = n-1\,I_{n-2}$$

$$I_n = \frac{n-1}{n}I_{n-2}.$$

Similarly $I_{n-2} = \frac{n-3}{n-2}I_{n-4}.$

$$\therefore I_n = \frac{n-1}{n}\frac{(n-3)}{n-2} I_{n-4}$$

Continuing in this way.

If n is even

we get $I_n = \frac{(n-1)(n-3)(n-5)(n-7)}{n(n-2)(n-4)(n-6)} \ldots I$

If n is even then $I = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$.

If n is odd then $I = \int_0^{\frac{\pi}{2}} \sin x\, dx$

$$= [-\cos x]_0^{\frac{\pi}{2}} = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \frac{1}{2}\cdot\frac{\pi}{2} \text{ n is even}$$

$$= \frac{(n-1)(n-3)(n-5)\ldots}{n(n-2)(n-4)\ldots} \frac{2}{3}\cdot 1 \text{ if n is odd}$$

Ex:

$$\int_0^{\frac{\pi}{2}} \sin^7 x\, dx = \frac{6\,4\,2}{7\,5\,3\,1} = \frac{48}{105} = \frac{16}{35}$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x\, dx = \frac{7\,5\,3\,1}{8\,6\,4\,2}\frac{\pi}{2} = \frac{35\pi}{256}.$$

Exercise $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$, find $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx$ and $\int_0^{\frac{\pi}{2}} \cos^6 x\, dx$.

Evaluate $\int \tan^n x\, dx$.

Soln

$$I_n = \int \tan^n x\, dx$$

$$= \int \tan^{n-2} x \cdot \tan^2 x\, dx.$$

$$= \int \tan^{n-2} x\,(\sec^2 x - 1)\, dx$$

$$= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx \quad \text{—(1)}$$

put $u = \tan x$, $du = \sec^2 x\, dx$

$$\therefore \int \tan^{n-2} x \sec^2 x = \int u^{n-2}\, du = \frac{u^{n-1}}{n-1}.$$

$$= \frac{\tan^{n-1} x}{n-1}$$

Sub in (1)

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}.$$

ie $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$

$n \neq 1.$

Exercise.

Evaluate $\int \sec^n x\, dx$

Ans. $\frac{1}{n-1} \tan x \sec^{n-2} x + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$

A.U. 2018 Evaluate $\int \tan^5\theta \sec^7\theta\, d\theta$

Soln $I = \int \tan^5\theta \sec^7\theta\, d\theta = \int \tan^4\theta \sec^6\theta (\sec\theta \tan\theta\, d\theta)$

$$I = \int (\sec^2\theta - 1)^2 \sec^6\theta \cdot (\sec\theta \tan\theta\, d\theta).$$

put $t = \sec\theta$ $\quad dt = \sec\theta \tan\theta\, d\theta.$

140. 18-1-19

$$I = \int (t^2-1)^2 t^6 \cdot dt$$

$$= \int (t^4 - 2t^2 + 1) t^6 \, dt$$

$$= \int t^{10} - 2t^8 + t^6 \, dt$$

$$= \int t^{10} - 2\int t^8 + \int t^6 \, dt$$

$$= \frac{t^{11}}{11} - 2\frac{t^9}{9} + \frac{t^7}{7} + C.$$

$$\int \tan^5\theta \sec^7\theta \, d\theta = \frac{\sec^{11}\theta}{11} - 2\frac{\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C.$$

Evaluate $\int \sin 4x \cos 5x \, dx$.

Soln $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$A = 4x, \; B = 5x$

$2\sin 4x \cos 5x = \sin(4x+5x) + \sin(4x-5x)$

$= \sin 9x + \sin(-x)$

$= \sin 9x - \sin x.$

$\therefore \int \sin 4x \cos 5x \, dx = \int \frac{1}{2}(\sin 9x - \sin x)\, dx$

$= \frac{1}{2}\int \sin 9x \, dx - \frac{1}{2}\int \sin x \, dx$

$= \frac{1}{2}\left(\frac{-\cos 9x}{9}\right) - \frac{1}{2}(-\cos x) + C$

$= \frac{1}{2}\left[\cos x - \frac{\cos 9x}{9}\right] + C.$

Evaluate $\int \tan^3 x\, dx$.

Soln: $\int \tan^3 x\, dx = \int \tan^2 x \tan x\, dx$.

$= \int (\sec^2 x - 1) \tan x\, dx$

$= \int \sec^2 x \tan x - \int \tan x\, dx$.

$= \frac{\tan^2 x}{2} - \log \sec x + C$.

Evaluate $\int \sec^3 x\, dx$.

Soln: $\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx$.

$u = \sec x \quad dv = \sec^2 x\, dx$.

$du = \sec x \tan x\, dx \quad v = \tan x$.

$\int \sec^3 x\, dx = \sec x \tan x - \int \tan x \sec x \tan x\, dx$.

$= \sec x \tan x - \int \tan^2 x \sec x\, dx$.

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx$

$= \sec x \tan x - \int \sec^3 x - \int \sec x\, dx$

$2\int \sec^3 x\, dx = \sec x \tan x - \int \sec x\, dx$.

$= \sec x \tan x - \log(\sec x + \tan x) + C$

$\int \sec^3 x = \frac{1}{2}\left[\sec x \tan x - \log(\sec x + \tan x)\right] + C$.

142

Evaluate $\int \sin^6 x \cos^3 x \, dx$.

Soln $\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx$

$= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx$

$= \int \sin^6 x \cos x - \int \sin^8 x \cos x \, dx$

$t = \sin x \quad dt = \cos x \, dx$

$= \int t^6 \, dt - \int t^8 \, dt$

$= \frac{t^7}{7} - \frac{t^9}{9} + c = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + c.$

A.U Jan. 2018 Evaluate $\int_0^{\pi/2} \cos^5 x \, dx$.

Soln $I = \int_0^{\pi/2} \cos^5 x \, dx = \int_0^{\pi/2} \cos^4 x \cos x \, dx$.

$= \int_0^{\pi/2} (\cos^2 x)^2 \cos x \, dx$

$= \int_0^{\pi/2} (1 - \sin^2 x)^2 \cos x \, dx$

$t = \sin x \quad dt = \cos x \, dx$

$x = 0, \quad t = 0$

$x = \frac{\pi}{2} \quad t = 1$

$\therefore I = \int_0^1 (1 - t^2)^2 \, dt = \int_0^1 1 + t^4 - 2t^2 \, dt$

$= \int_0^1 dt + \int_0^1 t^4 \, dt - 2 \int_0^1 t^2 \, dt$

$$= t\Big]_0^1 + \frac{t^5}{5}\Big]_0^1 - 2\,\frac{t^3}{3}\Big]_0^1$$

$$= 1 + \frac{1}{5} - \frac{2}{3} = \frac{15+3-10}{15} = \frac{8}{15}$$

Evaluate $\int_0^{\pi} \sin^2 x\, dx$.

$$\int_0^{\pi} \sin^2 x\, dx = \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \int_0^{\pi} \frac{1}{2}\, dx - \frac{1}{2}\int_0^{\pi} \cos 2x\, dx.$$

$$= \frac{1}{2}\left[x\right]_0^{\pi} - \frac{1}{2}\left(\frac{\sin 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{2}(\pi - 0) - \frac{1}{4}(0-0) = \frac{\pi}{2}$$

Exercise sums. P.T.

① $\int \sin^3 x \cos^2 x\, dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$

② ⓐ $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x\, dx = \frac{\pi}{16}$ ⓑ AU 2019 $\int_0^{\pi} \sin^2 x \cos^4 x\, dx = \frac{\pi}{16}$.

③ $\int \sin 8x \cos 5x\, dx = -\frac{1}{6}\cos 3x - \frac{1}{26}\cos 13x + C$

Integration by Trigonometric Substitution.

	Substitution	Identity
① $\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
② $\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
	$a\sinh\theta$	$1+\sinh^2\theta = \cosh^2\theta$
③ $\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$
	$a\cosh\theta$	$\cosh^2\theta - 1 = \sinh^2\theta$

EM
144

19-1-21

A.U 2019 Dec

Evaluate $\int\sqrt{a^2-x^2}\,dx$

<u>Soln</u> put $x = a\sin\theta$ $\quad dx = a\cos\theta\, d\theta$

$$\int\sqrt{a^2-x^2} = \int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\, d\theta$$

$$= \int a\cos\theta\; a\cos\theta\, d\theta$$

$$= \int a^2\cos^2\theta\, d\theta$$

$$= a^2\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{2}\int d\theta + \frac{a^2}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\frac{\sin 2\theta}{2}$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}$$

$$= \frac{a^2}{2}\theta + \frac{a\sin\theta\cdot a\cos\theta}{2}$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}$$

Evaluate $\int\sqrt{a^2+x^2}\,dx$.

<u>Soln</u> put $x = a\sinh\theta$

$dx = a\cosh\theta\cdot d\theta$.

$$\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\sinh^2\theta}\; a\cosh\theta\, d\theta$$

$$= \int a\sqrt{1+\sinh^2\theta}\; a\cosh\theta\, d\theta.$$

$$= a^2 \int \sqrt{\cosh^2\theta}\cdot\cosh\theta\, d\theta$$

$$= a^2 \int \cosh^2\theta\, d\theta$$

$$= a^2 \int \left(\frac{1+\cosh 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\int d\theta + \frac{a^2}{2}\int \cosh 2\theta\, d\theta.$$

$$= \frac{a^2}{2}(\theta) + \frac{a^2}{2}\frac{\sinh 2\theta}{2}.$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\frac{2\sinh\theta\cosh\theta}{2}$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sinh\theta\cosh\theta$$

$$= \frac{a^2}{2}\cdot\theta + \left[\frac{a\sinh\theta\cdot a\cosh\theta}{2.}\right]$$

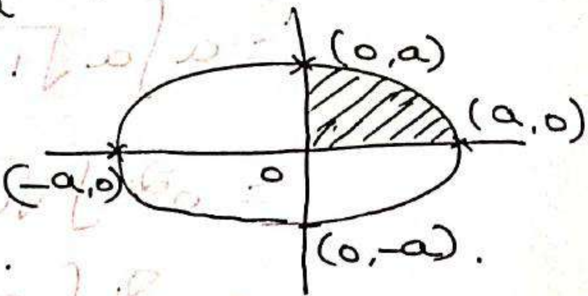
$$= \left[\frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x\sqrt{a^2+x^2}}{2}\right]$$

Exercise

Evaluate $\int \sqrt{x^2-a^2}\,dx$

Ans: $x\dfrac{\sqrt{x^2-a^2}}{2} - \dfrac{a^2}{2}\cosh^{-1}\dfrac{x}{a}$

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Soln: The ellipse is symmetrical about both axes.

Area of ellipse $= 4$ Area in I quadrant

$$= 4\int_0^a y\,dx.$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

$$\frac{y}{b}=\sqrt{\frac{a^2-x^2}{a^2}} \Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}.$$

$$\therefore \text{Area} = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\int_0^a \sqrt{a^2-x^2}\,dx\right]$$

$$= \frac{4b}{a}\left[\left(\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0^a + \left(x\sqrt{a^2-x^2}\right)_0^a\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\sin^{-1}\frac{a}{a} + 0\right]$$

$$= \frac{4b}{a}\cdot\frac{a^2}{2}\sin^{-1}1 = 2ab\left(\frac{\pi}{2}\right) = \pi ab.$$

Evaluate $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$.

Soln $x = 3\sec\theta$ $\quad dx = 3\sec\theta\tan\theta$.

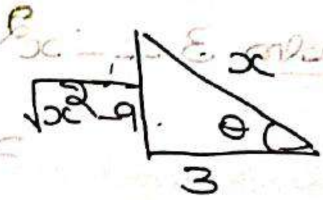
$$\int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}$$

$$= \int \frac{\tan\theta\,d\theta}{3\sec\theta\,3\sqrt{\sec^2\theta-1}}$$

$$= \int \frac{\tan\theta\,d\theta}{9\sec\theta\tan\theta} = \frac{1}{9}\int \cos\theta\,d\theta$$

III Unit in progress

$$= \frac{1}{9}\sin\theta.$$



$$\frac{x}{3} = \sec\theta.\quad \cos\theta = \frac{3}{x}$$

18/1/21

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} = \frac{\sqrt{x^2-9}}{9x} + C.$$

Exercise $\int \frac{x^2}{\sqrt{x^2+9}}\,dx$ sub. $x = 3\tan\theta$.

Ans. $\frac{1}{3}(x^2-18)\sqrt{x^2+9} + C$.

Evaluate $\int \sqrt{5+4x-x^2}\,dx$.

Soln $5+4x-x^2 = 9-(x^2-4x+4)$

$= 9-(x-2)^2$.

Put $t = x-2$ $\quad dt = dx$.

$$\int\sqrt{5+4x-x^2}\,dx = \int\sqrt{9-t^2}\,dt$$

$$= \int\sqrt{3^2-t^2}\,dt$$

$$= \frac{9}{2}\sin^{-1}\frac{t}{3} + \frac{1}{2}t\sqrt{9-t^2}$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\sqrt{5+4x-x^2} + C.$$

Evaluate $\int \frac{dx}{\sqrt{3x-x^2-2}}$

Soln $3x-x^2-2 = \frac{1}{4} - \left(x^2 - 3x + \frac{9}{4}\right)$

$$= \frac{1}{4} - \left(x-\frac{3}{2}\right)^2$$

$t = x - \frac{3}{2}$ $\quad dt = dx$

$$\therefore \int\frac{dx}{\sqrt{3x-x^2-2}} = \int\frac{dt}{\sqrt{\frac{1}{4}-t^2}}$$

$$= \int\frac{dt}{\sqrt{\left(\frac{1}{2}\right)^2-t^2}} = \sin^{-1}\frac{t}{\frac{1}{2}} + C$$

$$= \sin^{-1}2t + C$$

$$= \sin^{-1}2\left(x-\frac{3}{2}\right) + C.$$

$$= \sin^{-1}(2x-3) + C.$$

Exercise P.T. $\int\frac{dx}{\sqrt{3x^2+x-2}} = \frac{1}{\sqrt{3}}\cosh^{-1}\frac{6x+1}{5}+$

Integrals of type $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$.

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}} = A\int \frac{\frac{d}{dx}(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= 2A\sqrt{ax^2+bx+c} + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Evaluate $\int \frac{6x+5}{\sqrt{6+x-2x^2}}\,dx$.

$$6x+5 = A\frac{d}{dx}(6+x-2x^2)+B.$$

$$6x+5. = A(1-4x)+B$$

Comparing coeff of x on both sides.

$$6 = -4A \quad A = -\frac{6}{4} = -\frac{3}{2}.$$

Comparing constants on both sides.

$$5 = A+B.$$

$$5 = -\frac{3}{2}+B.$$

$$B = 5+\frac{3}{2} = \frac{13}{2}.$$

$$\int \frac{6x+5}{\sqrt{6+x-2x^2}}\,dx = -\frac{3}{2}\int \frac{\frac{d}{dx}(6+x-2x^2)}{\sqrt{6+x-2x^2}} + \frac{13}{2}\int \frac{dx}{\sqrt{6+x-2x^2}}$$

$$= -\frac{3}{2}\cdot 2\sqrt{6+x-2x^2} + \frac{13}{2}\int \frac{dx}{\sqrt{6+x-2x^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2}\frac{dx}{\sqrt{2\left(3+\frac{x}{2}-x^2\right)}}$$

$$I = -3\sqrt{6+x-x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{\frac{49}{16} - \left(x-\frac{1}{4}\right)^2}}$$

$$= -3\sqrt{6+x-x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{\left(\frac{7}{4}\right)^2 - \left(x-\frac{1}{4}\right)^2}}$$

$$= -3\sqrt{6+x-x^2} + \frac{13}{2\sqrt{2}} \sin^{-1} \frac{x-\frac{1}{4}}{\frac{7}{4}}$$

$$= -3\sqrt{6+x-x^2} + \frac{13}{2\sqrt{2}} \sin^{-1} \frac{4x-1}{7} + C.$$

Evaluate $\int \frac{\sqrt{5-x}}{\sqrt{x-2}}\, dx$

Soln: $\int \frac{\sqrt{(5-x)(5-x)}}{\sqrt{(x-2)(5-x)}}\, dx = \int \frac{5-x}{\sqrt{-10+7x-x^2}}$

$$5-x = A\frac{d}{dx}(-10+7x-x^2) + B.$$

$$5-x = A(7-2x) + B$$

Comparing Coef of x

$$-1 = -2A \qquad A = \frac{1}{2}$$

$$5 = 7A + B$$

$$5 = 7\left(\frac{1}{2}\right) + B$$

$$B = 5 - \frac{7}{2} = \frac{3}{2}$$

$$5-x = \frac{1}{2}(7-2x) + \frac{3}{2}$$

$$\int \frac{5-x}{\sqrt{-10+7x-x^2}}\,dx = \frac{1}{2}\int \frac{\frac{d}{dx}(-10+7x-x^2)}{\sqrt{-10+7x-x^2}}\,dx$$

$$+\frac{3}{2}\int \frac{dx}{\sqrt{-10+7x-x^2}}$$

$$= \frac{1}{2}\,2\sqrt{-10+7x-x^2} + \frac{3}{2}\,\frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{7}{2}\right)^2}}$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\sin^{-1}\frac{x-\frac{7}{2}}{\frac{3}{2}} + C$$

$$= \sqrt{-10+7x-x^2} + \frac{3}{2}\sin^{-1}\frac{2x-7}{3} + C.$$

Evaluate $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Soln put $x+1=\frac{1}{t}$ $dx = -\frac{1}{t^2}dt$

$$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1-t+t^2}{t^2}}}$$

$$= -\int \frac{dt}{\sqrt{t^2-t+1}}$$

$$= -\int \frac{dt}{\sqrt{\frac{3}{4}+\left(t-\frac{1}{2}\right)^2}} = -\sinh^{-1}\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C.$$

$$= -\sinh^{-1}\frac{\frac{1}{x+1}-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C.$$

$$= -\sinh^{-1}\frac{\frac{2-(x+1)}{2(x+1)}}{\frac{\sqrt{3}}{2}} + C.$$

$$= -\sinh^{-1}\frac{1-x}{\sqrt{3}(x+1)} + C.$$

Exercise ① $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}} = \frac{1}{\sqrt{3}}$.

② $\int \frac{dx}{(x+1)\sqrt{x^2+1}} = -\frac{1}{2}\sinh^{-1}\frac{1-x}{1+x} + C.$

③ $\int \frac{x}{\sqrt{x^2+x+1}} = \sqrt{x^2+x+1} - \frac{1}{2}\sinh^{-1}\frac{2x+1}{\sqrt{3}} + C$

Evaluate $\int \sqrt{x^2+2x+10}\, dx$.

Soln $x^2+2x+10 = (x+1)^2+9$.

$$\int \sqrt{x^2+2x+10}\, dx = \int \sqrt{(x+1)^2+3^2}\, dx.$$

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}.$$

$$\int \sqrt{(x+1)^2+9}\, dx = \frac{1}{2}(x+1)\sqrt{x^2+2x+10} + \frac{9}{2}\sinh^{-1}\frac{x+1}{3} + C.$$

Evaluate $\int \sqrt{1+x-2x^2}\, dx$.

Soln: $1+x-2x^2 = 2\left(\frac{1}{2}+\frac{x}{2}-x^2\right)$

$$= 2\left[\frac{9}{16}-\left(x-\frac{1}{4}\right)^2\right]$$

$$\sqrt{1+x-2x^2} = \sqrt{2}\sqrt{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}$$

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$\int\sqrt{1+x-2x^2}\,dx = \sqrt{2}\int\sqrt{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}$$

$$= \sqrt{2}\left[\frac{x-\frac{1}{4}}{2}\sqrt{\left(\frac{3}{4}\right)^2-\left(x-\frac{1}{4}\right)^2} + \frac{9}{2(16)}\sin^{-1}\frac{x-\frac{1}{4}}{\frac{3}{4}}\right]+C.$$

$$= \frac{4x-1}{8}\sqrt{1+x-2x^2} + \frac{9}{16\sqrt{2}}\sin^{-1}\frac{4x-1}{3} + C.$$

Integrals of type $\int px+q\sqrt{ax^2+bx+c}\,dx$.

$$\int(px+q)\sqrt{ax^2+bx+c}\,dx = A\int\sqrt{ax^2+bx+c}\,\frac{d}{dx}(ax^2+bx+c)$$

$$+ b\int\sqrt{ax^2+bx+c}\,dx.$$

$$= A\frac{(ax^2+bx+c)^{\frac{3}{2}}}{\frac{3}{2}} + B\int\sqrt{ax^2+bx+c}\,dx.$$

Second Integral in RHS Can be easily Evaluated.

Evaluate $\int(3x-2)\sqrt{x^2+x+1}\,dx$.

<u>Soln</u>

$$3x-2 = A\frac{d}{dx}(x^2+x+1) + B.$$

$$3x-2 = A(2x+1) + B.$$

Comparing Coef of x on bothsides.

$$3 = 2A \quad \therefore A = \frac{3}{2}$$

Comparing Constants:

$$-2 = A + B$$

$$-2 = \frac{3}{2} + B$$

$$B = -2 - \frac{3}{2} = -\frac{7}{2}.$$

$$I = \frac{3}{2}\int \sqrt{x^2+x+1}\,\frac{d}{dx}(x^2+x+1)\,dx - \frac{7}{2}\int\sqrt{x^2+x+1}$$

$$= \frac{3}{2}\frac{(x^2+x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{7}{2}\int\sqrt{\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}}\,dx.$$

$$= (x^2+x+1)^{\frac{3}{2}} - \frac{7}{2}\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} - \frac{3}{4}\frac{1}{2}\sinh^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+c$$

$$= (x^2+x+1)^{\frac{3}{2}} - \frac{7}{4}\left(\frac{2x+1}{2}\right)\sqrt{x^2+x+1}$$

$$-\frac{21}{16}\sinh^{-1}\frac{2x+1}{\sqrt{3}} + C.$$

Exercise

$$\int(x+1)\sqrt{x^2-2x+2}\,dx = \frac{1}{3}(x^2-2x+2)^{\frac{3}{2}}$$

$$+(x-1)\sqrt{x^2-2x+2} + \sinh^{-1}(x-1) + C.$$

Evaluate $\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$

Soln put $x = 3\sin\theta$ $dx = 3\cos\theta\,d\theta$

$$\frac{x^2}{\sqrt{9-x^2}} = \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} = \frac{9\sin^2\theta}{3\cos\theta}.$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta$$

$$= 9\int \sin^2\theta\,d\theta = 9\int \frac{1-\cos 2\theta}{2}\,d\theta.$$

$$= \frac{9}{2}\left[\int d\theta - \int \cos 2\theta\,d\theta\right]$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C.$$

$$= \frac{9}{2}\left[\theta - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right] + C$$

$$= \frac{9}{2}\theta - 3\sin\theta\, 3\cos\theta + C.$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - x\sqrt{9-x^2} + C.$$

Evaluate $\displaystyle\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}.$

Soln put $x = a\tan\theta \quad dx = a\sec^2\theta\,d\theta$

$x = 0, \quad \theta = 0$

$x = a \quad a = a\tan\theta \quad \tan\theta = 1, \theta = \frac{\pi}{4}.$

$$\int_0^a \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{4}} \frac{a\sec^2\theta\,d\theta}{(a^2+a^2\tan^2\theta)^{\frac{3}{2}}}$$

$$= a\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\,d\theta}{(a^2\sec^2\theta)^{\frac{3}{2}}} = \frac{a}{a^3}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta.$$

156

$$= \frac{1}{a^2}\int_0^{\pi/4} \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{a^2}\int_0^{\pi/4}\cos\theta\,d\theta = \frac{1}{a^2}\left(\sin\theta\right)_0^{\pi/4}$$

$$= \frac{1}{a^2}\left(\sin\frac{\pi}{4} - \sin 0\right)$$

$$= \frac{1}{a^2}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}a^2}.$$

Evaluate $\int_0^{2/3}\sqrt{4-9x^2}\,dx$.

<u>Soln</u> put $x = \frac{2}{3}\sin\theta$ $dx = \frac{2}{3}\cos\theta\,d\theta$.

$x = 0$, $\sin\theta = 0$, $\theta = 0$

$x = \frac{2}{3}$ $\frac{2}{3} = \frac{2}{3}\sin\theta$ $\sin\theta = 1$, $\theta = \frac{\pi}{2}$.

$$\int_0^{2/3}\sqrt{4-9x^2}\,dx = \int_0^{\pi/2}\sqrt{4-9\left(\frac{2}{3}\sin\theta\right)^2}\,\frac{2}{3}\cos\theta\,d\theta.$$

$$= \frac{2}{3}\int_0^{\pi/2}\sqrt{4-4\sin^2\theta}\,.\cos\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} 2\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= \frac{a^4}{3}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{4}{6}\left[(\theta)_0^{\frac{\pi}{2}} + \left(\frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{2}}\right]$$

$$= \frac{4}{6}\left[\frac{\pi}{2} + (0 - \sin 0)\right] = \frac{4}{6}\cdot\frac{\pi}{2} = \frac{\pi}{3}$$

Exercise ① $\int_0^{\frac{1}{2}} x\sqrt{1-4x^2}\, dx = \frac{1}{2}$.

② $\int_0^{a} x^2\sqrt{a^2-x^2}\, dx = \frac{\pi a^4}{16}$.

Trigonometric Formulae.

① $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$; $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

② $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$. $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Evaluate $\int \frac{dx}{\sqrt{x^2+2x+5}}$

Soln $\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x+1)^2+4}}$

$$= \int \frac{dx}{\sqrt{(x+1)^2+2^2}} = \sinh^{-1}\frac{x+1}{2} + C$$

$$= \log\left[\frac{x+1}{2} + \sqrt{\left(\frac{x+1}{2}\right)^2+1}\right] + C$$

$$= \log\left|\frac{x+1}{2} + \sqrt{\frac{x^2+2x+1+4}{2^2}}\right| + C$$

158

$= \log \left| x+1 + \sqrt{x^2+2x+5} \right| - \log 2 + C.$

$\left(\because \sinh^{-1} x = \log \left| x + \sqrt{x^2+1} \right| \right)$

Evaluate $\int \frac{dx}{a^2+x^2}$

Soln. $x = a \tan\theta \quad dx = a\sec^2\theta \cdot d\theta$

$$I = \int \frac{a\sec^2\theta \, d\theta}{a^2+a^2\tan^2\theta} = \int \frac{a\sec^2\theta}{a^2\sec^2\theta} d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

Integrals of type.

$$\int \frac{px+q}{ax^2+bx+c} dx$$

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B$$

find A & B.

Ex $\int \frac{2x+3}{x^2+x+1} dx.$

Soln $2x+3 = A(2x+1) + B.$

Comparing Coef of x on both sides.

$2 = 2A \implies A = 1$

Comparing Constants

$3 = A+B \quad 3 = 1+B \quad B = 2.$

$$\int \frac{2x+3}{x^2+x+1}\,dx = \int \frac{1\cdot(2x+1)}{x^2+x+1}\,dx + \int \frac{2\,dx}{x^2+x+1}$$

$$= \log(x^2+x+1) + \int \frac{2\,dx}{(x+\frac{1}{2})^2+\frac{3}{4}}.$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \log(x^2+x+1) + 2\,\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C.$$

$$= \log(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C.$$

Exercise $\displaystyle\int \frac{3x+1}{2x^2+x+3}\,dx = \frac{3}{4}\log(2x^2+x+3)$

$$+\frac{1}{2\sqrt{23}}\tan^{-1}\frac{4x+1}{\sqrt{23}} + C.$$

$$\int \frac{dx}{x^2+2x+5} = \frac{1}{2}\tan^{-1}\frac{x+1}{2}.$$

II Integrals of type.

$$\int \frac{dx}{Ax^2+B\sqrt{Cx^2+D}} \qquad \text{put } x = \frac{1}{t}$$

Evaluate $\displaystyle\int_0^1 \frac{dx}{1+x^2\sqrt{x^2+2}}$

Soln put $x = \frac{1}{t}$ $dx = -\frac{1}{t^2}\,dt$.

$$I = \int_0 \frac{dx}{1+x^2\sqrt{x^2+2}} = \int_\infty^1 \frac{-\frac{1}{t^2}\,dt}{t^2+1\sqrt{2+\frac{1}{t^2}}}$$

$x=0,\ t=\infty,\ x=1,\ t=1.$

$$= -\int_{\infty}^{1} \frac{\frac{1}{t^2}\,dt}{\left(1+\frac{1}{t^2}\right)\sqrt{\frac{1}{t^2}+2}}$$

$$= \int_{1}^{\infty} \frac{\frac{1}{t^2}\,dt}{\frac{t^2+1}{t^2}\sqrt{\frac{1+2t^2}{t}}}$$

$$I = \int_{1}^{\infty} \frac{t\,dt}{t^2+1\sqrt{2t^2+1}}.$$

$2t^2+1 = u^2 \quad 4t\,dt = 2u\,du$

$t\,dt = \frac{1}{2}u\,du.$

$t \to \infty,\ u \to \infty \quad t=1\ u=\sqrt{2+1}=\sqrt{3}.$

$$I = \int_{\sqrt{3}}^{\infty} \frac{\frac{u}{2}\,du}{\frac{u^2+1}{2}\sqrt{u^2}} = \int_{\sqrt{3}}^{\infty} \frac{\frac{u}{2}\,du}{\frac{(u^2+1)u}{2}}.$$

$$= \int_{\sqrt{3}}^{\infty} \frac{du}{u^2+1} = \left[\tan^{-1}u\right]_{\sqrt{3}}^{\infty} = \tan^{-1}\infty - \tan^{-1}\sqrt{3}$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

Integrals of type. $\int \sqrt{(x-\alpha)(\beta-x)}\,dx$

$$\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}} \qquad \int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx \quad \beta > \alpha.$$

Substitution is $x = \alpha\cos^2\theta + \beta\sin^2\theta$

Evaluate $\int \sqrt{(x-3)(7-x)}\, dx$.

<u>Soln</u> $x = 3\cos^2\theta + 7\sin^2\theta$.

$dx = [6\cos\theta(-\sin\theta) + 14\sin\theta\cos\theta]\, d\theta$.

$dx = 8\sin\theta\cos\theta\, d\theta$

$x-3 = 3\cos^2\theta + 7\sin^2\theta - 3$

$= 7\sin^2\theta + 3(\cos^2\theta - 1)$

$= 7\sin^2\theta + 3(-\sin^2\theta)$

$= 4\sin^2\theta$.

$7-x = 7-(3\cos^2\theta + 7\sin^2\theta)$

$= 7 - 7\sin^2\theta - 3\cos^2\theta$

$= 7(1-\sin^2\theta) - 3\cos^2\theta$

$= 7\cos^2\theta - 3\cos^2\theta$

$= 4\cos^2\theta$.

$\int \sqrt{(x-3)(7-x)}\, dx = \int \sqrt{4\sin^2\theta\, 4\cos^2\theta}\, .\, 8\sin\theta\cos\theta\, d\theta$

$= 32\int \sin^2\theta\cos^2\theta\, d\theta$

$= 32\int (\sin\theta\cos\theta)^2\, d\theta$

$= 8\int (2\sin\theta\cos\theta)^2\, d\theta$

$= 8\int (\sin 2\theta)^2\, d\theta = 8\int \sin^2 2\theta\, d\theta$.

$= 8\int \frac{1-\cos 4\theta}{2}\, d\theta$.

$= \frac{8}{2}\int (1-\cos 4\theta)\, d\theta$

$$= 4\left[\theta - \frac{\sin 4\theta}{4}\right] + C.$$

$$= 4\theta - \sin 4\theta + C.$$

$$= 4\theta - 2\sin 2\theta \cos 2\theta + C.$$

$$= 4\theta - 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1) + C.$$

$$= 4\sin^{-1}\sqrt{\frac{x-3}{4}} - (4\sin\theta\cos\theta)\left[2\cdot\left(\frac{7-x}{4}\right) - 1\right] + C$$

$$= 4\sin^{-1}\sqrt{\frac{x-3}{4}} - \sqrt{(x-3)(7-x)}\left[\frac{7-x}{2} - 1\right] + C$$

$$= 4\sin^{-1}\sqrt{\frac{x-3}{4}} - \sqrt{(x-3)(7-x)}\left(\frac{5-x}{2}\right) + C.$$

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3 Evaluate $\int \frac{dx}{\sqrt{(x-3)(7-x)}}$.

Soln

$$x = 3\cos^2\theta + 7\sin^2\theta$$

$$dx = [6\cos\theta(-\sin\theta) + 14\sin\theta\cos\theta]\,d\theta$$

$$= 8\sin\theta\cos\theta]\,d\theta.$$

$$x - 3 = 3\cos^2\theta + 7\sin^2\theta - 3$$

$$= 3(\cos^2\theta - 1) + 7\sin^2\theta$$

$$= 3(-\sin^2\theta) + 7\sin^2\theta$$

$$= 4\sin^2\theta$$

$$7 - x = 7 - (3\cos^2\theta + 7\sin^2\theta)$$

$$= 7(1 - \sin^2\theta) - 3\cos^2\theta$$

$$= 7\cos^2\theta - 3\cos^2\theta = 4\cos^2\theta.$$

$\sqrt{(x-3)(7-x)} = \sqrt{4\sin^2\theta \, 4\cos^2\theta} = 4\sin\theta\cos\theta.$

$$\int \frac{dx}{\sqrt{(x-3)(7-x)}} = \int \frac{8\sin\theta\cos\theta}{4\sin\theta\cos\theta}\,d\theta$$

$$= 2\int d\theta = 2\theta + C.$$

$$= 2\sin^{-1}\sqrt{\frac{x-3}{4}} + C$$

$$= 2\sin^{-1}\frac{\sqrt{x-3}}{2} + C.$$

$\left[\because \frac{x-3}{4} = \sin^2\theta\right.$

$\sin\theta = \sqrt{\frac{x-3}{4}}$

$\theta = \sin^{-1}\sqrt{\frac{x-3}{4}}$

Evaluate $\int \frac{\sqrt{5-x}}{\sqrt{x-2}}\,dx$. $\int \frac{dx}{\sqrt{(x-3)(7-x)}} = 2\sin^{-1}\frac{\sqrt{x-3}}{2} + C$

Ans. $-3\sin^{-1}\frac{\sqrt{5-x}}{3} + \sqrt{(5-x)(x-2)} + C$.

Integrand is of first degree in $\cos x$ and $\sin x$, substitution is

$t = \tan\frac{x}{2}$. $dt = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx$

$dt = \frac{1}{2}(1+t^2)\,dx$ $\quad dx = \frac{2\,dt}{1+t^2}$.

Example. Evaluate $\int_0^{\pi} \frac{dx}{5+4\cos x}$.

Soln: let $t = \tan\frac{x}{2}$. $dx = \frac{2\,dt}{1+t^2}$.

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$x = 0,\ t = 0,\quad x = \pi,\ t = \infty$

164.

$$\int_0^{\pi} \frac{dx}{5+4\cos x} = \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^{\infty} \frac{\frac{2dt}{1+t^2}}{\frac{5(1+t^2)+4(1-t^2)}{1+t^2}}$$

$$= \int_0^{\infty} \frac{2dt}{5+5t^2+4-4t^2}$$

$$= \int_0^{\infty} \frac{2dt}{9+t^2} = \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\infty}$$

$$= \frac{2}{3}\left(\frac{\pi}{2}-0\right) = \frac{2}{3}\frac{\pi}{2} = \frac{\pi}{3}.$$

Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{9\cos x+12\sin x}$.

Soln. Let $t = \tan\frac{x}{2}$ $dx = \frac{2dt}{1+t^2}$

when $x=0$, $t=0$, $x=\frac{\pi}{2}$, $t=1$

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{9\cos x+12\sin x} = \int_0^{1} \frac{\frac{2dt}{1+t^2}}{9\left(\frac{1-t^2}{1+t^2}\right)+12\left(\frac{2t}{1+t^2}\right)}$$

$$= \int_0^{1} \frac{2\,dt}{9(1-t^2)+24t}$$

$$= \frac{2}{3}\int_0^{1} \frac{dt}{3-3t^2+8t}$$

$$= \frac{2}{3}\int_0^1 \frac{dt}{(3-t)(3t+1)}.$$

$$\frac{1}{(3-t)(3t+1)} = \frac{A}{3-t} + \frac{B}{3t+1}$$

$$\frac{1}{(3-t)(3t+1)} = \frac{A(3t+1) + B(3-t)}{(3-t)(3t+1)}$$

$$1 = A(3t+1) + B(3-t)$$

Comparing coef of t on both sides

$0 = 3A - B$. — ①

put $t = 3$, $1 = A(9+1)$ $10A = 1$ $A = \frac{1}{10}$.

Sub in ①

$0 = 3\left(\frac{1}{10}\right) - B$ $B = \frac{3}{10} - 0 = \frac{3-0}{10} = \frac{3}{10}$.

$$\frac{1}{(3-t)(3t+1)} = \frac{1}{10(3-t)} + \frac{3}{10}\frac{1}{(3t+1)}$$

$$\therefore \quad I = \frac{2}{3}\int_0^1 \left[\frac{1}{10(3-t)} + \frac{3}{10}\frac{1}{3t+1}\right] dt.$$

$$= \frac{2}{3(10)}\int_0^1 \left[\frac{1}{3-t} + \frac{3}{3t+1}\right] dt$$

$$= \frac{1}{15}\left[-\log(3-t) + \log(3t+1)\right]_0^1$$

$$= \frac{1}{15}\left[\log\frac{3t+1}{3-t}\right]_0^1 = \frac{1}{15}\left[\log 2 - \log\frac{1}{3}\right]$$

$$= \frac{1}{15}\log\frac{2}{\frac{1}{3}} = \frac{1}{15}\log 6$$

166.

3.9. Integration By partial fraction.

$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$. by using partial fractions.

Soln

$$f(x) = \frac{x^2+2x-1}{2x^3+3x^2-2x}$$

$$2x^3+3x^2-2x = x(2x^2+3x-2)$$

$$= x(2x^2+4x-x-2)$$

$$= x[2x(x+2)-(x+2)]$$

$$= x(x+2)(2x-1).$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A(x+2)(2x-1)+Bx(2x-1)+C\,x(x+2)}{x(x+2)(2x-1)}$$

$$x^2+2x-1 = A(x+2)(2x-1)+Bx(2x-1)+Cx(x+$$

put $x=0$, $-1 = -2A$ $\quad A = \frac{1}{2}$

$x=-2$ $\quad 4-4-1 = B(-2)(-4-1)$

$$-1 = 10B$$

$$B = -\frac{1}{10}$$

Comparing Coef. x^2 on both sides.

$$1 = 2A + 2B + C.$$

$$1 = 2\left(\frac{1}{2}\right) + 2\left(-\frac{1}{10}\right) + C.$$

$$1 = 1 - \frac{1}{5} + C$$

$$C = \frac{1}{5}.$$

$$\int \frac{x^2+2x+1}{2x^3+3x^2-2x} = \int \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}\, dx.$$

$$= \frac{1}{2}\int \frac{dx}{x} - \frac{1}{10}\int \frac{dx}{x+2} + \frac{1}{5(2)}\int \frac{2dx}{2x-1}$$

$$= \frac{1}{2}\log|x| - \frac{1}{10}\log|x+2| + \frac{1}{10}\log|2x-1| + C$$

Evaluate $\int \frac{dx}{x^2-a^2}$

Soln

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}.$$

$$\frac{1}{(x-a)(x+a)} = \frac{A(x+a)+B(x-a)}{(x-a)(x+a)}.$$

$$1 = A(x+a) + B(x-a).$$

$$x = a; \quad 1 = A(2a) \Rightarrow A = \frac{1}{2a}$$

$$x = -a \quad 1 = B(-a-a) \Rightarrow B = -\frac{1}{2a}.$$

$$\int \frac{dx}{x^2-a^2} = \int \frac{1}{2a(x-a)}\, dx - \frac{1}{2a}\int \frac{dx}{x+a}.$$

$$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right]$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$$

168.

Evaluate $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c.$

Evaluate $\int \frac{x+4}{6x-7-x^2} dx.$

$$x+4 = A \frac{d}{dx}(6x-7-x^2) + B.$$

$$x+4 = A(6-2x) + B.$$

Comparing Coef of x

$$1 = -2A \quad A = -\frac{1}{2}$$

Comparing Constants.

$$4 = 6A + B$$

$$4 = 6\left(-\frac{1}{2}\right) + B.$$

$$4 = -3 + B \quad B = 7.$$

$$\therefore \int \frac{x+4}{6x-7-x^2} dx = -\int \frac{\frac{1}{2}(6-2x)}{6x-7-x^2} dx + 7 \int \frac{dx}{6x-7-x^2}$$

$$= -\frac{1}{2} \log(6x-7-x^2) + 7 \int \frac{dx}{2-(x-3)^2}$$

$$= -\frac{1}{2} \log(6x-7-x^2) + \frac{7}{2\sqrt{2}} \log \left| \frac{\sqrt{2}+(x-3)}{\sqrt{2}-(x-3)} \right| + C$$

Evaluate $\int \frac{dx\,(x+4)}{6x-7-x^2}$

Soln $I = \int \frac{(x+4)\,dx}{6x-7-x^2}$

$$x+4 = A \frac{d}{dx}(6x-7-x^2) + B$$

$$x+4 = A(6-2x) + B.$$

Comparing Coef of x on both sides.

$$1 = -2A \quad A = -\frac{1}{2}$$

Comparing consts. $4 = 6A + B$

$$4 = 6\left(-\frac{1}{2}\right) + B$$

$$4 = -3 + B$$

$$B = 7.$$

$$\int \frac{(x+4)\,dx}{6x-7-x^2} = -\frac{1}{2}\int \frac{(6-2x)}{6x-7-x^2}dx + 7\int \frac{dx}{6x-7-x^2}$$

$$= -\frac{1}{2}\int \frac{6-2x}{6x-7-x^2}dx + 7\int \frac{dx}{-7-(x^2-6x+9)+9}$$

$$= -\frac{1}{2}\int \frac{6-2x}{6x-7-x^2}dx + 7\int \frac{dx}{2-(x-3)^2}$$

$$= -\frac{1}{2}\log|6x-7-x^2| + \frac{7}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+(x-3)}{\sqrt{2}-x+3}\right| + C$$

$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}dx$ by using partial fractions.

Soln: The degree of numerator is higher than the degree of the denominator. Divide the numerator by denominator.

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4 - 2x^2 + 4x + 1 \\ & \underline{x^4 - x^3 - x^2 + x} \\ & x^3 - x^2 + 3x + 1 \\ & \underline{x^3 - x^2 - x + 1} \\ & 4x \end{array}$$

170.

$$\therefore \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

Factorise the denominator x^3-x^2-x+1

$$x^3-x^2-x+1 = (x-1)(x^2-1)$$

$$x^3-x^2-x+1 = (x-1)(x-1)(x+1)$$
$$= (x-1)^2(x+1)$$

$$x-1\overline{)\,x^3-x^2-x+1\,}\;\; x^2-1$$
$$-\;x^3-x^2$$
$$\overline{\quad -x+1}$$
$$-x+1$$
$$\overline{\quad 0}$$

$$\therefore \frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}.$$

$$\therefore \frac{4x}{x^3-x^2-x+1} = \frac{A(x-1)(x+1)+B(x+1)+C(x-1)^2}{(x-1)^2(x+1)}$$

$$4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2.$$

$x=1$, $4 = B(1+1)$ $\quad 2B = 4$ $\quad \boxed{B=2}$

$x=-1$, $-4 = C(-1-1)^2$ $\quad -4 = 4C$ $\quad \boxed{C=-1}$

Comp. Coef. of x^2 on both sides.

$0 = A+C$ $\quad$ sub $C=-1$, $\boxed{A=1}$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx + \int \frac{1}{x-1}\,dx + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x+1}.$$

$$= \frac{(x+1)^2}{2} + \log|x-1| - \frac{2}{x-1} - \log|x+1|$$

$$= \frac{(x+1)^2}{2} - \frac{2}{x-1} + \log\left|\frac{x-1}{x+1}\right| + C.$$

Evaluate $\int \frac{x^2+x-1}{x^2-x-1}\,dx$.

$$\frac{x^2+x-1}{x^2-x-1} = 1 + \frac{2x}{x^2-x-1}$$

$$\begin{array}{r|l} & 1 \\ \hline x^2-x-1 & x^2+x-1 \\ & x^2-x-1 \\ & \overline{\quad 2x \quad} \end{array}$$

quotient 1

Remainder $2x$.

$$2x = A\frac{d}{dx}(x^2-x-1) + B$$

$$2x = A(2x-1) + B.$$

Comp. Coef of x on both sides.

$$2 = 2A \Rightarrow A = 1$$

Comp. Const on both sides.

$$0 = -A + B \Rightarrow A = B = 1$$

$$\int \frac{x^2+x-1}{x^2-x-1}\,dx = \int dx + \int \frac{2x}{x^2-x-1}\,dx +$$

$$= \int dx + \int \frac{2x\,dx}{x^2-x-1} + \int \frac{dx}{(x-\frac{1}{2})^2-(\frac{\sqrt5}{2})^2} + \int \frac{dx}{(x-\frac{1}{2})^2-\frac{5}{4}}$$

$$= x + \log|x^2-x-1| + \frac{1}{2\frac{\sqrt5}{2}} \log\left|\frac{x-\frac{1}{2}-\frac{\sqrt5}{2}}{x-\frac{1}{2}+\frac{\sqrt5}{2}}\right| + C.$$

$$= x + \log|x^2-x-1| + \frac{1}{\sqrt5}\log\frac{2x-1-\sqrt5}{2x-1+\sqrt5} + C.$$

$$= x + \log|x^2-x-1| + \frac{1}{\sqrt5}\log\frac{2x-1-\sqrt5}{2x-1+\sqrt5} + C.$$

$$\int \frac{10}{(x-1)(x^2+9)}\,dx.$$

Soln

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}.$$

179
179.

$$= \frac{A(x^2+9) + (Bx+C)(x-1)}{(x-1)(x^2+9)}$$

$\therefore 10 = A(x^2+9) + (Bx+C)(x-1)$.

Comp. Coef of x^2 on both sides

$0 = A + B.$

Put $x=1$, $10 = A(1+9)$ $A = 1$

$B = -1.$

Equating Const on both sides.

$0 = 9A - C. \Rightarrow 0 = 9(1) - C$

$C = 9.$

$$\therefore \int \frac{10}{(x-1)(x^2+9)} dx = \int \frac{dx}{x-1} + \int \frac{-x+9}{x^2+9} dx.$$

$$= \int \frac{dx}{x-1} - \frac{1}{2}\int \frac{2x}{x^2+9} dx + 9\int \frac{dx}{x^2+9}.$$

$$= \log|x-1| - \frac{1}{2}\log|x^2+9| + 9\frac{1}{3}\tan^{-1}\frac{x}{3} +$$

$$= \log|x-1| - \frac{1}{2}\log|x^2+9| + 3\tan^{-1}\frac{x}{3} + C$$

$\int \operatorname{cosech} x \, dx$

Soln $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\operatorname{cosech} x = \frac{2}{e^x - \frac{1}{e^x}} = \frac{2e^x}{(e^x)^2 - 1}.$$

$$\int \operatorname{cosech} x \, dx = \int \frac{2e^x}{(e^x)^2 - 1} dx$$

put $t = e^x$. $dt = e^x dx$.

$$\therefore \int \operatorname{cosech} x \, dx = \int \frac{2\, dt}{t^2 - 1}$$

$$= 2 \cdot \frac{1}{2} \log \frac{t-1}{t+1}$$

$$= \log \left| \frac{e^x - 1}{e^x + 1} \right|$$

$$= \log \frac{e^{\frac{x}{2}} e^{\frac{x}{2}} - e^{\frac{x}{2}} e^{-\frac{x}{2}}}{e^{\frac{x}{2}} e^{\frac{x}{2}} + e^{\frac{x}{2}} e^{-\frac{x}{2}}}$$

$$= \log \frac{e^{\frac{x}{2}} \left[e^{\frac{x}{2}} - e^{-\frac{x}{2}} \right]}{e^{\frac{x}{2}} \left[e^{\frac{x}{2}} + e^{-\frac{x}{2}} \right]}$$

$$= \log \tanh \frac{x}{2}.$$

3.10 Improper Integrals.

The Improper Integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called Convergent if the corresponding limit exists and divergent if the limit does not exist.

$\int_{-\infty}^{\infty} f(x)\,dx$ is defined as.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx,$$

is Convergent if both terms on RHS. Convergent

and divergent if either term diverges

Example 1

Evaluate $\int_1^{\infty} \frac{1}{x}\,dx$ and determine whether the Integral is Convergent or divergent

Soln

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \Big[\log|x|\Big]_1^t$$

$$= \lim_{t\to\infty} \big[\log t - \log 1\big]$$

$$= \lim_{t\to\infty} \log t = \infty.$$

The limit does not exist as a finite number and so Improper Integral $\int_1^{\infty} \frac{1}{x}\,dx$.

is divergent.

Evaluate $\int_2^\infty \frac{1}{x^2}\,dx$ and determine whether the Integral is Convergent or divergent.

Soln $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x^2}\,dx$

$= \lim_{t\to\infty} \left[\left(-\frac{1}{x}\right)_2^t\right]$

$= \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{2}\right) = \frac{1}{2}$

The limit exist as a finite number and so the Integral $\int_2^\infty \frac{1}{x^2}\,dx$ is Convergent.

Evaluate $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$. check. Convergent or n

Soln. $\int_4^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_4^t \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to\infty} \left[2\sqrt{x}\right]_4^t$

$\lim_{t\to\infty} \left[2\sqrt{t} - 2\sqrt{4}\right] = \infty.$

∴ The limit does not exist as a

---

finite number and so the Integral $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent.

For what values of p is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent. use p – test.

Soln For $p=1$ the Integral is divergent (by exe: 1).

Assume $p \neq 1$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx.$$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{1}{(1-p)x^{p-1}}\right]_1^t.$$

$$= \lim_{t\to\infty} \frac{1}{(1-p)}\left[\frac{1}{t^{p-1}} - 1\right]$$

If $p>1$, $p-1>0$ $t^{p-1} \to \infty$ $\frac{1}{t^{p-1}} \to 0$.

$$\therefore \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}(-1) = \frac{1}{p-1}$$

$\therefore$ The Integral Converges if $p>1$.

If $p<1$ $p-1<0$ $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t\to\infty$ and so the Integral diverges.

$\therefore \int_1^\infty \frac{1}{x^p}$ diverges if $p \leq 1$.

177

$$\int_3^\infty \frac{dx}{(x-2)^{3/2}}$$ Check Convergent or no

Soln

$$\int_3^\infty \frac{dx}{(x-2)^{\frac{3}{2}}} = \int_3^\infty (x-2)^{-\frac{3}{2}} dx$$

$$= \left[\frac{(x-2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right]_3^\infty$$

$$= \left[\frac{(x-2)^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_3^\infty$$

$$= -2\left[\frac{1}{\sqrt{x-2}}\right]_3^\infty$$

$$= -2[0-1] = 2.$$

$$\int_0^\infty \frac{x^2}{\sqrt{1+x^3}} dx.$$

Soln. put $1+x^3 = t$ $\quad x=0, t=1$

$3x^2 dx = dt$ $\quad x=\infty \; t=\infty.$

$$\therefore \int_0^\infty \frac{x^2}{\sqrt{1+x^3}} dx = \int_1^\infty \frac{dt}{2\sqrt{t}} = \left[\sqrt{t}\right]_1^\infty = \infty$$

$\therefore$ The limit does not exist

$\therefore \int_0^\infty \frac{x^2}{\sqrt{1+x^3}} dx$ divergent.

Comparison test for Improper Integrals

Suppose f and g are Continuous functions with $0 \le g(x) \le f(x)$.

(a) If $\int_a^\infty f(x)\,dx$ is Convergent then $\int_a^\infty g(x)\,dx$ is Convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent then $\int_a^\infty f(x)\,dx$ is divergent.

Does the Integral $\int_1^\infty \frac{dx}{xe^x}$ Convergent.

$$0 < \frac{1}{xe^x} < \frac{1}{e^x}.$$

$$\int_1^\infty \frac{1}{e^x}dx = \int_1^\infty e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_1^\infty = \frac{e^{-\infty} - e^{-1}}{-1}.$$

$$= 0 + \frac{1}{e} = 0.3678.$$

Thus $\int_1^\infty \frac{1}{e^x}dx$ Convergent by Comparison test $\int_1^\infty \frac{dx}{xe^x}$ Convergent.

179

unit V

Multiple Integrals

1.1 Double Integration in Cartesian Coordinates

Let $f(x, y)$ be a single valued and double variable function which is defined for all points in the finite Region R of xy plane. If R is bounded by the curves $x = x_1, y = y_1$, $x = x_2, y = y_2$ then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$$

Evaluate $\int_0^1\int_0^1 (x^2+y^2)\,dx\,dy$

$$= \int_0^1 \left(\frac{x^3}{3}\right)_0^1 + y^2(x)_0^1\,dy$$

$$= \int_0^1\left(\frac{1}{3}+y^2\right)dy = \left(\frac{y}{3}\right)_0^1 + \left(\frac{y^3}{3}\right)_0^1$$

$$= \frac{1}{3}+\frac{1}{3} = \frac{2}{3}.$$

Evaluate $\int_0^3\int_0^2 e^{x+y}\,dy\,dx.$

Soln. $I = \int_0^3\int_0^2 e^{x+y}\,dy\,dx = \int_0^3\int_0^2 e^x e^y\,dy\,dx$

$$= \int_0^3 e^x \left[e^y\right]_0^2 dx$$

$$= \int_0^3 e^x (e^2 - 1)\, dx$$

$$= (e^2 - 1) \int_0^3 e^x\, dx$$

$$= (e^2 - 1)\left(e^x\right)_0^3 = (e^2 - 1)(e^3 - 1)$$

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} y\, dy\, dx$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2 - x^2}} dx$$

$$= \int_0^a \frac{a^2 - x^2}{2}\, dx$$

$$= \frac{1}{2}\left[a^2 (x)_0^a - \left(\frac{x^3}{3}\right)_0^a\right]$$

$$= \frac{1}{2}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{1}{2} \cdot \frac{2a^3}{3} = \frac{a^3}{3}$$

Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

$$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx = \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$$

181

$$= \int_0^1 x(e^x - 1)\,dx$$

$$= \int_0^1 x e^x dx - \int_0^1 x\,dx \quad \text{—— (1)}$$

First Integral $\int_0^1 \underset{u}{x}\, \underset{dv}{e^x dx}$ $u = x$, $dv = e^x dx$, $v = e^x$

$$= (x e^x)_0^1 - \int_0^1 e^x dx$$

$$= e - 0 - (e^x)_0^1 = 0 + 1$$

Second Integral $= \int_0^1 x\,dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$

sub in (1)

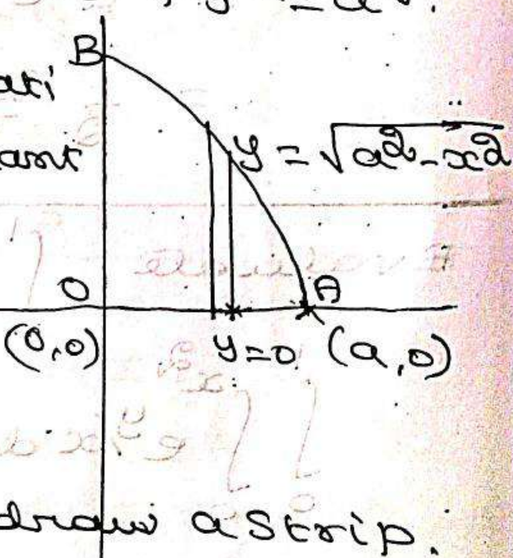
$$\int_0^1 \int_0^{x^2} e^{y/x}\,dy\,dx = 1 - \frac{1}{2} = \frac{1}{2}.$$

Dec. 2019 (A.U)

Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

Soln. The region of Integration is the positive quadrant of the circle $x^2+y^2=a^2$

Centre $(0,0)$ radius a



Keep x as const

x varies from 0 to a

To find limits of y draw a strip.

parallel to y-axis on this Strip.

y varies from 0 to $\sqrt{a^2-x^2}$.

$$\iint_R xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx.$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} y\,dy\right] x\,dx$$

$$= \int_0^a \left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} x\,dx$$

$$= \frac{1}{2}\int_0^a x(a^2-x^2)\,dx.$$

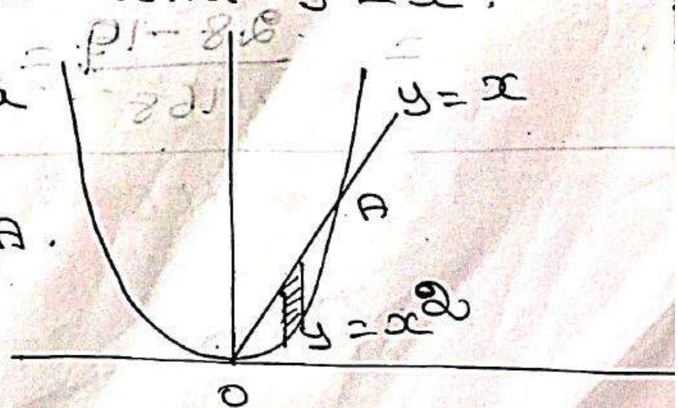
$$= \frac{1}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{1}{2}\left[a^2\left(\frac{x^2}{2}\right)_0^a - \left(\frac{x^4}{4}\right)_0^a\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{a^4}{8}.$$

Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$.



Soln The parabola $y=x^2$ and St. line $y=x$ Intersect at A. Consider the Strip. parallel to y axis

A is the pt of intersection of $y = x^2$ and $y = x$. $\therefore x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$\therefore$ y varies from x^2 to x

x varies from 0 to 1.

$$\int_0^1 \int_{x^2}^{x} (x^2 y + xy^2)\, dy\, dx$$

$$= \int_0^1 x^2 \left(\frac{y^2}{2}\right)_{x^2}^{x} + x\left(\frac{y^3}{3}\right)_{x^2}^{x} dx$$

$$= \int_0^1 \frac{x^2}{2}(x^2 - x^4) + \frac{x}{3}(x^3 - x^6)\, dx$$

$$= \frac{1}{2}\left(\frac{x^5}{5}\right)_0^1 - \frac{1}{2}\left(\frac{x^7}{7}\right)_0^1 + \frac{1}{3}\left(\frac{x^5}{5}\right)_0^1 - \frac{1}{3}\left(\frac{x^8}{8}\right)_0^1$$

$$= \frac{1}{10} - \frac{1}{14} + \frac{1}{15} - \frac{1}{24}$$

$$= \frac{1}{10} + \frac{1}{15} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3+2}{30} - \left(\frac{12+7}{168}\right)$$

$$= \frac{1}{6} - \frac{19}{168}$$

$$= \frac{28-19}{168} = \frac{9}{168} = \frac{3}{56}$$

Evaluate $\iint \frac{dx\,dy}{1+x^2+y^2}$ over the area lying in first quadrant bounded by $x=0$ $x=1$, $y=0$ and rectangular hyperbola $y^2-x^2=1$.

Soln: The region of Integration lies in the first quadrant bounded by $x=0$, $x=1$, $y=0$ and $y^2-x^2=1$.

The limits for x & y are

$x=0$ and $x=1$

$y=0$ to $y=\sqrt{1+x^2}$.

$$\iint \frac{dy\,dx}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{(\sqrt{1+x^2})^2+y^2}.$$

$\frac{dx}{\sqrt{a^2+x^2}}$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx. \qquad = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1} 1\, dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4}\left[\sinh^{-1} x\right]_0^1 = \frac{\pi}{4}\left[\log x+\sqrt{1+x^2}\right]_0^1$$

$$= \frac{\pi}{4} \log 1+\sqrt{2}$$

4.185

4.2. Double Integration (Polar Coordinates)

$$\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta = I$$

<u>Soln</u>. $I = \int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$

$$= \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi}\frac{\cos^2\theta}{2}\,d\theta = \frac{1}{2}\int_0^{\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\pi\right] = \frac{\pi}{4}.$$

3

Evaluate $\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$

$$\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2(1-\cos\theta)^2\,d\theta = \frac{a^2}{2}\int_0^{\pi} 1+\cos^2\theta - 2\cos\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 1+\left(\frac{1+\cos 2\theta}{2} - 2\cos\theta\right)d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi} 3+\cos 2\theta - 4\cos\theta\,d\theta$$

$$= \frac{a^2}{4}\left[3\theta + \frac{\sin 2\theta}{2} - 4\sin\theta\right]_0^{\pi}$$

$$= \frac{a^2}{4}[3\pi] = \frac{3\pi a^2}{4}.$$

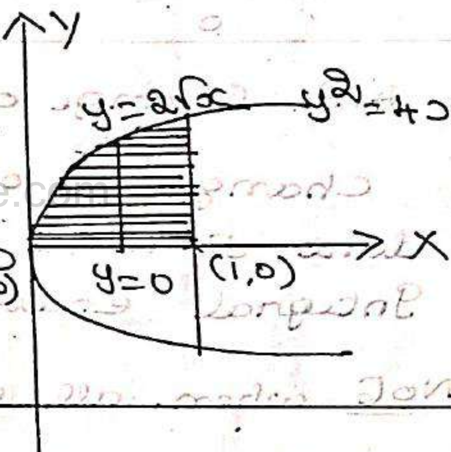
Find the limits of Integration in the double Integral $\iint_R f(x,y)\,dx\,dy$ where R is in the first quadrant and bounded by $x=1$, $y=0$, $y^2=4x$.

Soln. The region of Integration given by $y=0$, $x=1$, $y^2=4x$.

The limits of Integration.

$x=0$ to $x=1$

$y=0$ to $y=2\sqrt{x}$.



Evaluate $\int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\,dx\,dy$.

$$= \int_0^{\infty} \frac{e^{-y}}{y}[x]_0^{y}\,dy = \int_0^{\infty} \frac{e^{-y}}{y}\cdot y\,dy$$

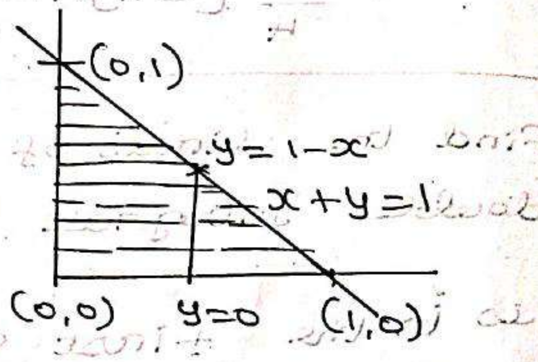
$$= \int_0^{\infty} e^{-y}\,dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= \frac{e^{-\infty} - e^{0}}{-1} = \frac{-1}{-1} = 1.$$

Find $\iint dx\,dy$ over the region bounded by $x \geq 0$, $y \geq 0$, $x+y \leq 1$.

Soln:

$$\iint_R dx\,dy = \int_0^1 \int_0^{1-x} dy\,dx$$



$$= \int_0^1 \left[y\right]_0^{1-x} dx$$

$$= \int_0^1 1-x\,dx = \left.\frac{(1-x)^2}{-2}\right]_0^1 = 0 - \frac{1}{-2} = \frac{1}{2}$$

4.3. Change of order of Integration.

Change of order of Integration is done to make the evaluation of the Integral easier.

Note when all limits are Constants we can change the order of Integration as we like.

Procedure to evaluate Double Integration by changing its order of Integration.

(i) using limits of Integration sketch the region of Integration

(ii) Find Intersecting points

(iii) If the limits of Inner Integral is a function of x change the limit as a function of y

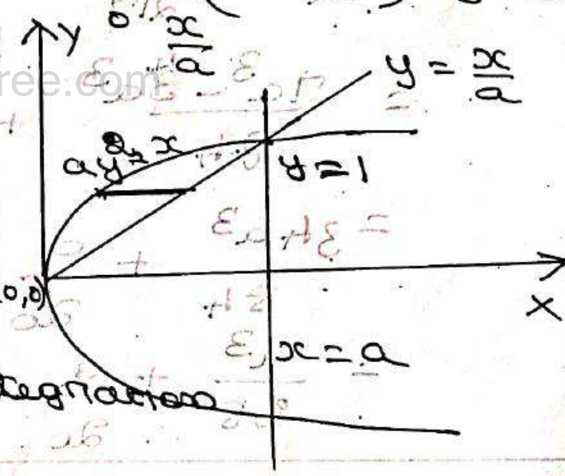
(IV) If the limits of Inner Integration is a fn of y change the limit as a fn of x.

(V) If the Integration is w.r.t. x. Consider horizontal strip and find the new limits.

(VI) If the Integration is w.r.t. y. Consider vertical strip and find new limits.

A.U. Dec 2019. Change the order of Integration for the given Integral $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dy\, dx$

Soln: x varies from 0 to a



y varies from $y = \frac{x}{a}$ to $y = \sqrt{\frac{x}{a}}$

(or) $y^2 = \frac{x}{a} \Rightarrow x = ay^2$

By change order of Integration first is w.r.t. x

∴ x varies from $x = ay^2$ to $x = ay$.

y varies from y = 0 to y = 1

$$\therefore \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\, dy\, dx = \int_0^1 \int_{ay^2}^{ay} (x^2+y^2)\, dx\, dy$$

$$= \int_0^1 \left(\frac{x^3}{3}\right)_{ay^2}^{ay} + y^2 (x)_{ay^2}^{ay}\, dy$$

189.

$$= \int_0^1 \frac{a^3y^3}{3} - \frac{a^3y^6}{3} + y^2(ay - ay^2)\,dy$$

$$= \int_0^1 \frac{a^3y^3}{3} - \frac{a^3y^6}{3} + ay^3 - ay^4\,dy$$

$$= \frac{a^3}{3}\left(\frac{y^4}{4}\right)_0^1 - \frac{a^3}{3}\left(\frac{y^7}{7}\right)_0^1 + a\left(\frac{y^4}{4}\right)_0^1 - a\left(\frac{y^5}{5}\right)_0^1$$

$$= \frac{a^3}{3}\left(\frac{1}{4}\right) - \frac{a^3}{21}(1) + \frac{a}{4}(1) - \frac{a}{5}(1)$$

$$= \frac{a^3}{12} - \frac{a^3}{21} + \frac{a}{4} - \frac{a}{5}$$

$$= \frac{7a^3 - 4a^3}{84} + \frac{5a - 4a}{20}$$

$$= \frac{3a^3}{84} + \frac{a}{20}$$

$$= \frac{a^3}{28} + \frac{a}{20}.$$

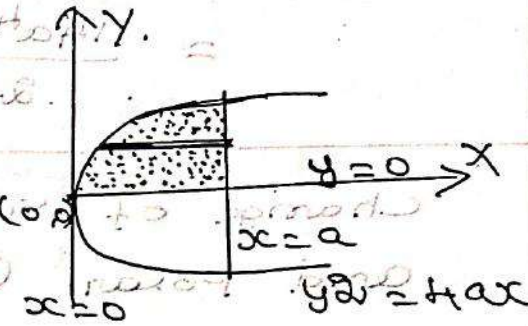
A.U. June 2018 Change the order of Integration in the Integral $\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx$ and evaluat

Soln first limit is y, y varies from 0 to $2\sqrt{ax}$.

second limit is x, x varies from 0 to a.

ie $y=0$, (x-axis), $y^2 = 4ax$ (parabola)

For change order of integration.



The first integration is w.r.t. x.

Draw the horizontal strip parallel to x-axis ie $x = \frac{y^2}{4a}$ to $x = a$.

y varies from 0 to $2a$

$y^2 = 4ax$

$y^2 = 4a(a)$

$y = 2a$.

$$\therefore \int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx = \int_0^{2a} \int_{\frac{y^2}{4a}}^{a} x^2\,dx\,dy$$

$$= \int_0^{2a} \left(\frac{x^3}{3}\right)_{\frac{y^2}{4a}}^{a} dy$$

$$= \int_0^{2a} \frac{a^3}{3} - \frac{1}{3}\left(\frac{y^2}{4a}\right)^3 dy$$

$$= \int_0^{2a} \frac{a^3}{3} - \frac{1}{192a^3}(y^6)\,dy$$

$$= \frac{a^3}{3}(y)_0^{2a} - \frac{1}{192a^3}\left(\frac{y^7}{7}\right)_0^{2a}$$

$$= \frac{2a^4}{3} - \frac{1}{192\times 7}\,\frac{2^7 a^7}{a^3}$$

$$= \frac{2a^4}{3} - \frac{2^4 \times 2^3 a^7}{192 \times 7a^3} = \frac{2a^4}{3} - \frac{2}{21}a^4$$

$=$

191

$$= \frac{14a^4 - 2a^4}{21} = \frac{12a^4}{21} = \frac{4a^4}{7}.$$

Change of variables between Cartesian and Polar Coordinates.

$x = r\cos\theta, \quad y = r\sin\theta.$

$dx\,dy \longrightarrow r\,dr\,d\theta$ ie $dx\,dy = |J|\,dr\,d\theta$

Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\,dy$ by Changing into polar Coordinates.

Soln: The limits of y is $y=0$ to $y=a$.

The limit of x is $x=0$ to $x=\sqrt{a^2-y^2}$

$x^2 = a^2-y^2, \quad x^2+y^2 = a^2.$

To change Cartesian to polar Coordinates

$x = r\cos\theta, \; y = r\sin\theta, \; dx\,dy = r\,dr\,d\theta.$

$\therefore$ The polar Eqn of Circle is

$r^2\cos^2\theta + r^2\sin^2\theta = a^2$

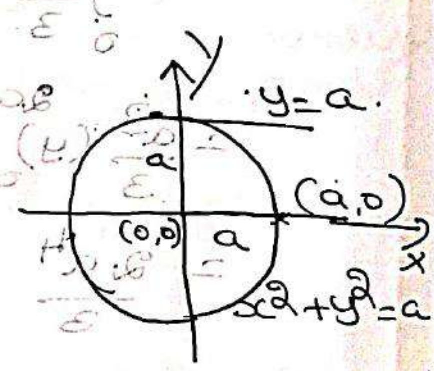
$r^2 = a^2$

$r = \pm a.$

$dx\,dy = r\,dr\,d\theta$

r varies from 0 to a

θ varies from 0 to $\frac{\pi}{2}$



The given Integral $= \int_0^{\frac{\pi}{2}} \int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta$.

put $a^2 - r^2 = t$

$-2r\,dr = dt \Rightarrow r\,dr = -\frac{dt}{2}$

$r = 0,\ t = a^2,\quad r = a,\ t = 0$

$= \int_0^{\frac{\pi}{2}} \int_{a^2}^{0} \sqrt{t}\left(-\frac{dt}{2}\right) d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^{a^2} t^{\frac{1}{2}}\, dt\, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{a^2} d\theta$

$= \frac{1}{3}\int_0^{\frac{\pi}{2}} (a^2)^{\frac{3}{2}}\, d\theta$

$= \frac{1}{3}\int_0^{\frac{\pi}{2}} a^3\, d\theta = \frac{1}{3} a^3 (\theta)_0^{\frac{\pi}{2}}$

$= \frac{1}{3} a^3 \frac{\pi}{2} = \frac{\pi a^3}{6}$.

Evaluate by changing to polar Coordinates $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$.

Soln. The limit of y is $y = 0$ to $y = a$

limit of x is $x = y$ to $x = a$.

By changing Cartesian to polar Coordinate

$x = r\cos\theta,\ y = r\sin\theta,\ dx\,dy = r\,dr\,d\theta$

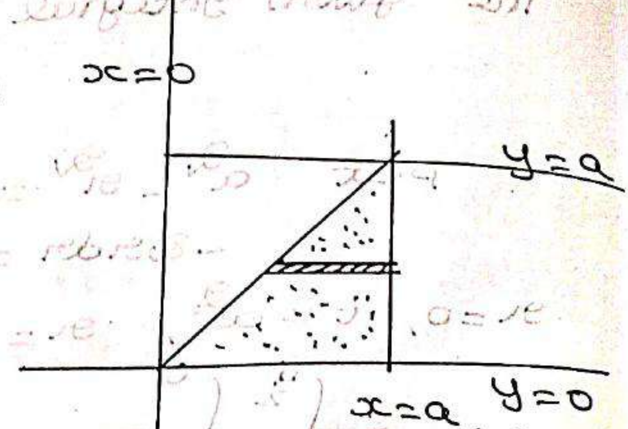
193

$x = r\cos\theta \quad y = r\sin\theta$

$r = 0, \quad x = a$

$r\cos\theta = a$

$r = \frac{a}{\cos\theta} = a\sec\theta$



r varies from 0 to $a\sec\theta$.

θ varies from 0 to $\frac{\pi}{4}$.

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r\cos\theta}{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos\theta}{r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \cos\theta . dr . d\theta$$

$$= \int_0^{\pi/4} \cos\theta \left[r\right]_0^{a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \cos\theta . (a\sec\theta)\, d\theta$$

$$= \int_0^{\pi/4} a\, d\theta = a(\theta)_0^{\pi/4} = \frac{a\pi}{4}.$$

Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy$ by polar coordinates

Soln. The limits are $x=0$, $x=\infty$, $y=0$, $y=\infty$. By changing given coordinates to polar coordinates. Put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.

r varies from 0 to ∞

θ varies from 0 to $\frac{\pi}{2}$

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\,dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r\,dr\,d\theta.$$

put $t = r^2$, $dt = 2r\,dr$

$r=0$, $t=0$

$r=\infty$, $t=\infty$

$$\therefore I = \int_0^{\pi/2} \int_0^\infty e^{-t} \frac{dt}{2} \, d\theta$$

$$= +\frac{1}{2} \int_0^{\pi/2} \left. \frac{e^{-t}}{-1} \right]_0^\infty d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} (0-1) \, d\theta \quad (\because e^{-\infty} = 0)$$

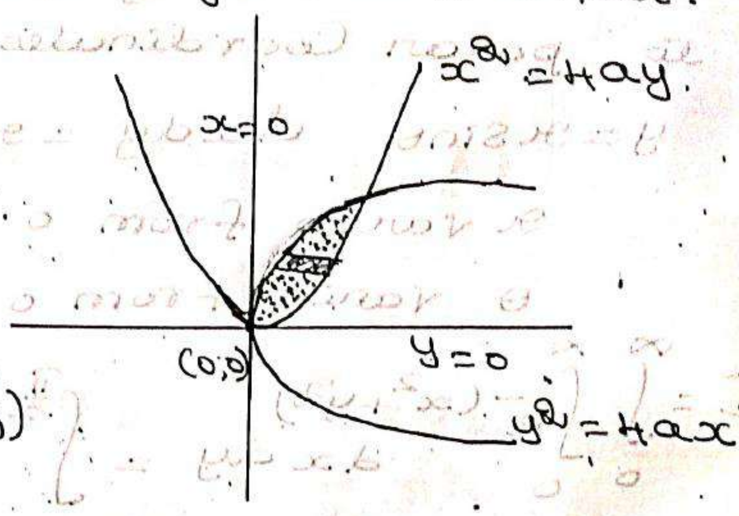
$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

195

Area as a Double Integral (Cartesian Coordinates)

Find the area common to the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

<u>Soln</u>



$x^2 = 4ay$

$y^2 = 4ax$

$y^2 = 4a\sqrt{4ay}$

$y^4 = 16a^2(4ay)$

$y^4 = 64a^3y$

$y^3 = 64a^3 = (4a)^3 \Rightarrow y = 4a$.

$\therefore$ y varies from 0 to $4a$

x varies from $\frac{y^2}{4a}$ to $2\sqrt{ay}$.

$$\text{Area} = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$= 2\sqrt{a}\int_0^{4a} y^{\frac{1}{2}}\,dy - \frac{1}{4a}\int_0^{4a} y^2\,dy$$

$$= 2\sqrt{a}\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right)_0^{4a} - \frac{1}{4a}\left(\frac{y^3}{3}\right)_0^{4a}.$$

$$= \frac{4}{3}\sqrt{a}\,(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3$$

$$= \frac{4\sqrt{a}}{3}(4a)\sqrt{4a} - \frac{64a^3}{12a}$$

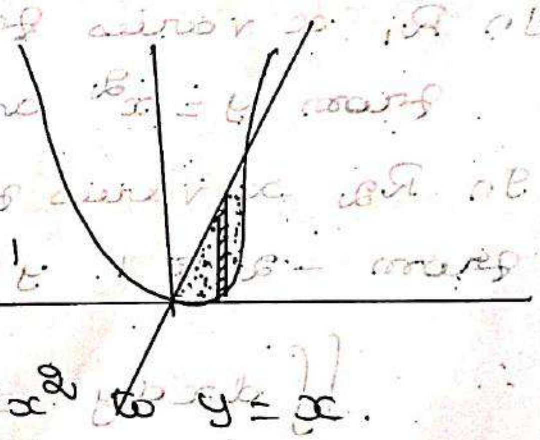
$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{(32-16)a^2}{3} = \frac{16a^2}{3}$$

Ex: Find the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Ans: $\frac{16}{3}$ (Hint: As in previous problem put $a = 1$.)

Using double integral, find the area bounded by $y = x$ and $y = x^2$.



Soln: $y = x$, $y = x^2$

$\therefore x = x^2$

$x(x-1) = 0$

x varies from 0 to 1

$x = 1,\ y = 1$.

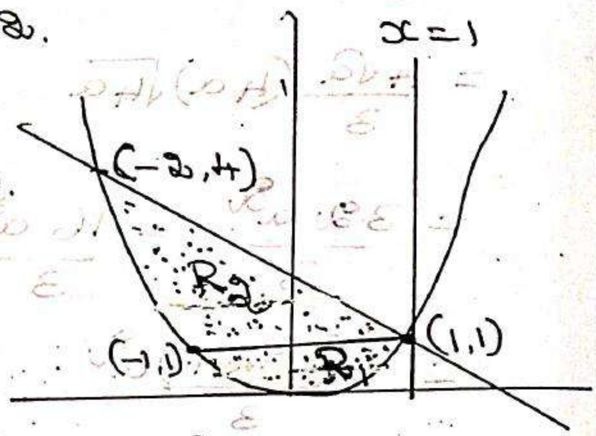
y varies from $y = x^2$ to $y = x$.

$$\text{Area} = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 y\Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

197

Find the area Enclosed by the curves $y = x^2$ and $x + y = 2$.



Soln: given

$y = x^2$ and $x + y = 2$.

Solving these two equations.

$y = 2 - x$.

$\therefore\ y = x^2 \Rightarrow 2 - x = x^2$

$x^2 + x - 2 = 0$.

$(x+2)(x-1) = 0 \Rightarrow x = 1, -2$

$x = 1,\ y = 2 - 1 = 1$ $\qquad (1,1)$

$x = -2 \quad y = 2 + 2 = 4$ $\qquad (-2,4)$

$(1,1)$ & $(-2,4)$ are the points of intersection of $x+y=2$ and $y = x^2$.

In R_1 x varies from $-1, 1$ & y varies from $y = x^2$ and $y = 1$

In R_2 x varies from -2 to 1 y varies from -2 to 1. y varies from $y = 1$, to $y = 2 - x$

$$\therefore \iint_{R_1} dx\,dy = \int_{-1}^{1}\int_{x^2}^{1} dy\,dx.$$

$$= \int_{-1}^{1} [y]_{x^2}^{1}\,dx = \int_{-1}^{1} 1 - x^2\,dx.$$

$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}.$$

$$\iint_{R_2} dx\,dy = \int_{-2}^{1}\int_{1}^{2-x} dy\,dx$$

$$= \int_{-2}^{1}[y]_{1}^{2-x}\,dx$$

$$= \int_{-2}^{1} 2-x-1\,dx = \int_{-2}^{1}(1-x)\,dx$$

$$= (x)_{-2}^{1} - \left(\frac{x^2}{2}\right)_{-2}^{1}$$

$$= 1+2-\frac{1}{2}(1-4)$$

$$= 3 - \frac{1}{2}(-3)$$

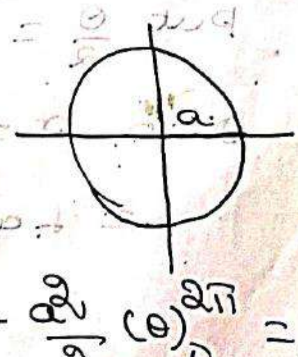
$$= 3+\frac{3}{2} = \frac{9}{2}$$

$$\iint_R dx\,dy = \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$= \frac{4}{3} + \frac{9}{2} = \frac{35}{6} \text{ sq. units}$$

Using double Integration find the area of the circle. $x^2+y^2=a^2$.

Soln



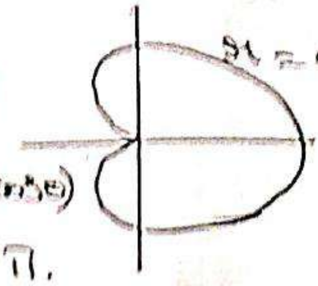
Area $\iint dx\,dy = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$

$$= \int_0^{2\pi}\left(\frac{r^2}{2}\right)_0^a d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta = \frac{a^2}{2}(\theta)_0^{2\pi} = \frac{\cancel{2}\pi a^2}{\cancel{2}}$$

179.

Find the area of Cardioid $r = a(1+\cos\theta)$

Area $= \iint r\,dr\,d\theta$



r varies from 0 to $a(1+\cos\theta)$

θ varies from $-\pi$ to π.

$$\therefore \text{Area} = \int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_{-\pi}^{\pi} \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \int_{-\pi}^{\pi} \frac{a^2}{2}(1+\cos\theta)^2\,d\theta$$

$1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta.$$

$$= \frac{a^2}{\cancel{2}}\cdot\cancel{2}\int_0^{\pi} 4\cos^4\frac{\theta}{2}\,d\theta$$

f is even

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$$= 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta.$$

put $\frac{\theta}{2} = x$, $\frac{d\theta}{2} = dx$

$\theta = 0$, $x = 0$, $\theta = \pi$ $x = \frac{\pi}{2}$

$$= 4a^2\int_0^{\pi/2}\cos^4 x\,2\,dx$$

$$= 8a^2 \int_0^{\pi/2} \cos^4 x \, dx$$

$$= 8a^2 \frac{3.1}{4.2} \frac{\pi}{2}$$

$$= \frac{3\pi a^2}{2}$$

Triple Integration (in Cartesian and polar Coordinates.

Compute the value of $\int_0^1 \int_0^2 \int_0^3 xyz \, dx\, dy\, dz$

$$= \int_0^1 \int_0^2 \left.\frac{x^2yz}{2}\right]_0^3 dy\, dz$$

$$= \frac{9}{2} \int_0^1 \left(\frac{y^2}{2}\right)_0^2 z \, dz$$

$$= \frac{9}{4}(4) \int_0^1 z \, dz = 9\left(\frac{z^2}{2}\right)_0^1 = \frac{9}{2}.$$

Express the region $x \geq 0$ $y \geq 0$ $z \geq 0$ $x^2+y^2+z^2 \leq 1$ by triple Integration.

Soln

z varies from 0 to $\sqrt{1-x^2-y^2}$

y varies from 0 to $\sqrt{1-y^2}$

x varies from 0 to 1.

Formulae in Integration.

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c.$$

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c.$$

(or)

$$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$$

$$= \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right|$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c \text{ (or) } -\cos^{-1}\frac{x}{a} + c.$$

$$\int \frac{dx}{\sqrt{x^2+x^2}} = \sinh^{-1}\frac{x}{a} + c \text{ (or) } \log\left|x+\sqrt{a^2+x^2}\right| + c.$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + c \text{ (or) } \log\left|x+\sqrt{x^2-a^2}\right|$$

(Q.01)

1 Volume as a Triple Integral.

Find the volume of the sphere $x^2+y^2+z^2=a^2$ using Triple Integration.

Soln

Volume = 8 × volume of the octant.

z varies from 0 to $\sqrt{a^2-x^2-y^2}$

y varies from 0 to $\sqrt{a^2-x^2}$

x varies from 0 to a.

$$V = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx.$$

3

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\; dy\,dx.$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2-x^2}\right)^2-y^2}\; dy\,dx.$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} + \frac{y\sqrt{a^2-x^2-y^2}}{2}\right]_0^{\sqrt{a^2-x^2}} dx.$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}(1) + 0\right] dx$$

$$= \frac{8}{4}\int_0^a (a^2-x^2)\frac{\pi}{2}\,dx$$

$$= \frac{8\pi}{4}\int_0^a (a^2-x^2)\,dx$$

$$= 2\pi\left[a^2(x)_0^a - \left(\frac{x^3}{3}\right)_0^a\right]$$

$$= 2\pi\left[a^3 - \frac{a^3}{3}\right] = 2\pi\left[\frac{3a^3-a^3}{3}\right]$$

$$= \frac{4\pi a^3}{3}$$

A.U 2018 Evaluate $\iiint xyz\,dx\,dy\,dz$ over the first octant of $x^2+y^2+z^2=a^2$.

Soln

$$\iiint xyz\,dx\,dy\,dz = \int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{a^2-x^2-y^2}} dy\,dx.$$

$$= \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2)\,dy\,dx.$$

$$= \frac{1}{2}\int_0^a\int_0^{\sqrt{a^2-x^2}} (xya^2 - x^3y - xy^3)\,dy\,dx.$$

$$= \frac{1}{2}\int_0^a x\,a^2\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} - x^3\left(\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} - x\left(\frac{y^4}{4}\right)_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x\frac{a^2(a^2-x^2)}{2} - \frac{x^3}{2}(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2\,dx$$

$$= \frac{1}{2}\int_0^a \frac{x}{2}(a^2-x^2)(a^2-x^2) - \frac{x}{4}(a^2-x^2)^2\,dx$$

$$= \frac{1}{2}\int_0^a \frac{x}{2}(a^2-x^2)^2 - \frac{x}{4}(a^2-x^2)^2\,dx$$

$$= \frac{1}{2}\int_0^a \frac{x}{4}(a^2-x^2)^2\,dx.$$

$$= \frac{1}{2}\int_0^a \frac{x}{4}(a^4+x^4-2a^2x^2)\,dx$$

$$= \frac{1}{8}\int_0^a a^4x + x^5 - 2a^2x^3\,dx$$

$$= \frac{1}{8}\left[a^4\left(\frac{x^2}{2}\right)_0^a + \left(\frac{x^6}{6}\right)_0^a - \cancel{2}a^2\left(\frac{x^4}{\cancel{4}_2}\right)_0^a\right]$$

$$= \frac{1}{8}\left[\frac{a^6}{2} + \frac{a^6}{6} - \frac{a^6}{2}\right]$$

$$= \frac{a^6}{48}$$

A.U 2017. Find the Volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Soln Here the region of Integration is the volume V of tetrahedron bounded by.

$0 \le x \le a$

$0 \le y \le b\left(1-\frac{x}{a}\right)$

$0 \le z \le c\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Required volume $= \iiint dx\,dy\,dz.$

$$V = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx.$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} (z)_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx.$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx.$$

$$= c\int_0^a \left(1-\frac{x}{a}\right)(y)_0^{b(1-\frac{x}{a})} - \frac{1}{b}\left(\frac{y^2}{2}\right)_0^{b(1-\frac{x}{a})} dx.$$

$$= c\int_0^a b\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right) - \frac{1}{2b} b^2\left(1-\frac{x}{a}\right)^2 dx.$$

$$= bc\int_0^a \left(1-\frac{x}{a}\right)^2 - \frac{1}{2}\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx = \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= \frac{abc}{6}\left[\frac{0-1}{-1}\right] = \frac{abc}{6}$$

2105

Evaluate $\iiint_V dx\,dy\,dz$ where V is the volume of the finite Region formed by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

<u>Soln:</u> As in previous problem, we get volume $= \frac{abc}{6}$.

Sperical polar Coordinates

$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$

r varies from 0 to a

ϕ varies from 0 to π

θ varies from 0 to 2π.

Dec. 2019 A.U. (2,06)

Evaluate $\int_0^{2a}\int_0^{x}\int_y^{x} xyz\, dz\, dy\, dx$.

<u>Soln</u> $\int_0^{2a}\int_0^{x}\int_y^{x} xyz\, dz\, dy\, dx$

$$= \int_0^{2a}\int_0^{x} xy\left(\frac{z^2}{2}\right)_y^x dy\, dx$$

$$= \frac{1}{2}\int_0^{2a}\int_0^{x} xy\,(x^2 - y^2)\, dy\, dx$$

$$= \frac{1}{2}\int_0^{2a}\int_0^{x} x^3y - xy^3\, dy\, dx.$$

$$= \frac{1}{2}\int_0^{2a} x^3\left(\frac{y^2}{2}\right)_0^x - x\left(\frac{y^4}{4}\right)_0^x dx.$$

$$= \frac{1}{2}\int_0^{2a} \frac{x^5}{2} - \frac{x^5}{4}\, dx$$

$$= \frac{1}{2}\int_0^{2a} \frac{x^5}{4}\, dx$$

$$= \frac{1}{8}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{2^6 a^6}{2^4 \cdot 3} = \frac{4a^6}{3}.$$

Cylindrical and rectangular Coordinates are related by $x = r\cos\theta$ $y = r\sin\theta$ $z = z$, $dx\,dy\,dz = r\,dz\,dr\,d\theta$

Ex By transforming into cylindrical Coordinates, evaluate the Integral $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the region of space defined by $x^2+y^2 \le 1$ and $0 \le z \le 1$.

Soln Here region of space is enclosed by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$

The radius of cylinder is 1

$x = r\cos\theta$ $y = r\sin\theta$ $z = z$ we have

$x^2+y^2=1 \Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 1$

$r^2 = 1 \Rightarrow r = \pm 1$

$x^2+y^2+z^2 = (r\cos\theta)^2 + (r\sin\theta)^2 + z^2$

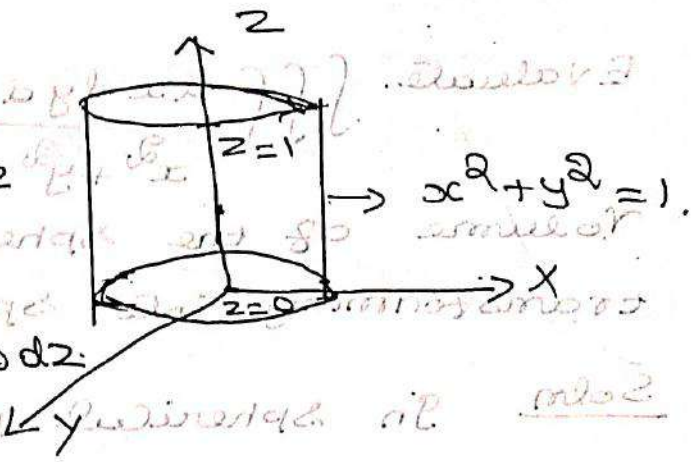
$= r^2 + z^2$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

r varies from 0 to 1

θ varies from 0 to 2π

z varies from 0 to 1.



$$\iiint x^2+y^2+z^2 \, dx\,dy\,dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 (r^2+z^2)\, r\,dr\,d\theta\,dz$$

$$= \int_0^1 \int_0^{2\pi} \int_0^1 r^3 + rz^2 \, dr\,d\theta\,dz$$

$$= \int_0^1 \int_0^{2\pi} \left(\frac{r^4}{4}\right)_0^1 + z^2\left(\frac{r^2}{2}\right)_0^1 d\theta\,dz$$

$$= \int_0^1 \int_0^{2\pi} \frac{1}{4} + \frac{z^2}{2} \, d\theta\,dz$$

$$= \int_0^1 \frac{1}{4}(\theta)_0^{2\pi} + \frac{z^2}{2}(\theta)_0^{2\pi} \, dz$$

$$= \int_0^1 \frac{\cancel{2}\pi}{\cancel{4}\,2} + \frac{z^2}{\cancel{2}}(\cancel{2}\pi) \, dz$$

$$= \frac{\pi}{2}(z)_0^1 + \pi\left(\frac{z^3}{3}\right)_0^1$$

$$= \frac{\pi}{2} + \frac{\pi}{3}$$

$$= \frac{5\pi}{6}.$$

(209)

Evaluate $\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ throughout the volume of the sphere $x^2+y^2+z^2=a^2$. transforming into spherical Coordinates.

Soln In spherical polar Coordinates

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$.

given $x^2+y^2+z^2 = a^2$

we have $dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

ρ	$\rho = 0$	$\rho = a$
θ	$\theta = 0$	$\theta = 2\pi$
ϕ	$\phi = 0$	$\phi = \pi$

$$\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\rho^2 \sin\phi}{\rho^2} \, d\rho \, d\phi \, d\theta$$

$$= \int_0^a d\rho \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta$$

$$= (\rho)_0^a \, (-\cos\phi)_0^{\pi} \, (\theta)_0^{2\pi}$$

$$= a(1+1)2\pi = 4\pi a.$$

Express the region. $x \geq 0$, $y \geq 0$, $z \geq 0$, $x^2+y^2+z^2 \leq 1$ by triple Integration.

Soln The region of Integration is given by $x \geq 0$, $y \geq 0$, $z \geq 0$.

x varies from 0 to 1

y varies from 0 to $\sqrt{1-x^2}$

z varies from 0 to $\sqrt{1-x^2-y^2}$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx.$$

unit $\underline{V}$

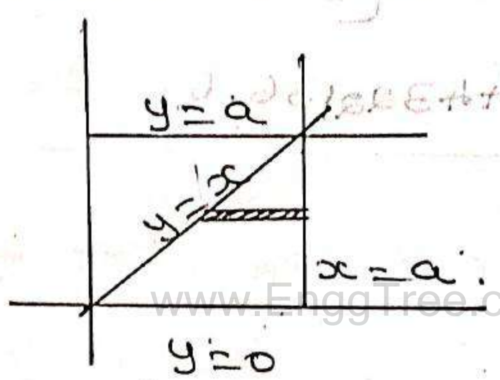
By changing to polar coordinates Find the value of $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx dy$

$\underline{\text{Soln}}$ The limits of y is from 0 to a

x is from y to a

$x = r\cos\theta$, $y = r\sin\theta$

$dxdy \Rightarrow r\,dr\,d\theta$.



One end is at $r=0$ and the other end is at $x=a$,

$r\cos\theta = a$

$\Rightarrow r = a\sec\theta$.

r varies from 0 to $a\sec\theta$

θ varies from 0 to $\frac{\pi}{4}$

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dx dy = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2 \cos^2\theta}{r} r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \left(\frac{r^3}{3}\right)_0^{a\sec\theta} \cos^2\theta \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \sec^3\theta \cos^2\theta \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$= \frac{a^3}{3} \log (\sec\theta + \tan\theta)_0^{\frac{\pi}{4}}$$

$$= \frac{a^3}{3} \left[\log \sqrt{2}+1 - 0 \right]$$

$$= \frac{a^3}{3} \log \sqrt{2}+1 .$$

Applications of multiple Integrals.

moments and centre of mass:

The moments of entire lamina about x-axis is given by,

$$M_x = \iint_D y\,\rho(x,y)\,dA.$$

Similarly the moment about the y-axis is given by

$$M_y = \iint_D x\,\rho(x,y)\,dA.$$

where D is the corresponding region and $\rho(x,y)$ is the density function.

The coordinates $(\bar{x}, \bar{y})$ of the centre of mass of a lamina has the region D and the density function $\rho(x,y)$ are given by.

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA \text{ and}$$

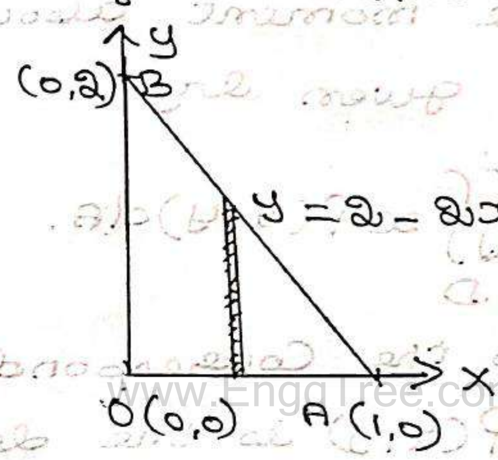
$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA. \text{ where}$$

mass m is given by $m = \iint \rho(x,y)\,dA.$

Find the mass, centre of mass of a triangular lamina with vertices (0,0), (1,0) and (0,2) if the density function is

$\rho(x,y) = 1+3x+y$

Soln First find the value of m, mass of the lamina.



Equation of straight line AB is given as follows

Equation of straight line joining (x_1,y_1) and (x_2,y_2) is

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

Here $(x_1,y_1) = (1,0)$ $(x_2,y_2) = (0,2)$

$$\therefore \frac{y-0}{0-2} = \frac{x-1}{1-0}$$

$$\frac{y}{-2} = \frac{x-1}{1}$$

$$y = -2(x-1).$$

$\therefore y = 2 - 2x.$

To find the limits for x and y.

x varies from 0 to 1

For find the limits for y.

Draw a vertical strip parallel to y-axis

$\therefore$ y varies from 0 to $2-2x$.

The mass m of the lamina

$$m = \iint_D \rho(x,y)\, dA.$$

$$= \int_0^1 \int_0^{2-2x} (1+3x+y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_0^{2-2x} dx.$$

$$= \int_0^1 2-2x + 3x(2-2x) + \frac{(2-2x)^2}{2}\, dx$$

$$= \int_0^1 2-2x+6x-6x^2 + \frac{1}{2}(4+4x^2-8x)\, dx.$$

$$= \int_0^1 2+4x-6x^2+2+2x^2-4x\, dx.$$

$$= \int_0^1 (4-4x^2)\, dx$$

$$= 4(x)_0^1 - 4\left(\frac{x^3}{3}\right)_0^1$$

$$= 4 - \frac{4}{3} = \frac{12-4}{3} = \frac{8}{3}.$$

Now find $\bar{x}, \bar{y}$

$$\bar{x} = \frac{1}{m}\iint_D x\, \rho(x,y)\, dA.$$

$$= \frac{1}{8/3}\int_0^1\int_0^{2-2x} x(1+3x+y)\, dy\, dx.$$

$$= \frac{3}{8}\int_0^1\int_0^{2-2x} x+3x^2+xy\, dy\, dx.$$

$$= \frac{3}{8}\int_0^1 \left[xy + 3x^2(y) + x\frac{y^2}{2}\right]_0^{2-2x} dx.$$

$$= \frac{3}{8}\int_0^1 x(2-2x) + 3x^2(2-2x) + \frac{x}{2}(2-2x)^2\, dx$$

$$= \frac{3}{8}\int_0^1 2x - 2x^2 + 6x^2 - 6x^3 + \frac{x}{2}(4+4x^2-8x)\, dx$$

$$= \frac{3}{8}\int_0^1 2x + 4x^2 - 6x^3 + x(2+2x^2-4x)\, dx$$

$$= \frac{3}{8}\int_0^1 (2x+4x^2-6x^3+2x+2x^3-4x^2)\, dx$$

$$= \frac{3}{8}\int_0^1 (-4x^3 + 4x)\,dx$$

$$= \frac{3}{8}\left[-\cancel{4}\frac{(x^4)_0^1}{\cancel{4}} + \cancel{4}^{2}\frac{(x^2)_0^1}{\cancel{2}}\right]$$

$$= \frac{3}{8}\left[-1 + 2\right] = \frac{3}{8}$$

$$y) \quad \bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA.$$

$$= \frac{1}{\frac{3}{8}}\int_0^1\int_0^{2-2x} y(1+3x+y)\,dy\,dx.$$

$$= \frac{3}{8}\int_0^1\int_0^{2-2x} (y + 3xy + y^2)\,dy\,dx$$

$$= \frac{3}{8}\int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3}\right]_0^{2-2x} dx$$

$$= \frac{3}{8}\int_0^1 \frac{(2-2x)^2}{2} + \frac{3}{2}x(2-2x)^2 + \frac{1}{3}(2-2x)^3\,dx$$

$$= \frac{3}{8}\int_0^1 \cancel{4}^{2}\frac{(1-x)^2}{\cancel{2}} + 3x\,\cancel{4}^{2}\frac{(1-x)^2}{\cancel{2}} + \frac{1}{3}2^3(1-x)^3\,dx$$

$$= \frac{3}{8}\int_0^1 2(1+x^2-2x) + 6x(1+x^2-2x) + \frac{8}{3}(1+3x^2-3x-x^3)\,dx$$

$$= \frac{3}{8}\int_0^1 \Big[2 + 2x^2 - 4x + 6x + 6x^3 - 12x^2 + \frac{8}{3} + 8x^2 - 8x - \frac{8}{3}x^3 \Big] dx$$

$$= \frac{3}{8}\int_0^1 \Big[\left(6 - \frac{8}{3}\right)x^3 - 2x^2 - 6x + \frac{14}{3}\Big] dx$$

$$= \frac{3}{8}\int_0^1 \frac{10}{3}x^3 - 2x^2 - 6x + \frac{14}{3}\, dx$$

$$= \frac{3}{8}\Big[\frac{10}{3}\left(\frac{x^4}{4}\right) - 2\left(\frac{x^3}{3}\right)_0^1 - 6\left(\frac{x^2}{2}\right)_0^1 + \frac{14}{3}(x)\Big]_0^1$$

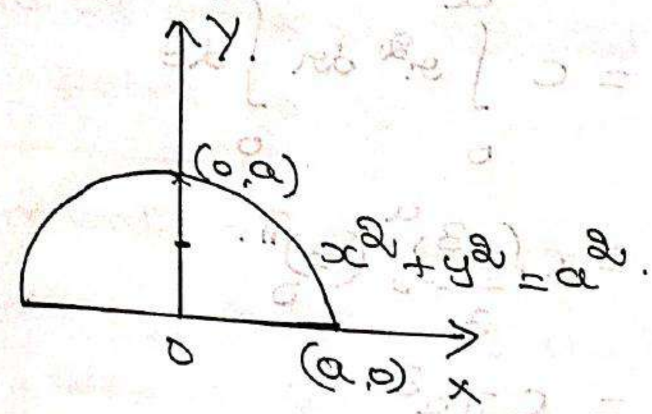
$$= \frac{3}{8}\Big[\frac{5}{6} - \frac{2}{3} - 3 + \frac{14}{3}\Big]$$

$$= \frac{3}{8}\Big[\frac{5 - 4 - 18 + 28}{6}\Big]$$

$$= \frac{3}{8}\,\frac{11}{6} = \frac{11}{16}$$

$\therefore$ The centre of mass at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$

The density at any point on a semicircular lamina is proportional to the distance from the centre of the circle. Find the centre of mass of the lamina.



Soln Let us Consider the lamina as the upper half of the circle $x^2+y^2=a^2$.

The distance from a point (x,y) to the centre of the circle is $\sqrt{x^2+y^2}$

$\therefore \rho(x,y) \propto \sqrt{y^2+x^2}$

$\therefore \rho(x,y) = c\sqrt{x^2+y^2}$

Convert into polar Coordinates

$x = r\cos\theta$, $y = r\sin\theta$

r varies from 0 to a

θ varies from 0 to π.

m - mass of the lamina

$$m = \iint_D \rho(x,y)\, dA.$$

$$= \iint_D c\sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_0^{\pi}\int_0^{a} c\, r.r.\, dr\, d\theta.$$

$$= c \int_0^a r^2\, dr \int_0^{\pi} d\theta$$

$$= c \cdot \left(\frac{r^3}{3}\right)_0^a (\theta)_0^{\pi}.$$

$$m = \frac{c\pi a^3}{3}.$$

Here both the lamina and the density function are symmetric with respect to y-axis, the centre of mass lie on y-axis

i.e. $\bar{x} = 0$.

$$\bar{y} = \frac{1}{m} \iint_D y \cdot \rho(x, y)\, dA.$$

$$= \frac{1}{\frac{c\pi a^3}{3}} \int_0^{\pi} \int_0^a r \sin\theta\, (c \cdot r) \cdot r\, dr\, d\theta$$

$$= \frac{3}{c\pi a^3} \int_0^{\pi} \sin\theta\, d\theta \int_0^a r^3\, dr.$$

$$= \frac{3}{c\pi a^3} (-\cos\theta)_0^{\pi} \left(\frac{r^4}{4}\right)_0^a.$$

$$= \frac{3a^4}{4c\pi a^3} \left[-\cos\pi + \cos 0\right]$$

$$= \frac{3a}{4\pi c} [1+1] = \frac{3a \cdot 2}{4\pi c}$$

$$\therefore \bar{y} = \frac{3a}{2\pi}$$

The centre of mass is located at the point $\left(0, \frac{3a}{2\pi}\right)$

Moments of Inertia:

The moment of Inertia of the lamina about x-axis is given by.

$$I_x = \iint_D y^2 \rho(x,y)\, dA.$$

The moment of Inertia about y-axis is given by.

$$I_y = \iint_D x^2 \rho(x,y)\, dA.$$

I_0 = moment of Inertia about origin (polar moment of Inertia

$$I_0 = \iint_D (x^2 + y^2) \rho(x,y)\, dA$$

$$I_0 = I_x + I_y.$$

Find the moments of Inertia I_x, I_y, I_0 of a homogeneous disk D with density $\rho(x,y) = C$. Centre the origin and radius a.

Soln The boundary of D is the circle $x^2 + y^2 = a^2$. In polar coordinates the given region D is bounded by $0 \le \theta \le 2\pi$, $0 \le r \le a$.

$$I_0 = \iint_D (x^2+y^2)\, \rho(x,y)\, dA$$

$$= \int_0^{2\pi}\int_0^a r^2\, c\, r\, dr\, d\theta.$$

$$= c\int_0^{2\pi} d\theta \int_0^a r^3\, dr.$$

$$= c\,(\theta)_0^{2\pi} \left(\frac{r^4}{4}\right)_0^a$$

$$= c\,(2\pi)\frac{(a^4)}{4}$$

$$= \frac{c\pi a^4}{2}$$

Here $I_x = I_y$

$I_x + I_y = I_0$

$\therefore\ 2I_x = I_0$

$$I_x = \frac{I_0}{2} = \frac{1}{2}\left(\frac{c\pi a^4}{2}\right)$$

$$\therefore\ I_x = I_y = \frac{c\pi a^4}{4}.$$

Here mass of the disc is

mass = density x Area.

$\therefore m = \sigma \pi a^2.$

$\therefore I_0 = \frac{\sigma \pi a^4}{2} = \frac{1}{2}(\sigma \pi a^2) a^2$

$I_0 = \frac{1}{2} m a^2.$