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DISCRETE FOURIER TRANSFORM :-

* The DFT is obtained by sampling one period of the Fourier Transform $X(\omega)$ of the signal $x(n)$ at a finite n.o of frequency points.

* The sampling is performed at N equally spaced points $0 \le \omega \le 2\pi$

i.e $\omega_K = \frac{2\pi k}{N}$.

* The DFT converts the continuous function of ω to a discrete function of ω.

Applications of DFT :

(i) It is used to determine the frequency content of a signal i.e to perform spectral analysis

(ii) It is used to perform filtering operations in the frequency domain.

The N-pt DFT of a finite duration sequence $x(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} \quad \text{for } k = 0,1,2 \ldots N-1.$$

The inverse DFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi nk}{N}} \quad \text{for } n = 0,1,2 \ldots N-1$$

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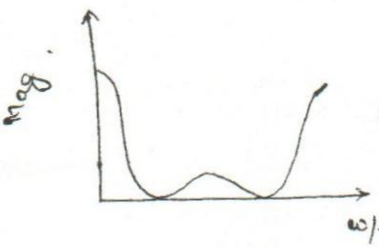
Zero Padding :-

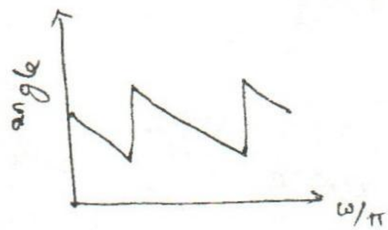
Let the sequence $x(n)$ has length 'L'.

To find N pt DFT of the sequence, add N-L zeros to the sequence if $N > L$. This is known as zero padding.

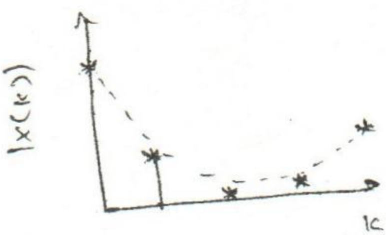
For eg. ($L=5$)

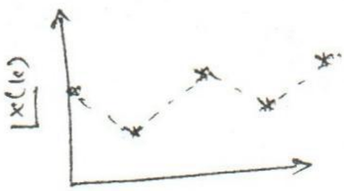
Let the actual magnitude & phase spectrum of $x(n)$ be.





If 5 pt DFT is done then the response will be





If 16 pt DFT is done then the response will be

[$N-L = 16-5$
$= 11$ zeros are added]

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Due to zero-padding, the frequency spectrum resolution is improved.

* The zero padded DFT can be used in linear filtering.

Problem No 1:

Find the DFT of a sequence $x(n) = \{1,1,0,0\}$ and find IDFT of $X(K) = \{1,0,1,0\}$.

1) $x(n) = \{1,1,0,0\}$

$N = 4$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} \quad \text{where } K = 0 \text{ to } N-1$$

Substitute $N = 4$

$$X(k) = \sum_{n=0}^{3} x(n) e^{\frac{-j\pi nk}{2}} \quad K = 0 \text{ to } 3.$$

To find $X(0)$:

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j\pi/2 \, n \cdot 0} \qquad \because e^0 = 1$$

$$= \sum_{n=0}^{3} x(n) e^0 = \sum_{n=0}^{3} x(n)$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1+1+0+0$$

$$\boxed{X(0) = 2}.$$

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To find X(1):

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi n}{2} \cdot 1}$$

$\because e^{j\theta} = \cos\theta + j\sin\theta$
$e^{-j\theta} = \cos\theta - j\sin\theta$.

$$= \sum_{n=0}^{3} x(n) e^{-j\pi/2 n}$$

$$= x(0)e^{0} + x(1) e^{-j\pi/2} + \cancel{x(2)}e^{-j\pi} + \cancel{x(3)}e^{-j3\pi/2}$$

$$= x(0) + x(1)\left[\cos \pi/2 - j\sin \pi/2\right]$$

$$= x(0) + x(1)[-j]$$

$$X(1) = \boxed{1-j}$$

To find X(2):

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= x(0)e^{0} + x(1)e^{-j\pi} + \cancel{x(2)}e^{-j2\pi} + \cancel{x(3)}e^{-j3\pi}$$

$$= x(0) + x(1)\left[\cos\pi - j\sin\pi\right]$$

$$= 1-1$$

$$\boxed{X(2) = 0}$$

To find X(3):

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi n}{2}}$$

$$= x(0)e^{0} + x(1) e^{-j3\pi/2}$$

$$= 1 + \cos 3\pi/2 - j\sin 3\pi/2.$$

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$$\boxed{X(3) = 1+j}$$

$$X(k) = \{2, 1-j, 0, 1+j\}.$$

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(ii) $X(k) = \{1, 0, 1, 0\}$ find IDFT.

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi nk}{N}}$$

To find $x(0)$:

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{\frac{j2\pi nk}{4}}$$

$$= \frac{1}{4} \sum_{K=0}^{3} X(K) e^{0}$$

$$= \frac{1}{4} \left[X(0) + X(1) + X(2) + X(3) \right]$$

$$= \frac{1}{4} \left[1 + 0 + 1 + 0 \right]$$

$$= 2/4$$

$$\boxed{x(0) = 0.5}$$

To find $x(1)$:

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{\frac{j\pi k \cdot 1}{2}}$$

$$= \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\pi/2 k}$$

$$= \frac{1}{4} \left[X(0) e^{0} + X(1) e^{j\pi/2} + X(2) e^{j\pi} + X(3) e^{j3\pi/2} \right]$$

$$= \frac{1}{4} \left[X(0) + X(2) \left[\cos\pi + j\sin\pi \right] \right]$$

$$= \frac{1}{4} \left[1 - 1 \right]$$

$$\boxed{x(1) = 0}$$

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To find $x(2)$:

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi/2\,2k}$$

$$= \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k}$$

$$= \frac{1}{4}\left[X(0)e^{0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[1 + 1\left[\cos 2\pi + j\sin 2\pi\right]\right]$$

$$= \frac{1}{4}\cdot 2$$

$$\boxed{x(2) = 0.5}$$

To find $x(3)$:

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j3\pi/2\,k}$$

$$= \frac{1}{4}\left[X(0)e^{0} + \cancel{X(1)}^{0}e^{j3\pi/2\cdot 1} + X(2)e^{j3\pi/2\cdot 2} + \cancel{X(3)}^{0}e^{j3\pi/2\cdot 3}\right]$$

$$= \frac{1}{4}\left[1 + 1\left[\cos 3\pi + j\sin 3\pi\right]\right]$$

$$= \frac{1}{4}\left[1 - 1\right]$$

$$\boxed{x(3) = 0}$$

$$x(n) = \{0.5, 0, 0.5, 0\}.$$

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Problem No 2:

Find the DFT of the given sequence

$$x(n) = 1 \quad \text{for} \quad 0 \le n \le 2$$
$$= 0 \quad \text{otherwise}$$

N = 8. Plot the Magnitude $|X(K)|$ and phase spectrum $\angle X(K)$.

$L = 3$, $N = 8$ so add $N - L = 8 - 3 = 5$ zeros

$x(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$.

Solution:

find $X(0)$:

$$X(0) = \sum_{n=0}^{7} x(n) e^{0}$$

$$= x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7)$$

$$= 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0$$

$$\boxed{X(0) = 3}$$

$|X(K)| = \sqrt{3^2} = 3$

$\angle X(k) = \tan^{-1}(0/3)$

$= \tan^{-1} 0$

$= 0$.

To find $|X(k)|$

$a + ib$

$= \sqrt{a^2 + b^2}$

$\angle X(k) = \tan^{-1} b/a$

find $X(1)$:

$$X(1) = \sum_{n=0}^{7} x(n) e^{-j\pi/4 \, n}$$

$$= x(0) e^{-j\pi/4 \cdot 0} + x(1) e^{-j\pi/4} + x(2) e^{-j\pi/2}.$$

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$= 1 + 1\left[\cos\pi/4 - j\sin\pi/4\right] + 1\left[\cos\pi/2 - j\sin\pi/2\right]$

$= 1 + [0.707 - j0.707] + (-j)$

$\boxed{X(1) = 1.707 - j1.707}$

$|X(1)| = \sqrt{(1.707)^2 + (1.707)^2}$

$= 2.414$

$\angle X(1) = \tan^{-1}\left(\frac{-1.707}{1.707}\right)$

$= \tan^{-1}(-1)$

$\angle X(1) = -\pi/4$

To find $X(2)$:

$X(2) = \sum_{n=0}^{7} x(n) e^{-j\pi/4 \cdot 2 \cdot n}$

$= \sum_{n=0}^{7} x(n) e^{-j\frac{\pi n}{2}}$

$= x(0)e^{0} + x(1)e^{-j\pi/2} + x(2)e^{-j\pi}$

$= 1 + \cos\pi/2 - j\sin\pi/2 + \cos\pi - j\sin\pi$

$= 1 - j - 1$

$\boxed{X(2) = -j}$

$\angle X(2) = \tan^{-1}\left(\frac{-1}{0}\right)$

$= \tan^{-1}(-\infty)$

$= -\pi/2$

$|X(2)| = \sqrt{1^2}$

$= 1$

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To find $X(3)$:-

$$X(3) = \sum_{n=0}^{7} x(n) e^{-j3\pi n/4}$$

$$= x(0)e^{0} + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2}$$

$$= 1 + \cos 3\pi/4 - j\sin 3\pi/4 + \cos 3\pi/2 - j\sin 3\pi/2$$

$$= 1 - 0.707 - j0.707 + j$$

$$\boxed{X(3) = 0.293 + j0.293}$$

$$|X(3)| = \sqrt{(0.293)^2 + (0.293)^2}$$

$$= 0.414$$

$$\angle X(3) = \tan^{-1}\left(\frac{0.293}{0.293}\right)$$

$$= \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \tan^{-1} 1 = \pi/4$$

To find $X(4)$:

$$X(4) = \sum_{n=0}^{7} x(n) e^{-j4\pi n/4}$$

$$= x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi}$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$$

$$= 1 - 1 + 1$$

$$\boxed{X(4) = 1}$$

$$|X(4)| = 1 \qquad \tan^{-1}\left(\frac{0}{1}\right) = \tan^{-1} = 0$$

$$\angle X(4) = 0$$

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To find $X(5)$:

$$X(5) = \sum_{n=0}^{7} x(n) e^{-j\pi n \frac{5}{4}}$$

$$= x(0)e^{0} + x(1)e^{-j5\pi/4} + x(2)e^{-j5\pi/2}$$

$$= 1 + \cos 5\pi/4 \; \cancel{\cos} - j\sin \frac{5\pi}{4} + \cos 5\pi/2 - j\sin 5\pi/2$$

$$= 1 - 0.707 + j0.707 - j$$

$$X(5) = 0.293 - j0.293$$

$$|X(5)| = \sqrt{(0.293)^2 + (0.293)^2} = 0.414$$

$$\angle X(5) = \tan\left(\frac{-0.293}{0.293}\right) = \tan^{-1}(-1) = -\pi/4.$$

To find $X(6)$:

$$X(6) = \sum_{n=0}^{7} x(n) e^{-j6\pi/4 \, n}$$

$$X(6) = \sum_{n=0}^{7} x(n) e^{-j3\pi/2 \, n}$$

$$= x(0)e^{0} + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi}$$

$$= 1 + \cos 3\pi/2 - j\sin 3\pi/2 + \cos 3\pi - j\sin 3\pi$$

$$= 1 + j - 1$$

$$\boxed{X(6) = j}$$

$$|X(6)| = \sqrt{1^2} = 1$$

$$\angle X(6) = \tan^{-1}(1/0) = \tan^{-1}(\infty) = \pi/2.$$

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find $X(7)$:

$$X(7) = \sum_{n=0}^{7} x(n)\, e^{-j\frac{7\pi n}{4}}$$

$$= x(0)e^{0} + x(1)\, e^{-j7\pi/4} + x(2)e^{-j14\pi/4}$$

$$= 1 + \cos 7\pi/4 - j\sin 7\pi/4 + \cos 7\pi/2 - j\sin 7\pi/2$$

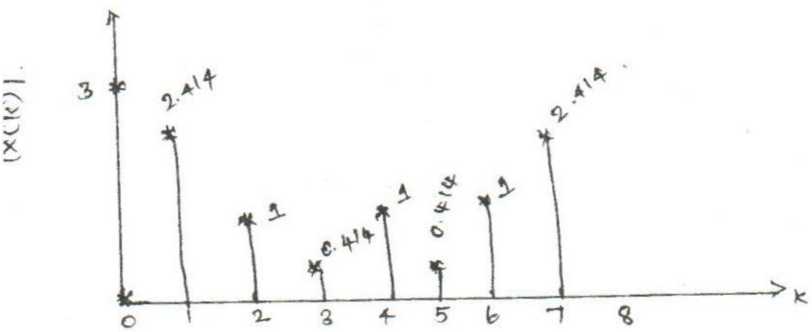
$$= 1 + 0.707 + j0.707 + j$$

$$X(7) = 1.707 + j1.707$$

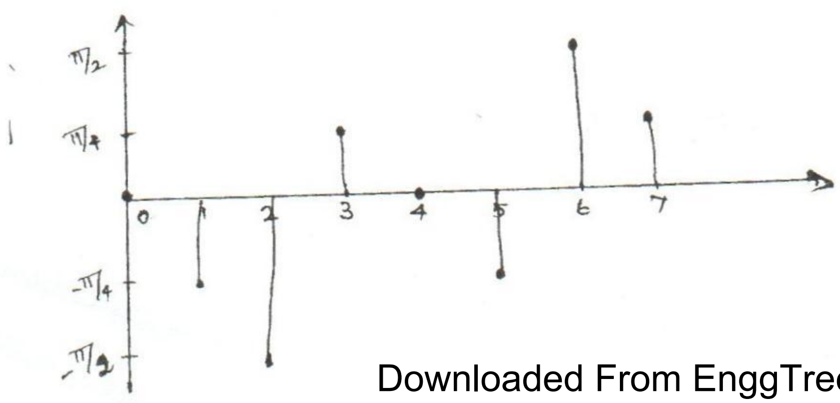
$$|X(7)| = \sqrt{(1.707)^2 + (1.707)^2} = 2.414.$$

$$\angle X(7) = \tan^{-1}\left(\frac{1.707}{1.707}\right) = \tan^{-1}(1) = \pi/4.$$

Magnitude spectrum.



Phase spectrum.



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Comparision of n.o of complex mulltiplication for DFT Vs FFT.

	N.o of stages	Num of points	DFT N^2	FFT $N/2 \log_2 N$	Speed improvement factor $\frac{N^2}{N/2 \log_2 N}$
1.	2	4	16	4	4
2.	4	16	256	32	8
3.	3	8	64	12	5.33
4.	5	32	1024	80	12.8
5.	10	~~10,48,5~~ 1024	10,48,576	5,120	204.8

Hence the FFT reduces the computation time required to compute DFT.

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Fast Fourier Transform :-

The FFT is a method of computing the DFT with reduced number of calculations. It uses the two basic properties of the twiddle factor

(i) $W_N^{k+N/2} = -W_N^k$ → Symmetry property

(ii) $W_N^{k+N} = W_N^k$ → Periodicity property

and reduces the n.o f complex multiplications and additions.

i.e N^2 multiplications are reduced to $\underline{\underline{N/2 \log_2 N}}$

$N(N-1)$ additions are reduced to $\underline{\underline{N \log_2 N}}$.

* FFT algorithms are based on the fundamental principle of decomposing the computation of DFT of a sequence of length N into successively smaller DFT.

* The FFT algorithm is also known as Radix-2 FFT algorithm since the number of o/p points N is expressed as a power of 2 i.e $N = 2^m$ where m is an integer.

FFT algorithms:

1. Decimation-in-time FFT algorithm
2. Decimation-in-frequency algorithm.

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Decimation-in-Time algorithm

In this algorithm, the time domain sequence $x(n)$ is decimated and smaller point DFT's are performed. The results of smaller point DFTs are combined to get the result of N-point DFT.

Let $x(n)$ is an N-point sequence, where N is a power of 2.

* Decimate or break this sequence into two sequences of length $N/2$. one is even-indexed values of $x(n)$ and the other of odd-indexed values.

i.e

$$\left.\begin{aligned} x_e(n) &\Rightarrow x(0), x(2), x(4), x(6) \\ x_o(n) &\Rightarrow x(1), x(3), x(5), x(7) \end{aligned}\right\} \text{ if } N \text{ is } 8.$$

This can be written as

$$x_e(n) = x(2n) \qquad n = 0, 1 \ldots N/2 - 1$$

$$x_o(n) = x(2n+1) \qquad n = 0, 1 \ldots N/2 - 1$$

The N-point DFT of $x(n)$ can be written as

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nK} \qquad k = 0, 1 \ldots N-1 \qquad \text{—— (1)}$$

separate into even & odd

$$X(K) = \sum_{n=0}^{N/2-1} x(n) W_N^{nK} + \sum_{n=0}^{N/2-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{N/2-1} x(2n) W_N^{2nk} + \sum_{n=0}^{N/2-1} x(2n+1) W_N^{(2n+1)k} \qquad \text{—— (2)}$$

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Expand and bring the constant outside the summation

$$X(k) = \sum_{n=0}^{N/2-1} x(2n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{N/2-1} x(2n+1) W_N^{2nk}$$

$$X(k) = \sum_{n=0}^{N/2-1} x_e(n) W_N^{2nk} + W_N^{k} \sum_{n=0}^{N/2-1} x_o(n) W_N^{2nk} \quad \text{——(3)}$$

$$W_N^{2nk} = W_{N/2}^{nk}.$$

Using above the expression becomes

$$X(k) = \underbrace{\sum_{n=0}^{N/2-1} x_e(n) W_{N/2}^{nk}}_{N/2 \text{ pt DFT of Even}} + W_N^{k} \underbrace{\sum_{n=0}^{N/2-1} x_o(n) W_{N/2}^{nk}}_{N/2 \text{ pt odd DFT}} \quad \text{——(4)}$$

$$X(k) = X_e(k) + W_N^{k} X_o(k) \quad \text{——(5).}$$

where $k = 0, 1 \ldots N/2-1$

For $k \geq N/2$

use the symmetrical property

$$W_N^{k} = -W_N^{k-N/2}.$$

$$X(k) = X_e(k-N/2) - W_N^{k-N/2} X_o(k-N/2)$$

where $k = N/2 \ldots N-1$.

——(6).

If $N=8$; then

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$$X(K) = X_e(K) + W_8^K X_o(K) \qquad 0 \le K \le 3$$

$$X(K) = X_e(K-4) - W_8^{K-4} X_o(K-4) \qquad 4 \le K \le 7.$$

By substituting different values of K

$K=0 \quad X(0) = X_e(0) + W_8^0 X_o(0)$

$K=1 \quad X(1) = X_e(1) + W_8^1 X_o(1)$

$K=2 \quad X(2) = X_e(2) + W_8^2 X_o(2)$

$K=3 \quad X(3) = X_e(3) + W_8^3 X_o(3)$

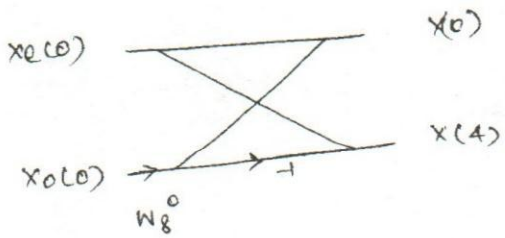
$K=4 \quad X(4) = X_e(0) - W_8^0 X_o(0)$

$K=5 \quad X(5) = X_e(1) - W_8^1 X_o(1)$

$K=6 \quad X(6) = X_e(2) - W_8^2 X_o(2)$

$K=7 \quad X(7) = X_e(3) - W_8^3 X_o(3)$

The above operation can be represented by a butterfly diagram



The 8 pt DFT flowgraph can be constructed from two - 4 pt DFT.

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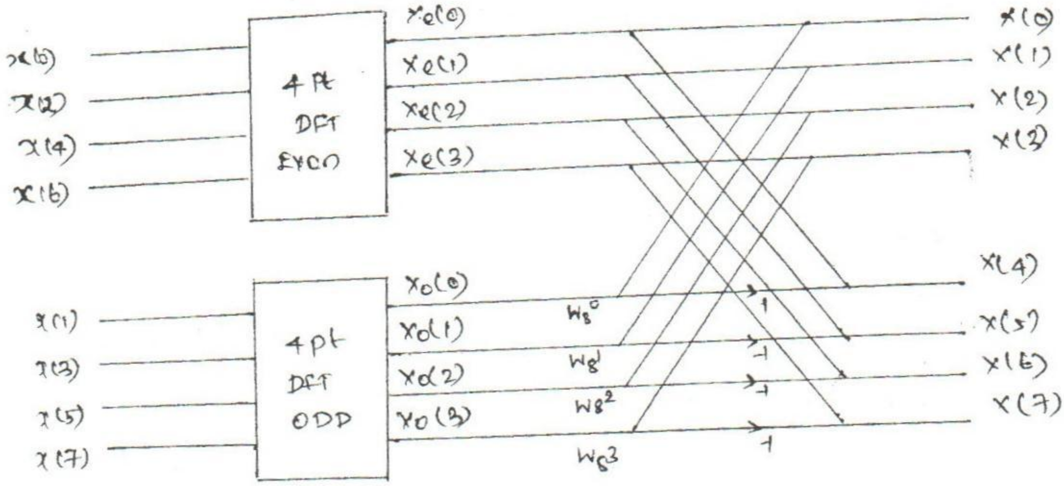

* The N/2 pt DFTs can be expressed as a combination of N/4 pt DFT.

* Divide the sequence $x_e(n)$ and $x_o(n)$ into two sequences.

* $X_e(K)$ can be written as

$$X_e(K) = \sum_{n=0}^{N/2-1} x_e(n) W_{N/2}^{nk} \quad \text{where } k = 0 \text{ to } N/2-1$$

* $x_{ee}(n) = x_e(2n)$

$x_{eo}(n) = x_e(2n+1)$

$$X_e(K) = \sum_{n=0}^{N/4-1} x_e(2n) W_{N/2}^{2nK} + \sum_{n=0}^{N/4-1} x_e(2n+1) W_{N/2}^{(2n+1)K} \quad (7)$$

$\because W_{N/2}^{2nK} = W_{N/4}^{nK}$

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$$x_e(k) = \underbrace{\sum_{n=0}^{N/4-1} x_{ee}(n) W_{N/4}^{nk}}_{2 \text{ pt DFT}} + W_N \underbrace{\sum_{n=0}^{N/4-1} x_{eo}(n) W_{N/4}^{nk}}_{2 \text{ pt DFT}}$$

$$x_e(k) = x_{ee}(k) + W_N^{2k} x_{eo}(k) \quad \text{for} \quad k = 0 \text{ to } N/4-1$$

For $k \geq N/4$

$$x_e(k) = x_{ee}(k-N/4) - W_N^{2(k-N/4)} x_{eo}(k-N/4)$$

for $k = N/4-1$ to $N/2-1$

$k=0 \quad x_e(0) = x_{ee}(0) + W_8^0 x_{eo}(0)$

$k=1 \quad x_e(1) = x_{ee}(1) + W_8^2 x_{eo}(1)$

$k=2 \quad x_e(2) = x_{ee}(0) - W_8^0 x_{eo}(0)$

$k=3 \quad x_e(3) = x_{ee}(1) - W_8^2 x_{eo}(0)$

III^{ly}

For odd sequence $[x_o(n)]$

$$x_o(k) = x_{oe}(k) + W_N^{2k} x_{oo}(k) \quad \text{for} \quad k = 0 \text{ to } N/4-1$$

$$x_o(k) = x_{oe}(k-N/4) - W_N^{2(k-N/4)} x_{oo}(k-N/4)$$

for $k = N/4-1$ to $N/2-1$

Let $k=0 \quad x_o(0) = x_{oe}(0) + W_8^0 x_{oo}(0)$

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$x_o(1) = x_{oe}(1) + W_8^2 x_{oo}(1)$

$x_o(2) = x_{oe}(0) - W_8^0 x_{oo}(0)$

$x_o(3) = x_{oe}(1) - W_8^2 x_{oo}(1)$

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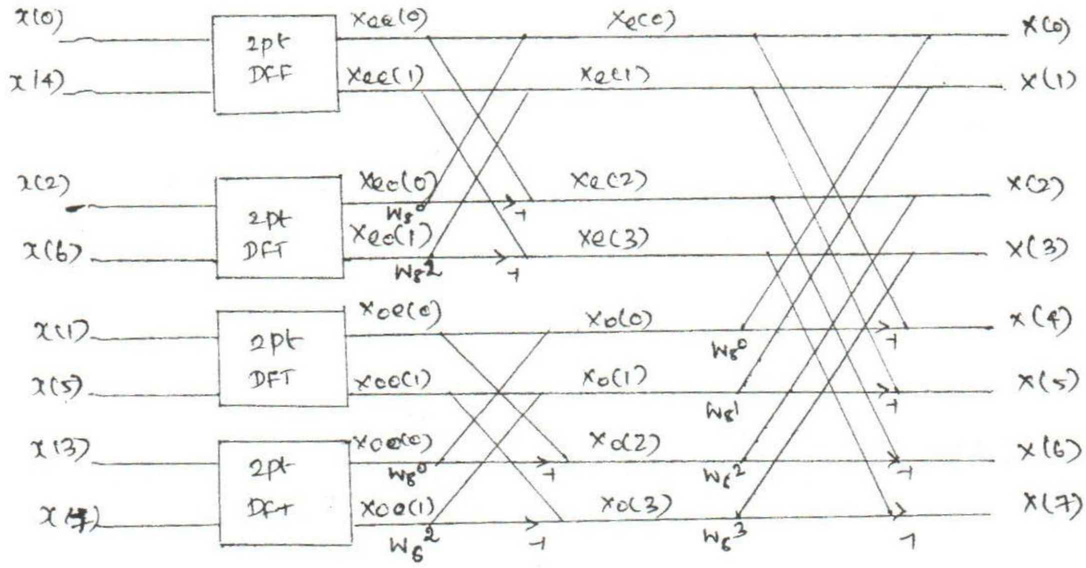
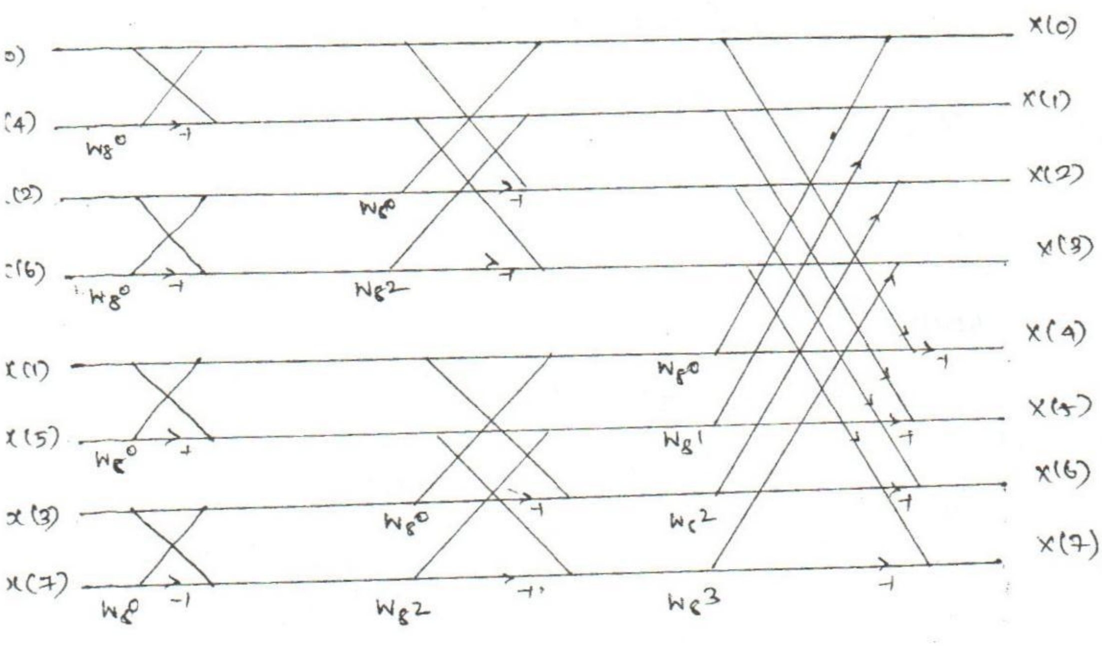


Final Flow graph: N = 8



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from the flow graph.

(i) Bit-reversal

(ii) Basic operation.

Bit Reversal :-

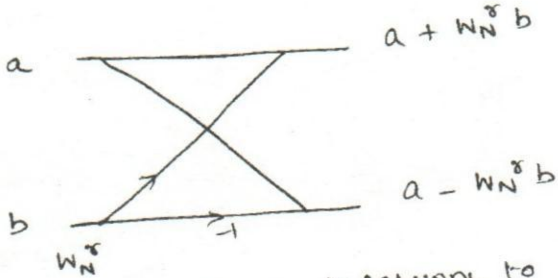
In DIT algorithm, the i/p sequence is given in a shuffled order i.e in bit-reversal order. The o/p is computed in natural order.

I/p sample	Binary value	Bit reversed Binary value	Sample index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7.

Bit reversed order

Basic Operation :

The basic computation block in the diagram is 'butterfly' in which two i/p's are combined to give o/p's.



An algorithm that uses the same locations to store both the i/p and o/p sequences is called an in-place algorithm.

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Steps of Radix - 2 DIT . FFT Algorithm -

Let N be the no of i/p samples

$N = 2^M$ where M is an integer.

. i/p sequence is shuffled through bit - reversal.

. The n.o of stages in the flowgraph is M.

for eg $N=8$ $\qquad$ $N=16$

$N = 2^3$ $\qquad$ $N = 2^4$

so three stages. $\qquad$ 4 stages

4. N.o of complex multiplications is given by $N/2 \log_2 N$.

. N.o of complex additions is given by $N \log_2 N$.

5. The twiddle factor exponents are a function of the stage index m and is given by

$$k = \frac{Nt}{2^m} \quad t = 0, 1, 2 \ldots (2^{m-1} - 1).$$

where m is the stage index.

for eg. $N = 8$, $m = 2$, $t = 0, \ldots 2^{2-1} - 1$

$t = 0, \ldots 2-1$

$t = 0, 1$

$$k = \frac{8 \times 0}{2^2} = 0$$

$$k = \frac{8 \times 1}{2^2} = \frac{8}{4} = 2.$$

$\Rightarrow W_8^0 , W_8^2$.

7. ERF (Exponent Repeat factor) → N.o of times the exponent sequence is repeated is given by 2^{M-m} for each stage.

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Draw the flow graph of 4 point DIT-FFT

1. $N = 4$.

2.

00	$\rightarrow$ 00	$x(0)$
01	$\rightarrow$ 10	$x(2)$
10	$\rightarrow$ 01	$x(1)$
11	$\rightarrow$ 11	$x(3)$

3. $N = 2^2$, $M = 2$, $m = 1, 2$

4. Multiplications $4/2 \log_2 2^2 \Rightarrow 4$

5. Additions $N \log_2 2^2 \Rightarrow 8$.

6. $k = \dfrac{Nt}{2^m}$ $\quad t = 0, \; 2^{1-1} - 1$ $\quad \Big\} \; m = 1$

$t = 0$

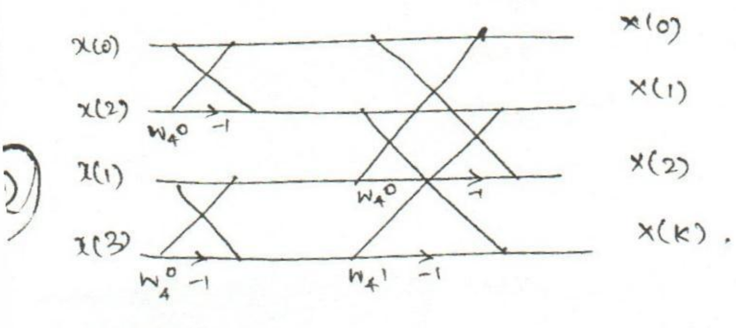
$k = 0$ $\quad W_4^0$

$k = \dfrac{4 \times 0}{2^2}$ $\quad t = 0, \; 2^{2-1} - 1$ $\quad \Big\} \; m = 2$

$= 0$ $\quad t = 0, 1$

$k = \dfrac{4 \times 1}{2^2}$ $\quad W_4^0$

$k = 1$. $\quad W_4^1$.



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Problem No 1 :-

Find the DFT of a sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIT algorithm.

$N = 8$

The twiddle factors associated with the flow graph

$$W_8^0 = e^{-j2\pi/8 \cdot 0} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi/8 \cdot 1} = e^{-j\pi/4} = \cos\pi/4 - j\sin\pi/4$$

$$= 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi/8 \cdot 2} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2$$

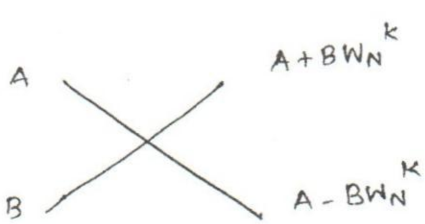
$$= 0 - j$$

$$W_8^2 = -j$$

$$W_8^3 = e^{-j2\pi/8 \cdot 3} = e^{-j3\pi/4} = \cos 3\pi/4 - j\sin 3\pi/4$$

$$= -0.707 - j0.707.$$

Basic operation



Bit reversal i/p

$x(0)$

$x(4)$

$x(2)$

$x(6)$

$x(1)$

$x(5)$

$x(3)$

$x(7)$.

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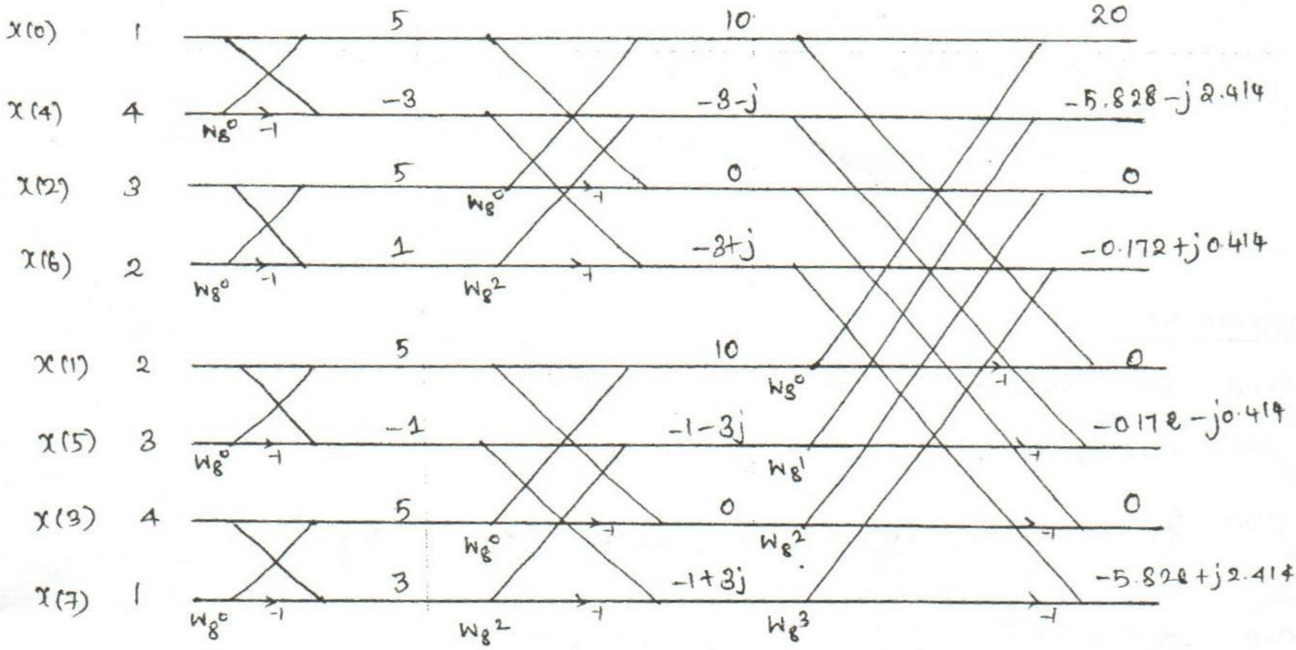

I/P	stage 1	stage 2	stage 3
1	$1+W_8^0 4$ $1+4 = 5$	$5+W_8^0 5 = 10$	$10+W_8^0 10 = 20$
4	$1-4 = -3$	$-3+W_8^2 = -3-j$	$-3-j+(-1-3j)(0.707-j0.707)$ $= -5.828-j2.414$
3	$3+W_8^0 2$ $3+2 = 5$	$5-5 = 0$	0
2	$3-W_8^0 2 = 1$	$-3-W_8^2 \cdot 1 = -3+j$	$-3+j+(-1+3j)(-0.707-j0.707)$ $= -0.172+j0.414$
2	$2+3W_8^0 = 5$	$5+W_8^0 5 = 10$	$10-W_8^0 10 = 0$
3	$2-3 = -1$	$-1+W_8^2 3 = -1-3j$	$(-3-j)-(-1-3j)(0.707-j0.707)$ $= -0.1428+j0.414$
4	$4+W_8^0 = 5$	$5-W_8^0 5 = 0$	0
1	$4-1 = 3$	$-1-3W_8^2 = -1+3j$	$(-3+j)-(-1+3j)(-0.707-j0.707)$ $= -5.828-j2.414$

3

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$$x(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}.$$

Problem No: 2.

Find the 8pt DFT using DIT Algorithm

$x(n) = \cos n\pi/4 \quad 0 \le n \le 7.$

$x(n) = \{1, 0.707, 0, -0.707, -1, -0.707, 0, 0.707\}.$

$n=0 \quad \cos 0 = 1$

$n=1 \quad \cos \pi/4 = 1/\sqrt{2}$

$n=2 \quad \cos \frac{2\pi}{4} = \cos \pi/2 = 0$

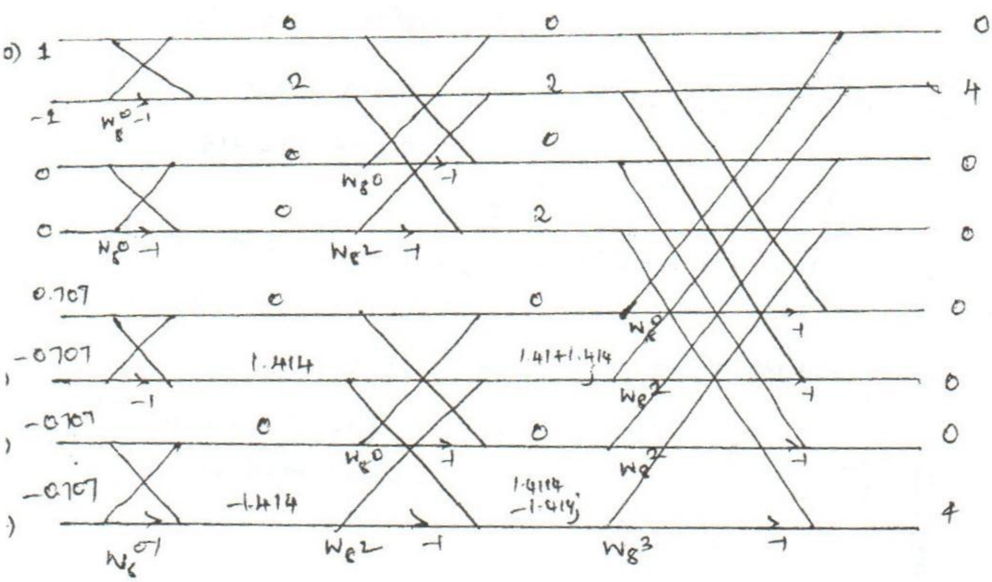
$n=3 \quad \cos 3\pi/4 = -0.707$

$n=4 \quad \cos \pi = -1$

$n=5 \quad \cos 5\pi/4 = -0.707$

$n=6 \quad \cos 3\pi/2 = 0$

$n=7 \quad \cos 7\pi/4 = 0.707.$



$x(k) = \{0, 4, 0, 0, 0, 0, 0, 4\}.$

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IDFT using DIT Algorithm

To find the IDFT

* Take the complex conjugate of $X(k)$
* Apply as Bit reversed i/p
* Compute DFT using DIT algorithm
* O/p will be in the form of $N x^*(n)$
* To get $x(n)$
  → Divide by N & take conjugate

IDFT formula.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+j\frac{2\pi nk}{N}}$$

Take conjugate on both sides

$$N\, x^*(n) = \underbrace{\sum_{k=0}^{N-1} X^*(K) e^{-j\frac{2\pi nk}{N}}}_{DFT\,[X^*(k)]}$$

So $$x^*(n) = \frac{DFT\{X^*(k)\}}{N}.$$

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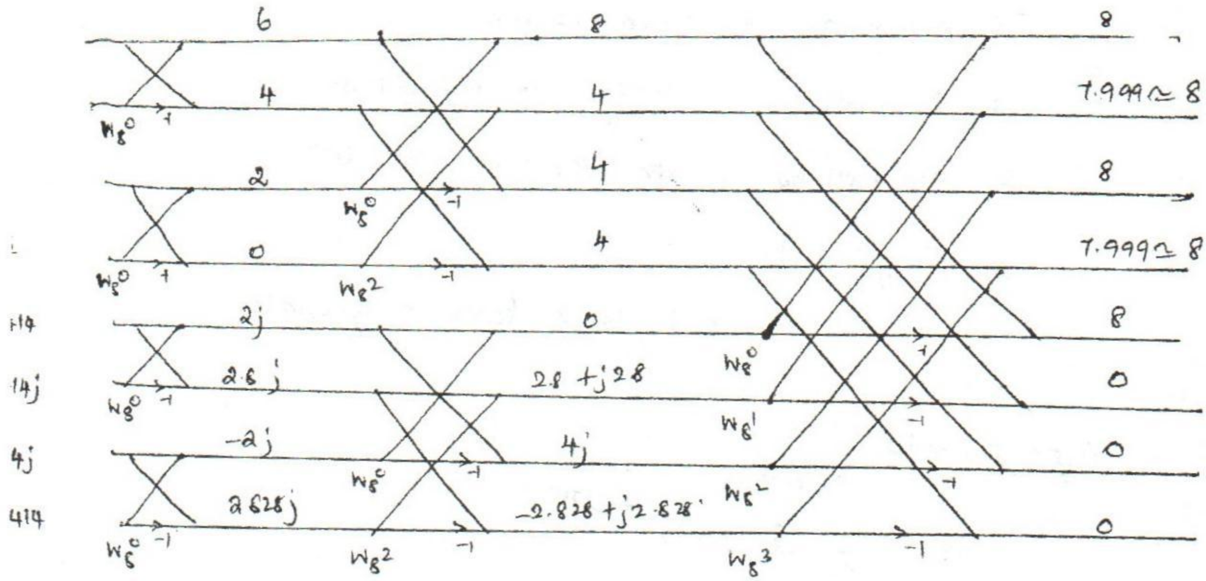
Problem NO 3:

Find the IDFT of $X(k) = \{5, -j2.414, 1, -0.414j, 1, 0.414j, 1, j2.414\}$.



I/P	Stage 1	Stage 2	Stage 3
5	$5+1 = 6$	$6+2 = 8$	$8+0 = 8$
1	$5-1 = 4$	$4+0 = 4$	$4 + (2.8+j2.8)(+0.707 - j0.707) = 7.999$
1	$1+1 = 2$	$6-2 = 4$	$4 + 4j(-j) = 8$
1	$1-1 = 0$	$4-0 = 4$	$4 + (-2.828 + j2.828)(-0.707 - j0.707) = 7.99$
$j2.414$	$j2.414 - 0.414j = 2j$	$2j - 2j = 0$	$8 - 0 = 8$
$0.414j$	$j2.414 + 0.414j = 2.828j$	$(2.828j + 2.828j)(-j) = 2.8 + j2.8$	$4 - (2.8+j2.8)(0.707 - j0.707) = 0$
$1.414j$	$0.414j - j2.414 = -2j$	$4j$	$4 - 4j(-j) = 0$
$j2.414$	$0.414j + j2.414 = 2.828j$	$-2.828 + j2.828$	$4 - (-2.828 + j2.828)(-0.707 - j0.707) = 0$

$x(n) = \{1, 1 \ldots \}$

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Problem No4 !

Use the 4pt inverse FFT and findout $x(n)$

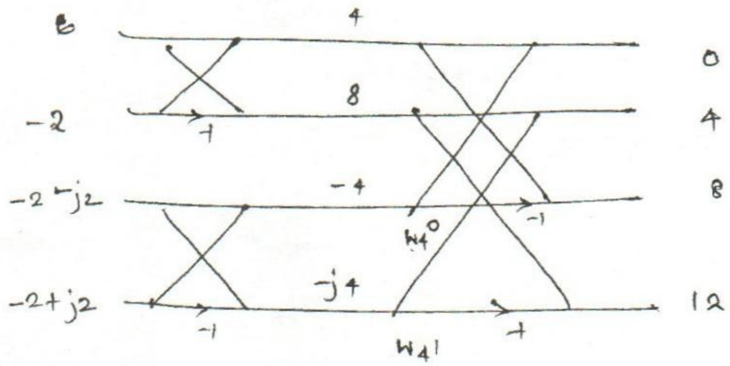
$X(k) = \{6, -2+j2, -2, -2-j2\}$.

$W_4^0 = e^{-j\frac{2\pi}{4}\cdot 0} = e^0 = 1$

$W_4^1 = e^{-j2\pi/4 \cdot 1} = e^{-j\pi/2} = -j$

$X^*(k) = \{6, -2-j2, -2, -2+j2\}$

$x^*(0) \quad x^*(1) \quad x^*(2) \quad x^*(3)$



Stage 1   Stage 2   Divide by N

Stage 1	Stage 2	Divide by N
$6-2=4$	$4-4=0$	0
$6-(-2)=8$	$8-(-j4)(-j)=8-4=4$	1
$-2-j2 \; -2+j2 = -4$	$4-(-4)W_4^0 = 4+4=8$	2
$-2-j2 \; +2-j2 = -j4$	$8-(-j4)(-j) = 8+4 \; 12$	3

5

$x(n) = \{0, 1, 2, 3\}$

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Decimation-in-frequency Algorithm:-

In this algorithm, the frequency samples of DFT are divided into smaller and smaller subsequences.

* The i/p $x(n)$ is divided into the first half and last half of points.

The N-pt DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=N/2}^{n=N-1} x(n) W_N^{nk} \quad \text{---(1)}$$

sub $n = n + N/2$ in the second term

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n+N/2=N/2}^{n+N/2=N-1} x(n+N/2) W_N^{(n+N/2)k}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=0}^{n=N/2-1} x(n+N/2) W_N^{nk} W_N^{N/2k}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + W_N^{N/2k} \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{nk} \quad \text{---(2)}$$

Let

$$W_N^{N/2k} = e^{\frac{-j2\pi N/2 K}{N}} = e^{-j\pi K} = (-1)^K .$$

Sub this in (2)

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + (-1)^k \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{nk} \quad \text{---(3)}$$

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Taking the summation outside.

$$x(k) = \sum_{n=0}^{N/2-1} [x(n) + (-1)^k x(n+N/2)] W_N^{nk} \quad \text{——} (4)$$

Two equations can be formed depending on whether k is even or odd.

when k is even $(-1)^k = 1$

so $$x(2k) = \sum_{n=0}^{N/2-1} [x(n) + x(n+N/2)] W_N^{2nk} \quad 0 \le k \le N/2-1$$

—— (5)

when k is odd $(-1)^k = -1$

$$x(2k+1) = \sum_{n=0}^{N/2-1} [x(n) - x(n+N/2)] W_N^{n(2k+1)}$$

—— (6)

$$x(2k+1) = \sum_{n=0}^{N/2-1} [x(n) - x(n+N/2)] W_N^{2nk} \cdot W_N^{n}$$

$$x(2k+1) = \sum_{n=0}^{N/2-1} [x(n) - x(n+N/2)] W_N^{n} \cdot W_{N/2}^{nk}$$

—— (7).

Let $g(n) = x(n) + x(n+N/2)$

$h(n) = x(n) - x(n+N/2)$

$g(0) = x(0) + x(4)$
$g(1) = x(1) + x(5)$
$g(2) = x(2) + x(6)$
$g(3) = x(3) + x(7)$

$h(0) = x(0) - x(4)$
$h(1) = x(1) - x(5)$
$h(2) = x(2) - x(6)$
$h(3) = x(3) - x(7)$

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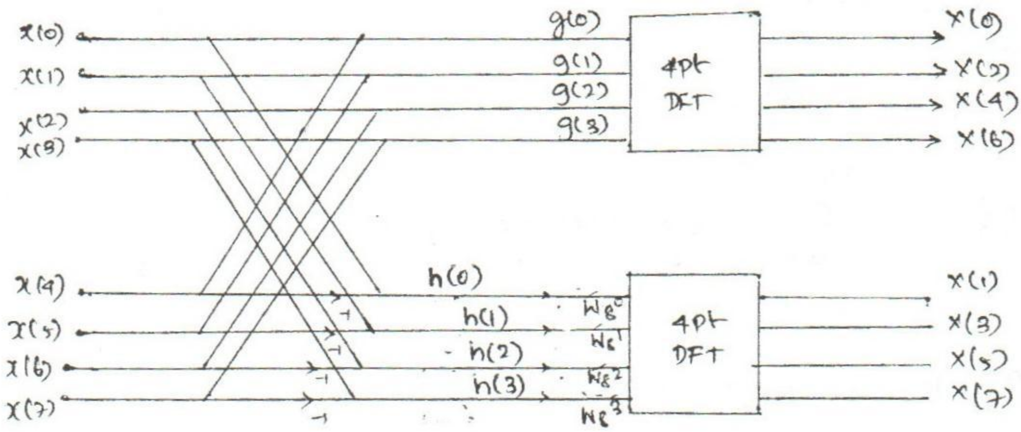
So Now

$$X(2k) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{nk}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} \left(h(n)\, W_N^{n}\right) W_{N/2}^{nk}.$$



The Even indexed N/2 pt DFT can be decimated into N/4 pt DFT.

$$X(2k) = \sum_{n=0}^{N/2-1} g(n)\, W_{N/2}^{nk} \qquad 0 \le k \le N/2-1 \qquad \text{--- (8)}$$

This is splitted

$$X(2k) = \sum_{n=0}^{N/4-1} g(n)\, W_{N/2}^{nk} + \sum_{n=N/4}^{n=N/2-1} g(n)\, W_{N/2}^{nk} \qquad \text{--- (9)}$$

Let $n = n + N/4$ in second term.

$$X(2k) = \sum_{n=0}^{N/4-1} g(n)\, W_{N/2}^{nk} + \sum_{n+N/4=N/4}^{n+N/4=N/2-1} \cancel{g(n)\ldots}\; g(n+N/4)\, W_{N/2}^{(n+N/4)k}$$

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$$X(2k) = \sum_{n=0}^{N/4-1} g(n) W_{N/2}^{nk} + \sum_{n=0}^{N/4-1} g(n+N/4) W_{N/2}^{N/4 k} W_{N/2}^{nk} \quad \leftarrow (10)$$

$$W_{N/2}^{N/4 k} = e^{\frac{-j2\pi}{N/2} \cdot N/4 k} = e^{-j\pi k} = (-1)^k.$$

sub the value in (10)

$$X(2K) = \sum_{n=0}^{N/4-1} g(n) W_{N/2}^{nk} + (-1)^k \sum_{n=0}^{N/4-1} g(n+N/4) W_{N/2}^{nk} \quad \leftarrow (11)$$

Take the summation outside

$$X(2k) = \sum_{n=0}^{N/4-1} [g(n) + (-1)^k g(n+N/4)] W_{N/2}^{nk}$$

Two expressions can be formed depending on k.

when k is even $(-1)^k = 1$, sub $k = 2k$

$$X(4K) = \sum_{n=0}^{N/4-1} [g(n) + g(n+N/4)] W_{N/2}^{2nk}. \quad \leftarrow (12)$$

$$X(4k) = \sum_{n=0}^{N/4-1} \underbrace{[g(n) + g(n+N/4)]}_{2\text{ pt DFT}} W_{N/4}^{nk}.$$

when k is odd $(-1)^k = -1$, sub $k = 2k+1$

$$X(2(2k+1)) = \sum_{n=0}^{N/4-1} [g(n) - g(n-N/4)] W_{N/2}^{n(2k+1)} \quad \leftarrow (13)$$

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$$x(4k+2) = \sum_{n=0}^{N/4-1} [g(n) - g(n+N/4)] \, W_{N/2}^{2nk} \, W_{N/2}^{2n}$$

$$x(4k+2) = \sum_{n=0}^{N/4-1} [g(n) - g(n+N/4)] \, W_{N/4}^{nk} \, W_N^{2n} \quad \text{———} (14)$$

Let $a(n) = g(n) + g(n+N/4)$

$b(n) = g(n) - g(n+N/4)$

$a(0) = g(0) + g(2)$ $\quad$ $b(0) = g(0) - g(2)$

$a(1) = g(1) + g(3)$ $\quad$ $b(1) = g(1) - g(3)$

Now

$$x(4k) = \underbrace{\sum_{n=0}^{N/4-1} a(n) \, W_{N/4}^{nk}}_{2 \text{ pt DFT}}$$

$$x(4k+2) = \underbrace{\sum_{n=0}^{N/4-1} (b(n) \, W_N^{2n}) \, W_{N/4}^{nk}}_{2 \text{ pt DFT}}$$

$|||^{ly}$. $x(2k+1)$ can be splitted into two pt dft

$$x(2k+1) = \sum_{n=0}^{N/4-1} [h(n) + (-1)^k h(n+N/4) \, W_{N/2}] \, W_N^{(2k+1)n}$$

$$x(2k+1) = \sum_{n=0}^{N/4-1} \left[h(n) + (-1)^k h(n+N/4)\right] W_N^{2n} \, W_{N/2}^{nk}.$$

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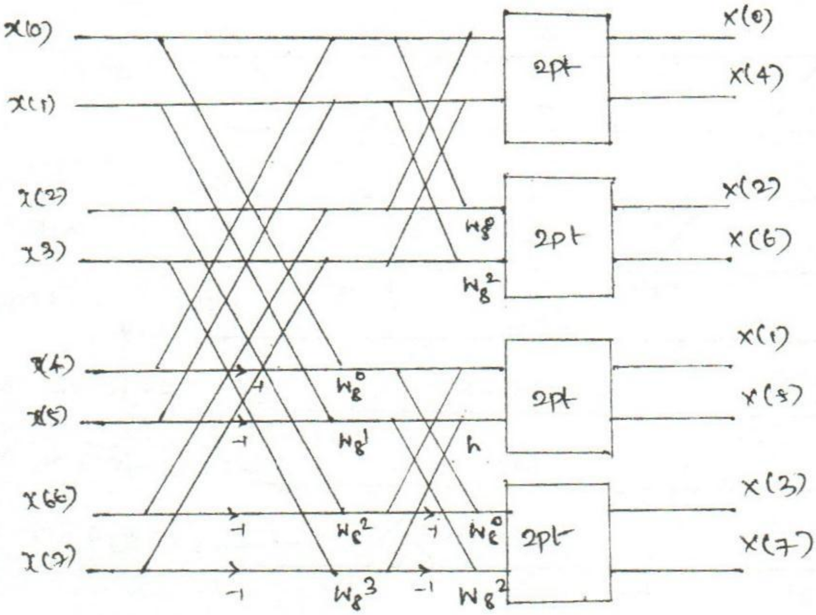
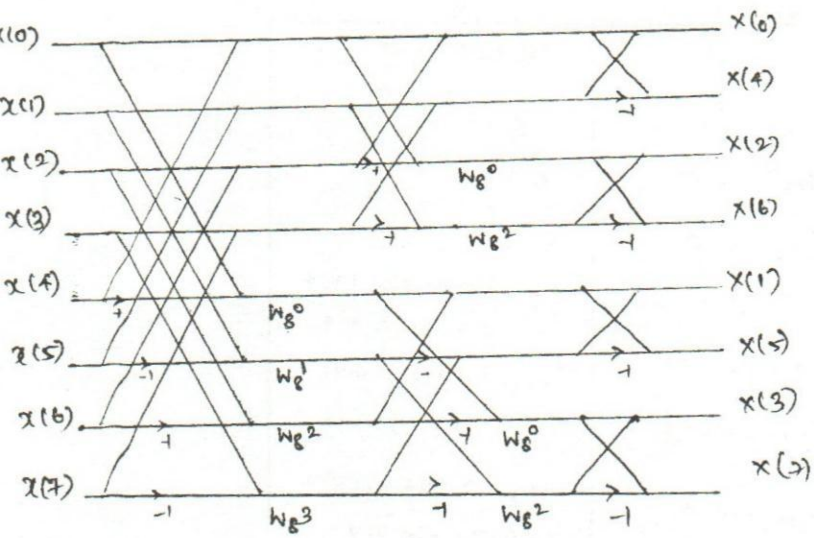
where.

$$c(n) = h(n) + h(n+N/4)$$

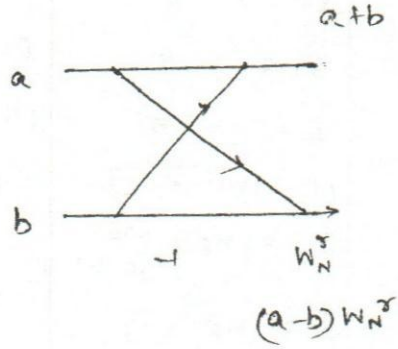
$$d(n) = h(n) - h(n-N/4).$$



Final Flow Graph :-



Basic Butterfly diagram.



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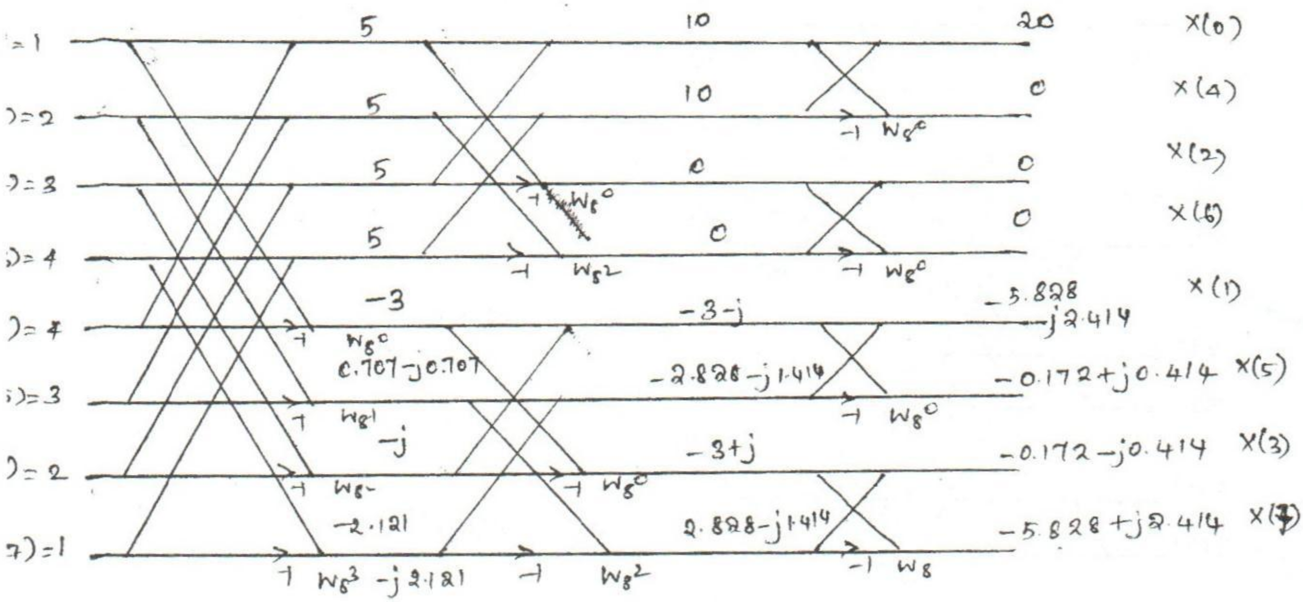
Problem No 1 :-

Find $X(k)$ using DIF FFT algorithm for the following sequence.

$$x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}.$$

$N = 8$ $\quad W_8^0 = 1 \quad W_8^1 = 0.707 - j0.707 \quad W_8^2 = -j \quad W_8^3 = -0.707 - j0.707$



I/p	stage 1	stage 2	stage 3.
1	$1+4 = 5$	$5+5 = 10$	$10+10 = 20$
2	$2+3 = 5$	$5+5 = 10$	$10-10 = 0$
3	$3+2 = 5$	$(5-5)W_8^0 = 0$	$0+0 = 0$
4	$4+1 = 5$	$(5-5)W_8^2 = 0$	$0-0 = 0$
4	$(1-4)W_8^0 = -3$	$-3-j$	$-3-j+(-2.828)-j1.414 = -5.828-j2.414$
3	$(2-3)W_8^1 = 0.707 - j0.707$	$-2.828-j1.414$	$-3-j-(-2.828-j1.414) = -0.172+j0.414$
2	$(3-2)W_8^2 = -j$	$-3-(-j) = -3+j$	$-3+j+2.828-j1.414 = -0.172-j0.414$
1	$(4-1)W_8^3 = -2.121-j2.121$	$2.828-j1.414$	$-3+j-(2.828-j1.414) = -5.828+j2.414$

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$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$.

Problem NO 2:

Compute DFTs of the sequence $x(n) = \cos n\pi/2$ where $N = 4$ using DIF FFT

$x(n) = \cos n\pi/2$

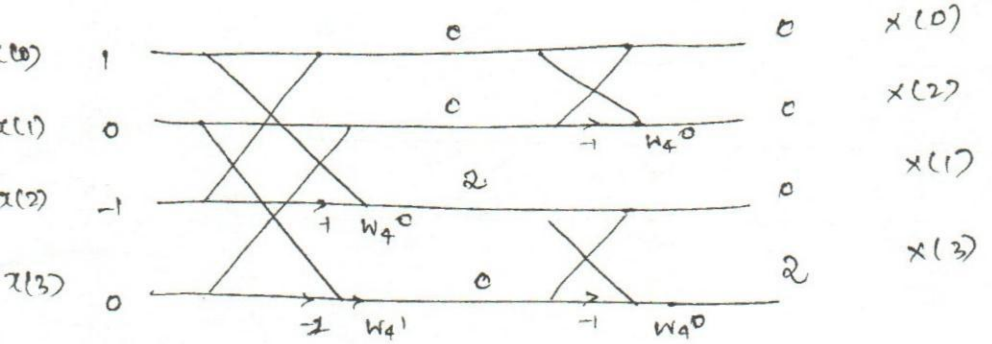
$n=0 \quad x(0) = \cos 0 = 1.$

$x(1) = \cos \pi/2 = 0$

$x(2) = \cos \pi = -1$

$x(3) = \cos 3\pi/2 = 0.$

$x(n) = \{1, 0, -1, 0\}.$



$W_4^0 = 1 \quad W_4^1 = e^{-j\frac{2\pi}{4}\cdot 1} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j.$

Stage 1

$1 + (-1) = 0$

$0 + 0 = 0$

$[1 - (-1)] W_4^0 = 2$

$[0 - 0] W_4^1 = 0$

Stage 2.

$0 + 0 = 0$

$(0 - 0) W_4^0 = 0$

$2 + 0 = 2$

$(2 - 0) W_4^0 = 2.$

$\therefore X(k) = \{0, 0, 0, 2\}$

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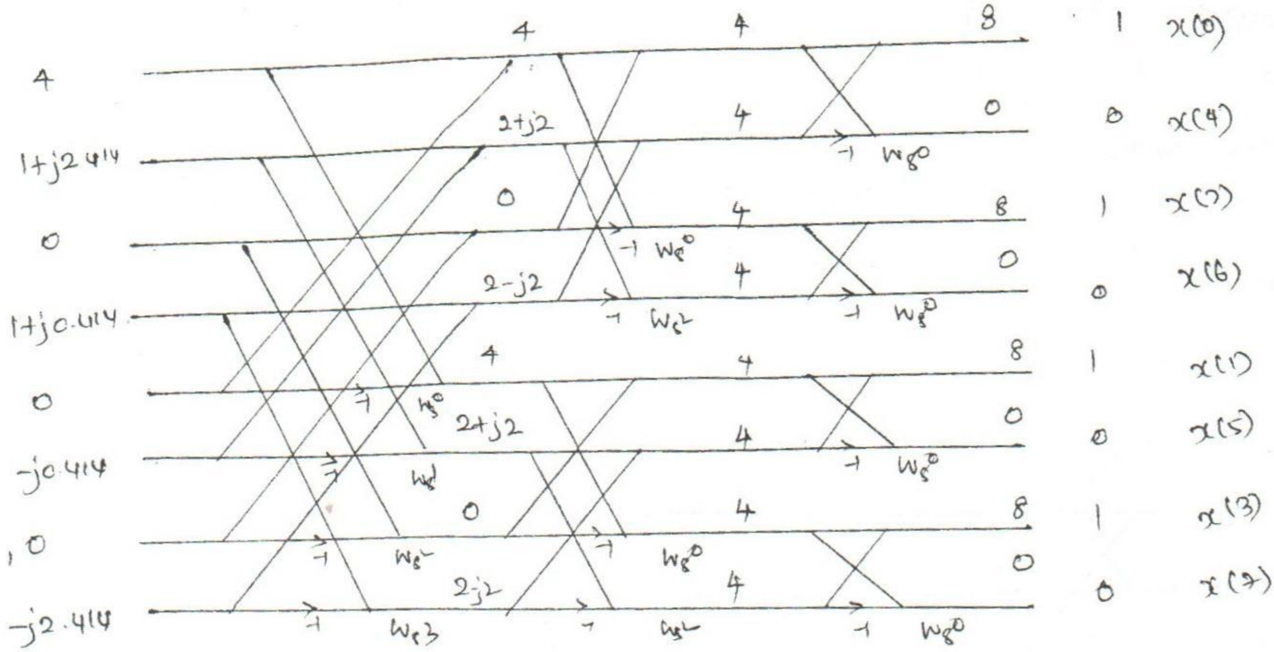
oblem No 3:

Find the IDFT of the sequence $X(K) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$ using DIF algorithm.



The o/p $8x^*(n)$ is in bit reversal.

Divide by N and take conjugate

$$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}.$$

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$X(K) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$.

Problem NO 2:

Compute DFTs of the sequence $x(n) = \cos n\pi/2$ where $N = 4$ using DIF FFT

$x(n) = \cos n\pi/2$

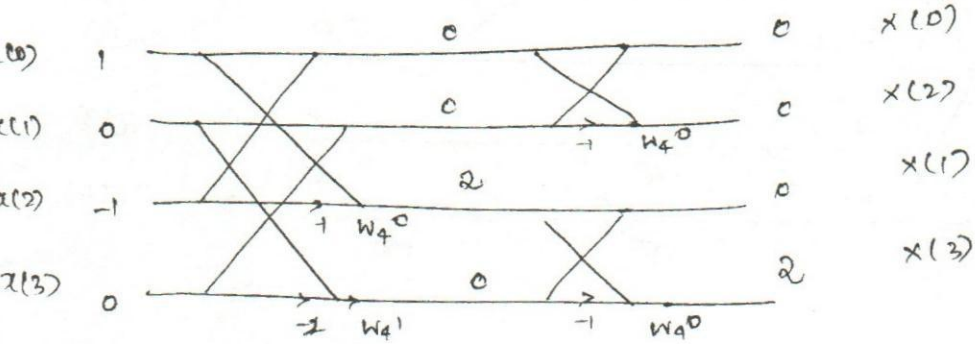
$n=0 \quad x(0) = \cos 0 = 1.$

$x(1) = \cos \pi/2 = 0$

$x(2) = \cos \pi = -1$

$x(3) = \cos 3\pi/2 = 0$

$x(n) = \{1, 0, -1, 0\}$.



$W_4^0 = 1 \quad W_4^1 = e^{-j\frac{2\pi}{4}\cdot 1} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j.$

Stage 1

$1 + (-1) = 0$

$0 + 0 = 0$

$[1 - (-1)] W_4^0 = 2$

$[0 - 0] W_4^1 = 0$

Stage 2.

$0 + 0 = 0$

$(0 - 0) W_4^0 = 0$

$2 + 0 = 2$

$(2 - 0) W_4^0 = 2$

$X(K) = \{0, 0, 0, 2\}$

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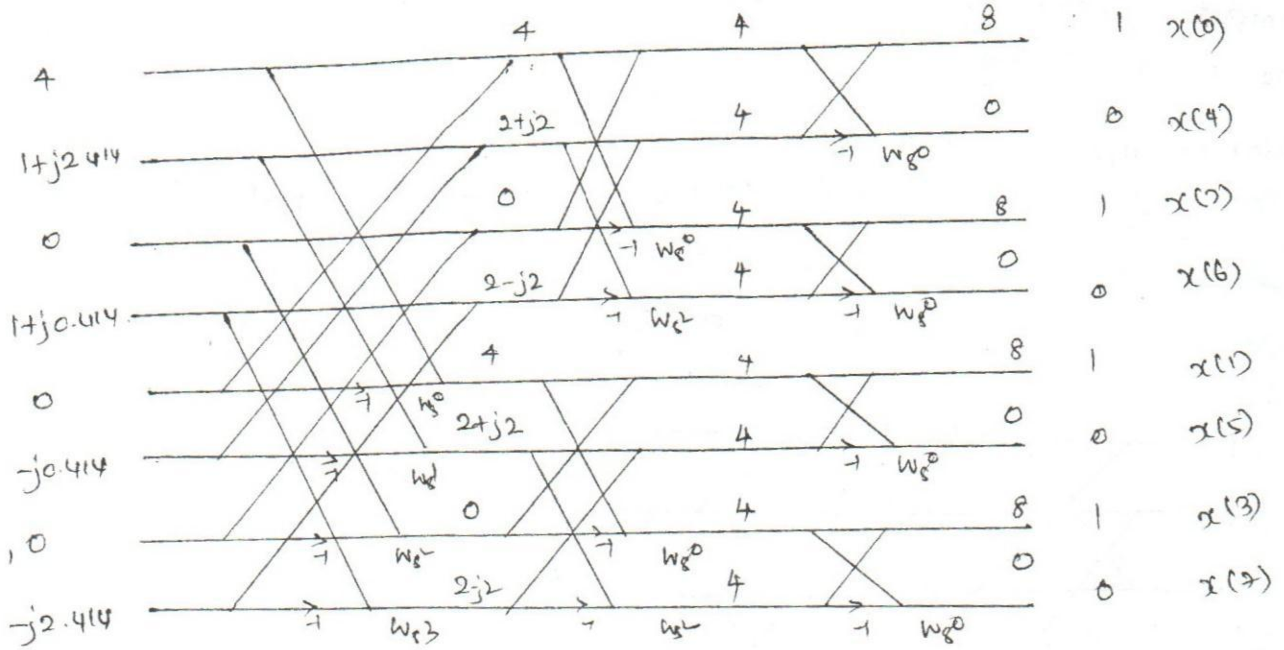
oblem No 3:

Find the IDFT of the sequence $x(k) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$ using DIF algorithm.



The o/p $8x^*(n)$ is in bit reversal.

Divide by N and take conjugate

$$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}.$$

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PROPERTIES OF DFT :

1. Circular convolution
2. Time reversal
3. Circular Time shift
4. Circular frequency shift
5. Periodicity
6. Linearity
7. Multiplication
8. Parseval's Theorem
9. Circular Correlation
10. Conjugate symmetry.

Periodicity :

If a sequence $x(n)$ is periodic with periodicity N samples, then N-point DFT is also periodic.

$x(n) = x(n+N)$ for all n

then $X(K) = X(K+N)$ for all K.

Proof :

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$$

Sub $k = K+N$, then

$$X(K+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi n(K+N)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}} \cdot e^{-j\frac{2\pi n N}{N}}$$

$\because e^{-j2\pi n} = 1.$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$$

$X(K+N) = X(K)$ Hence proved.

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LINEARITY:

If $DFT\{x_1(n)\} = X_1(k)$

$DFT\{x_2(n)\} = X_2(k)$

Then for any real-valued constant a_1 and a_2

$$DFT[a_1x_1(n) + a_2x_2(n)] = a_1X_1(k) + a_2X_2(k).$$

Proof:

Let the i/p sequence be $a_1x_1(n) + a_2x_2(n)$

$$DFT[a_1x_1(n) + a_2x_2(n)] = \sum_{n=0}^{N-1}[a_1x_1(n) + a_2x_2(n)]\,e^{-j\frac{2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} a_1x_1(n)e^{-j\frac{2\pi nk}{N}} + \sum_{n=0}^{N-1} a_2x_2(n)e^{-j\frac{2\pi nk}{N}}$$

$$= a_1X_1(k) + a_2X_2(k)$$

Hence proved.

TIME REVERSAL:

The time reversal of the sequence is obtained by graphing the sequence around the circle in clockwise direction

Let $x(n)$ be the sequence

$$x(-n)_N \Rightarrow x(N-n)$$

If DFT of $x(n) \Leftrightarrow X(k)$ then

$$DFT\{x(N-n)\} \Leftrightarrow X(N-K).$$

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Proof:

$$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) e^{-j\frac{2\pi kn}{N}}$$

changing the index from n to m.

$N-n = m$

$n = N-m$.

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi \frac{k(N-m)}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j2\pi k} e^{j\frac{2\pi km}{N}} \qquad \because e^{-j2\pi k} = 1$$

$$= \sum_{m=0}^{N-1} x(m) e^{j\frac{2\pi km}{N}}$$

Multiply by $e^{-j\frac{2\pi mN}{N}}$

$$= \sum_{m=0}^{N-1} x(m) e^{j\frac{2\pi km}{N}} e^{-j\frac{2\pi mN}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) e^{-j\frac{2\pi m(N-k)}{N}}$$

$$DFT\{x(N-n)\} = X(N-k).$$

Hence proved.

CIRCULAR FREQUENCY SHIFT:

If $DFT[x(n)] = X(k)$ then

$$DFT\{e^{j\frac{2\pi n l}{N}} x(n)\} = X(k-l)_N.$$

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Proof :

$$DFT\left\{x(n)e^{\frac{j2\pi nl}{N}}\right\} = \sum_{n=0}^{N-1} x(n)e^{\frac{j2\pi nl}{N}} \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\,e^{-j\frac{2\pi n(k-l)}{N}}$$

$\times$ by $e^{-j2\pi n N/N} = 1$

$$= \sum_{n=0}^{N-1} x(n)\,e^{-j\frac{2\pi n(k-l)}{N}}\,e^{-j2\pi nN/N}$$

$$= \sum_{n=0}^{N-1} x(n)\,e^{-j\frac{2\pi n(N+k-l)}{N}}$$

$$= X(N+k-l)$$

$$DFT\left\{x(n)e^{\frac{j2\pi nl}{N}}\right\} = X(k-l)_N$$

Hence proved.

COMPLEX CONJUGATE PROPERTY :

If $DFT\{x(n)\} = X(K)$

then $DFT[x^*(n)] = X^*(N-K)$.

Proof :

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\,e^{-j2\pi nK/N}$$

Multiply by $e^{-j2\pi nN/N} = 1$

$$X^*(K) = \sum_{n=0}^{N-1} x^*(n)\,e^{-j2\pi nK/N}\,e^{-j2\pi nN/N}$$

$$= \sum_{n=0}^{N-1} x^*(n)\,e^{-j2\pi n(N-k}$$

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Complex Conjugate Property:

$DFT\{x(n)\} = X(k)$ then

$DFT\{x^*(n)\} = X^*(N-k)$

Proof:

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}\right]^*$$

Multiply by $e^{-j2\pi nN/N}$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N} e^{-j2\pi nN/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n(N-k)}{N}}\right]^*$$

$$DFT\{x^*(n)\} = X^*(N-k)$$

Hence proved.

CIRCULAR CONVOLUTION:

If $DFT\{x_1(n)\} = X_1(k)$

$DFT\{x_2(n)\} = X_2(k)$

then $DFT\{x_1(n) \circledN x_2(n)\} = X_1(k) X_2(k)$.

Proof:

Let $X_1(k) = \sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi mk}{N}}$

Let $X_2(k) = \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi kl}{N}}$

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Let the product be $x_3(k)$

$$x_3(k) = x_1(k)\, x_2(k)$$

$$= \sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi mk}{N}} \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi kl}{N}}$$

$$x_3(n) = \text{IDFT}\{x_3(k)\}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi mk}{N}} \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi kl}{N}} \right\} e^{j\frac{2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{-j\frac{2\pi k(n-m-l)}{N}}$$

when $l = n-m$, $e^{-j\frac{2\pi k(n-m-l)}{N}} = e^0 = 1$

$$x_3(n) = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2(n-m) \sum_{k=0}^{N-1} e^0$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m).\ N$$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)$$

Hence proved.

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CIRCULAR CORRELATION :

$$DFT\{x(n)\} = X(k)$$

$$DFT\{y(n)\} = Y(k)$$

$$DFT\{r_{xy}(l)\} = DFT\left[\sum_{n=0}^{N-1} x(n)\, y^*(n-l)\right]$$

$$DFT\{r_{xy}(l)\} = X(k)\, Y^*(k)$$

MULTIPLICATION OF TWO SEQUENCES :

The DFT of multiplication of two sequences is given by the convolution of DFT's.

$$DFT\{x(n)\,y(n)\} = \frac{1}{N}\left[X(k) \circledast Y(k)\right]$$

Let $x_3(n) = x(n)\,y(n)$.

$$DFT[x_3(n)] = \sum_{n=0}^{N-1} x_3(n)\, e^{\frac{-j2\pi nk}{N}}$$

Let

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}}$$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi nk}{N}}$$

$$DFT[x_3(n)] = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi nm}{N}} \cdot \frac{1}{N}\sum_{l=0}^{N-1} Y(l)\, e^{\frac{j2\pi nl}{N}}\right] e^{-j2\pi nk}$$

$$= \frac{1}{N^2}\sum_{m=0}^{N-1} x(m) \sum_{l=0}^{N-1} Y(l)\, e^{\frac{-j2\pi n(k-m-l)}{N}}$$

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whe $l = k-m$, then

$$e^{-j2\pi n(k-m-l)} = e^{0} = 1$$

$$DFT\{x_3(n)\} = \frac{1}{N^2}\sum_{m=0}^{N-1} x(m) \sum_{l=0}^{N-1} y(l) \sum_{n=0}^{N-1} e^{0}$$

$$= \frac{1}{N^2}\sum_{m=0}^{N-1} X(m)\, Y(k-m)\, (1.N)$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X(m)\, Y(k-m)$$

$$DFT\{x_3(n)\} = \frac{1}{N}\left[X(k) \,Ⓝ\, Y(k)\right].$$

Hence proved.

$$DFT\{x(n)y(n)\} = \frac{1}{N}\left[X(k) \,Ⓝ\, Y(k)\right].$$

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PARSEVAL'S THEOREM:

If $x(n) \xleftrightarrow{DFT} X(k)$

$y(n) \xleftrightarrow{DFT} Y(k)$ then

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k).$$

If $x(n) = y(n)$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Proof:

Using correlation property

$$DFT[r_{xy}(l)] = X(k)\, Y^*(k)$$

Take IDFT on both sides

$$r_{xy}(l) = IDFT[X(k)\, Y^*(k)]$$

$$r_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)\, e^{+j\frac{2\pi lk}{N}} \quad \text{——— (1)}$$

Using formula

$$r_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*(n-l) \quad \text{——— (2)}$$

Apply $l = 0$ to both equations and equating them

$$r_{xy}(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k) = \sum_{n=0}^{N-1} x(n)\, y^*(n)$$

Hence proved.

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DFT of shifted sequence : [EnggTree.com]

If $DFT\{x(n)\} = X(k)$ then

$$DFT\{x(n-n_0)\} = X(k)e^{-j\frac{2\pi k n_0}{N}}$$

Proof:

$$IDFT\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi nk}{N}}$$

Let $n = n - n_0$.

$$x(n-n_0) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\frac{j2\pi k(n-n_0)}{N}}$$

$$x(n-n_0) = \underbrace{\frac{1}{N}\sum_{k=0}^{N-1}\left(X(k)e^{-j\frac{2\pi k n_0}{N}}\right)e^{j\frac{2\pi nk}{N}}}_{\text{IDFT formula}}$$

$$x(n-n_0) = IDFT\left\{X(k)e^{-j\frac{2\pi k n_0}{N}}\right\}.$$

Taking DFT on both sides

$$DFT\{x(n-n_0)\} = X(k)e^{-j\frac{2\pi k n_0}{N}}.$$

Hence proved.

CIRCULAR TIME SHIFT:

$DFT\{x(n)\} = X(k)$ then

$$DFT\{x(n-m)_N\} = e^{-j\frac{2\pi km}{N}} X(k).$$

Proof:

$$x(n-m)_N = x(N+n-m).$$

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$$DFT\{x(n-m)_N\} = \sum_{n=0}^{N-1} x(n-m)e^{\frac{-j2\pi kn}{N}}$$

Splitting the summation

$$DFT\{x(n-m)_N\} = \underbrace{\sum_{n=0}^{m-1} x(n-m)e^{-j\frac{2\pi kn}{N}}}_{\text{first part}} + \underbrace{\sum_{n=m}^{N-1} x(n-m)e^{-j\frac{2\pi kn}{N}}}_{\text{second part}}$$

first part :

$$\sum_{n=0}^{m-1} x(n-m)e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{m-1} x(N-m+n)\, e^{-j\frac{2\pi kn}{N}}$$

Let $l = N-m+n$.

$$= \sum_{N-m}^{N-1} x(l)\, e^{-j\frac{2\pi k}{N}(l-N+m)}$$

$$= \sum_{l=N-m}^{N-1} x(l)e^{-j\frac{2\pi kl}{N}}\, e^{j\frac{2\pi kN}{N}}\, e^{-j\frac{2\pi km}{N}}$$

$$= \sum_{l=N-m}^{N-1} x(l)e^{-j\frac{2\pi k(l+m)}{N}}$$

Second part :

$$\sum_{n=m}^{N-1} x(n-m)e^{-j\frac{2\pi kn}{N}} = \sum_{l=0}^{N-m-1} x(l)\, e^{-j\,2\pi k\,(l+m)}$$

$\to$ letting $l = n-m$.

Adding first & second part.

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$$\mathrm{DFT}\{x(n-m)_N\} = \sum_{l=0}^{N-m-1} x(l)\, e^{\frac{-j2\pi k(l+m)}{N}} + \sum_{N-m}^{N-1} x(l)\, e^{\frac{-j2\pi k(l+m)}{N}}$$

$$= \sum_{l=0}^{N-1} x(l)\, e^{\frac{-j2\pi k(l+m)}{N}}$$

$$= \underbrace{\sum_{l=0}^{N-1} x(l)\, e^{\frac{-j2\pi kl}{N}}} \; e^{\frac{-j2\pi km}{N}}$$

$$\mathrm{DFT}\{x(n-m)\}_N = e^{\frac{-j2\pi km}{N}} X(k).$$

Hence proved.

Note:

If $x(n)$ is a real sequence, then $X_R(k) = X_R(N-k)$ and $X_I(k) = -X_I(N-k)$. that is real part is even function and imaginary part is odd function.

Proof:

$$\mathrm{DFT}\{x^*(n)\} = X^*(N-k)$$

If $x(n)$ is real, then $x(n) = x^*(n)$

So $\mathrm{DFT}\{x(n)\} = X^*(N-k)$

$$X(k) = X^*(N-k)$$

$$X_R(k) + jX_I(k) = X_R(N-k) - jX_I(N-k)$$

So $X_R(k) = X_R(N-k)$ & $X_I(k) = -X_I(N-k)$

Hence proved.

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Problem No: 1

The first 5 DFT coefficients of a sequence x(n) are

$X(0) = 20$ $\quad$ $X(3) = 0.2 + j0.4$

$X(1) = 5 + j2$ $\quad$ $X(4) = 0$.

$X(2) = 0$ $\quad$ Determine the other coefficients

$X(k) = X^*(N-k)$ $\quad \because N = 8$

$X(5) = X^*(8-5)$

$= X^*(3)$

$X(5) = 0.2 - j0.4$

$X(6) = X^*(8-6) = X^*(2) = 0$

$X(7) = X^*(8-7) = X^*(1) = 5 - j2$.

Problem No: 2.

Perform the circular convolution of the following sequences. $x(n) = \{1, 1, 2, 1\}$ $\quad h(n) = \{1, 2, 3, 4\}$.

$y(n) = x(n) \circledast h(n)$

So using the convolution property

$Y(k) = X(k)\, H(k)$.

To find X(k):

$x(n) = \{1, 1, 2, 1\}$ $\quad N = 4$

$X(0) = \sum_{n=0}^{3} x(n) = 1 + 1 + 1 + 2 = 5$

$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi n \cdot 1}{4}} = \sum_{n=0}^{3} x(n) e^{-j\pi/2\, n}$.

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$$X(1) = \sum_{n=0}^{3} e^{-j\pi/2 n} \cdot x(n)$$

$$X(1) = 1 - j - 2 + j = -1$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi n \cdot 2}{4}}$$

$$= \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= 1 - 1 + 2 - 1$$

$$X(2) = 1$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi n \cdot 3}{4}}$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j 3\pi/2 n}$$

$$= 1 + j - 2 - j$$

$$= -1$$

$$X(k) = \{5, -1, 1, -1\}$$

To find $H(k)$.

$$H(0) = \sum_{n=0}^{3} y(n) e^{0}$$

$$= \sum_{n=0}^{3} y(n)$$

$$= 1 + 2 + 3 + 4$$

$$= 10$$

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$$H(1) = \sum_{n=0}^{3} y(n)\, e^{-j\frac{2\pi n \cdot 1}{4}}$$

$$= \sum_{n=0}^{3} y(n)\, e^{-j\pi n/2}$$

$$= 1 + 2(-j) + 3(-1) + 4(j)$$

$$= -2 + j2 \;//$$

$$H(2) = \sum_{n=0}^{3} y(n)\, e^{-j\frac{2\pi n \cdot 2}{4}}$$

$$= \sum_{n=0}^{3} y(n)\, e^{-j\pi n}$$

$$= 1 + 2(-1) + 3(1) + 4(-1)$$

$$= 2$$

$$H(3) = \sum_{n=0}^{3} y(n)\, e^{-j 3\pi/2 n}$$

$$= 1 + 2(j) + 3(-1) + 4(-j)$$

$$= -2 - j2.$$

So $H(k) = \{10, -2+j2, -2, -2-2j\}$.

To find $y(n)$:-

$$y(n) = \text{IDFT}\{Y(k)\}.$$

$$Y(k) = X(k)\, H(k).$$

$$Y(k) = \{50, 2-j2, -2, 2+j2\}.$$

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To find $y(n)$:

$$y(0) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{0}$$

$$= \frac{1}{4}\left[50 + 2-j2 - 2 + 2 + j2\right]$$

$$= 52/4 = 13.$$

To find $y(1)$:

$$y(1) = 1/4 \sum_{k=0}^{3} Y(k)e^{j\pi k/2}$$

$$= \frac{1}{4}\left[50 + (2-j2)j + (-2)(-1) + (2+j2)(-j)\right]$$

$$= 14.$$

$$y(2) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{j\pi k}$$

$$= \frac{1}{4}\left[Y(k)e^{j\pi k}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)(-1) - 2 + (2+j2)(-1)\right]$$

$$= \frac{44}{4} = 11$$

$$y(3) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{\frac{j3\pi k}{2}}$$

$$= \frac{1}{4}\left[50 + (2-j2)(-j) + (-2)(-1) + (2+j2)(j)\right]$$

$$= 48/4$$

$$= 12.$$

So.
$$y(n) = \{13, 14, 11, 12\}.$$

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Filtering of Long Data Sequences:

In real time signal processing applications, the input sequence x(n) is often a very long sequence. It would not be practical to store all the data before performing linear convolution, due to limited memory of a digital computer.

Therefore the input sequence must be divided into blocks. The successive blocks are processed separately one at a time. Then the results are combined together to form the overall output signal sequence.

There are two methods

(i) Overlap Add Method

(ii) Overlap Save Method.

Overlap Add Method:

Let x(n) be the input sequence and its' length is Ls.

Let h(n) be the impulse response and its' length is M.

* The i/p data sequence is segmented into blocks of length 'L'.

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* The size of the input data block is

$N = L + M - 1$.

* To each data block, $M-1$ zeros are appended.

* Then the data blocks are

$$x_1(n) = \{\underbrace{x(0), x(1) \ldots x(L-1)}_{L \text{ data points}}, \underbrace{0, 0 \ldots 0}_{M-1 \text{ zeros}}\}$$

$$x_2(n) = \{\underbrace{x(L), x(L+1) \ldots x(2L-1)}_{\text{Next } L \text{ data points}}, \underbrace{0, 0 \ldots 0}_{M-1 \text{ zeros}}\}$$

* Now $L-1$ zeros are added to the impulse response $h(n)$ to make length 'N'.

* The N-point circular convolution is performed.

* The last $M-1$ points from each block must be overlapped and added to the first $M-1$ points of the succeeding block. Hence this method is called overlap add method.

$y_1(n) = y_1(0), y_1(1) \ldots y_1(N-1)$

$y_2(n) = y_2(0), y_2(1) \ldots y_2(N-1)$.

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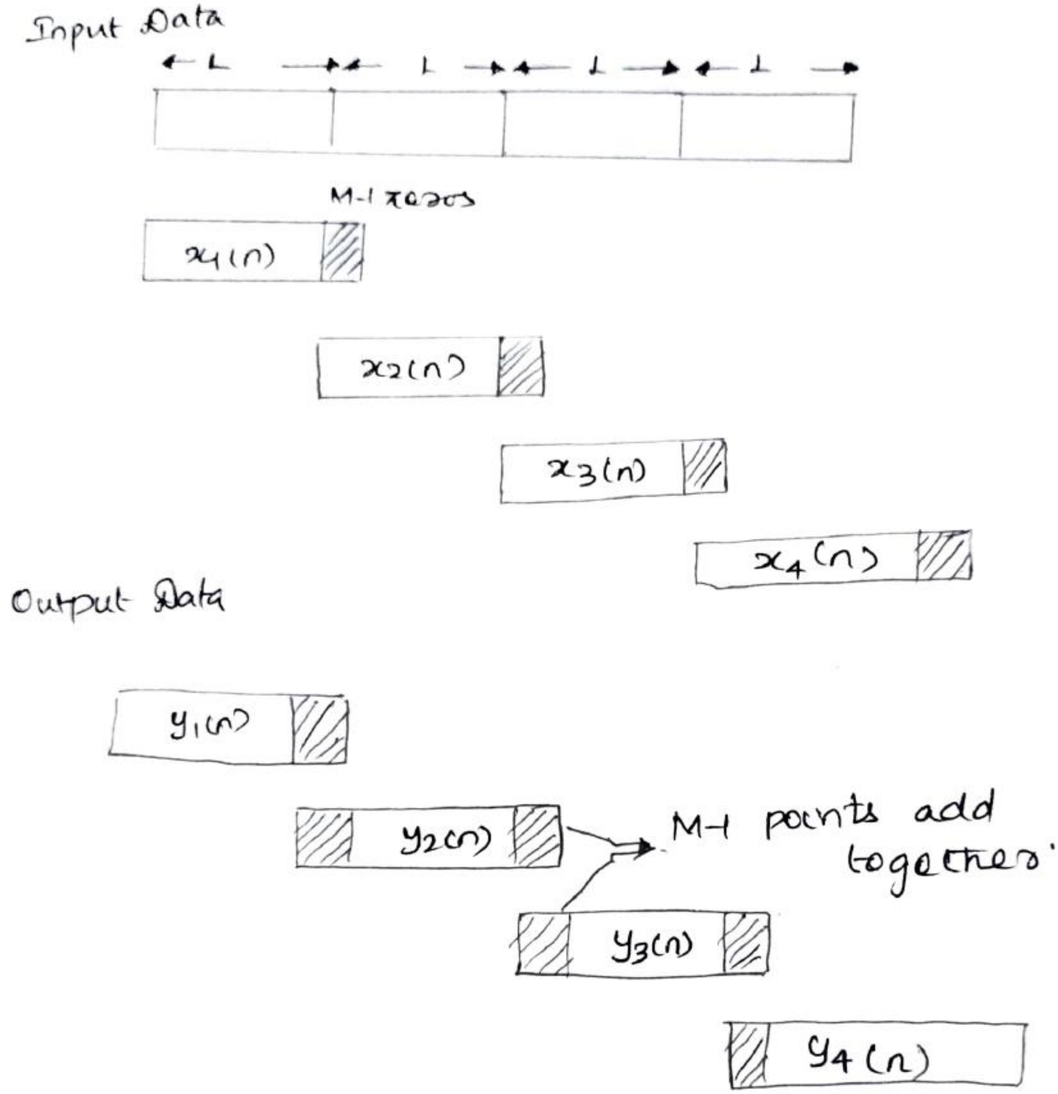
Overlapp add Method



Problem No 1 :

Find the o/p $y(n)$ of a filter whose $h(n)=\{1,1,1\}$ and the input signal $x(n)=\{3,-1,0,1,3,2,0,1,2,1\}$

Assume $L=3$, $M=3$

$$x(n)=\{\underbrace{3,-1,0}_{x_1(n)},\ \underbrace{1,3,2}_{x_2(n)},\ \underbrace{0,1,2}_{x_3(n)},\ \underbrace{1}_{x_4(n)}\}$$

add two zero's to $x_4(n)=\{1,0,0\}$. to Make same length.

$$N=L+M-1=3+3-1=5$$

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Step 2 :-

Add M-1 zeros to segmented i/p block.

$x_1(n) = \{3, -1, 0, 0, 0\}$

$x_2(n) = \{1, 3, 2, 0, 0\}$

$x_3(n) = \{0, 1, 2, 0, 0\}$

$x_4(n) = \{1, 0, 0, 0, 0\}$

Step 3 :-

Add L-1 zeros to h(n)

$h(n) = \{1, 1, 1, 0, 0\}$

Step 4 :

Perform 5 point circular convolution

$y_1(n) = x_1(n) \circledN h(n)$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$y_2(n) = x_2(n) \circledN h(n)$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 5 \\ 2 \end{bmatrix}$$

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$$y_3(n) = x_3(n) \, Ⓝ \, h(n)$$

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$$y_4(n) = x_4(n) \, Ⓝ \, h(n)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$y_1(n)$	3	2	2	-1	0									
$y_2(n)$				1	4	6	5	2						
$y_3(n)$							0	1	3	3	2			
$y_4(n)$										1	1	1	0	0
$y(n)$	3	2	2	0	4	6	5	3	3	4	3	1	0	0

$$\boxed{y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}}$$

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Overlap save Method :-

Let $x(n)$ be the i/p sequence and its length is Ls. Let $h(n)$ be the impulse response and its length is M.

The input data sequence is segmented into blocks of length 'L'. For the first block, append M-1 zeros at the front. Then the second block consists of the last M-1 data points of the previous data block followed by L new data points.

$$x_1(n) = \{ \underbrace{0, 0 \dots 0}_{\substack{M-1 \\ \text{zeros}}}, \underbrace{x(0), x(1) \dots x(L-1)}_{L \text{ data points}} \}$$

$$x_2(n) = \{ \underbrace{x(L-M+1) \dots x(L-1)}_{\substack{M-1 \text{ data points} \\ \text{from } x_1(n)}}, \underbrace{x(L) \dots x(2L-1)}_{\substack{L \text{ new data} \\ \text{points}}} \}$$

Add L-1 zeros to $h(n)$, to make length 'N'

where $N = L + M - 1$.

Then N point circular convolution is performed.

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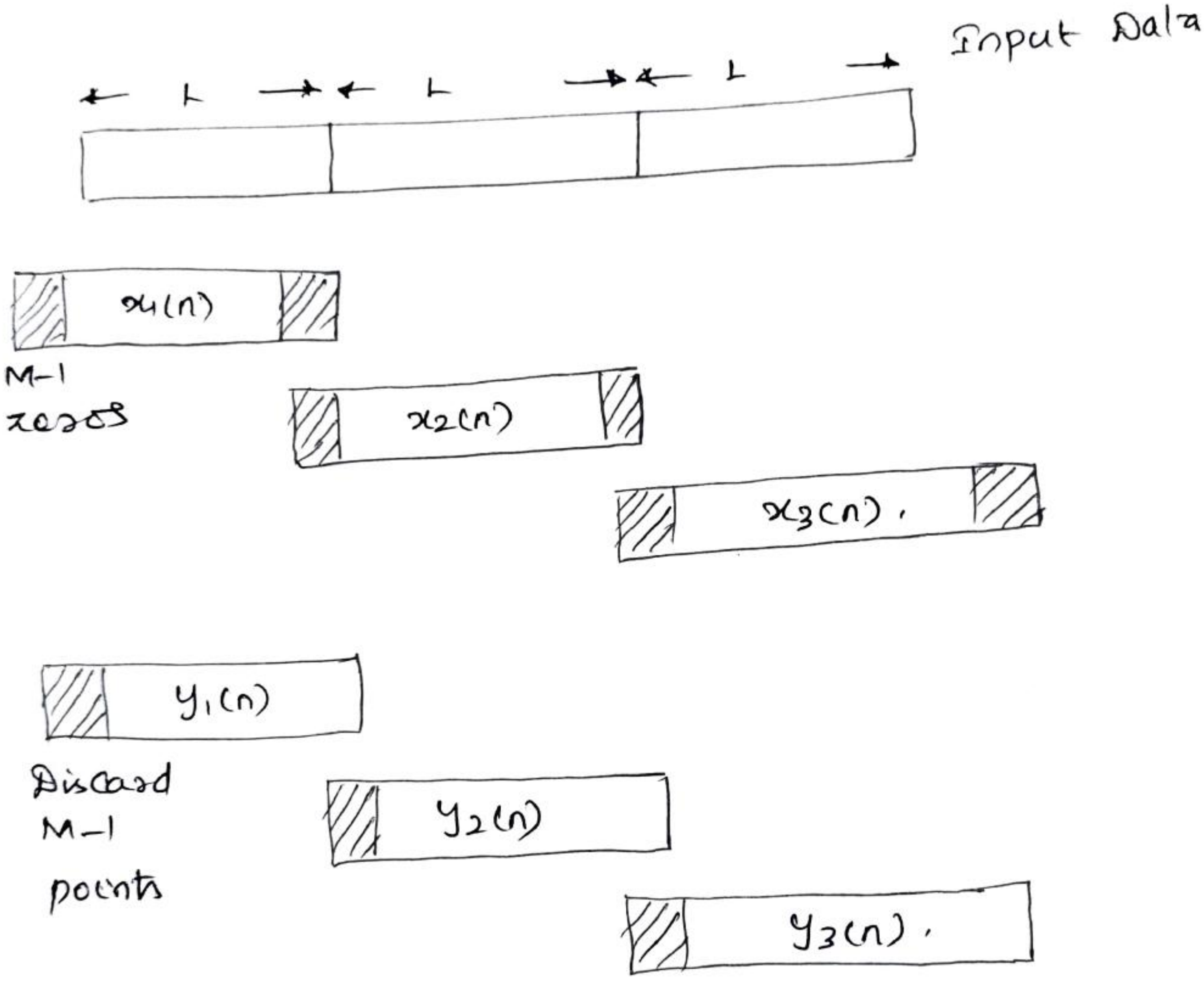
The first M-1 points are discarded due to aliasing and the remaining 'L' points constitute the desired result from linear convolution.



Problem No: 2

Find the filter output for $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ and $h(n) = \{1, 1, 1\}$

Let $L = 3$

$M = 3$

$N = L + M - 1 = 3 + 3 - 1$

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$x(n) = \{ 3, -1, 0, \; 1, 3, 2, \; 0, 1, 2, \; 1 \}$

$x_1(n) \quad x_2(n) \quad x_3(n) \quad x_4(n)$

$\hookrightarrow \{1, 0, 0\}$

add 2 zeros to make length 'L'

$x_1(n) = \{0, 0, 3, -1, 0\}$

$x_2(n) = \{-1, 0, 1, 3, 2\}$ $\qquad h(n) = \{1, 1, 1, 0, 0\}$

$x_3(n) = \{3, 2, 0, 1, 2\}$

$x_4(n) = \{1, 2, 1, 0, 0\}$

$y_1(n) = x_1(n) \; (N) \; h(n) = \{-1, 0, 3, 2, 2\}$

$y_2(n) = x_2(n) \; (N) \; h(n) = \{4, 1, 0, 4, 6\}$

$y_3(n) = \{x_3(n) \; (N) \; h(n)\} = \{6, 7, 5, 3, 3\}$

$y_4(n) = \{x_4(n) \; (N) \; h(n)\} = \{1, 3, 4, 3, 1\}$

$y_1(n) =$	$\cancel{-1}$	$\cancel{0}$	3	2	2										
$y_2(n) =$				$\cancel{4}$	$\cancel{1}$	0	4	6							
$y_3(n) =$							$\cancel{6}$	$\cancel{7}$	5	3	3				
$y_4(n) =$										$\cancel{1}$	$\cancel{3}$	4	3		
$y(n)$			3	2	2	0	4	6	5	3	3	4	3		

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UNIT - II

Infinite Impulse Response Filter

Review of design of analog Butterworth & chebyshev filter - frequency transformation in analog domain - Design of IIR digital filters using Impulse Invariant technique - Design of digital filters using bilinear transformation - Prewarping - Realization using Direct, Cascade and parallel forms.

Introduction.

The IIR filters are recursive type, whereby the present output sample depends on the present input, past input samples and output samples.

IIR digital filters have the transfer function of the form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

For a realizable filter $h(n) = 0$ for $n \leq 0$

and for a stable filter $\sum_{n=0}^{\infty} |h(n)| < \infty$.

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Bilinear Transformation Method :-

IIR filter design by means of impulse invariance method has a serios limitation in that it can be applied only for lowpass filters and a few limited bandpass filters. This limitation is overcome by bilinear transformation.

The bilinear transformation is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.

In this mapping all points in the left half of s-plane are mapped inside the unit circle in the z-plane and all points in the right half of s-plane are mapped outside the unit circle in the z-plane.

Let us consider an analog filter with the following system function

$$H(s) = \frac{Y(s)}{H(s)} = \frac{b}{s+a} \quad \text{--- (1)}$$

$$sY(s) + aY(s) = bX(s)$$

This can be characterised by differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \quad \text{--- (2)}$$

Integrate the limit from $nT - T$ to nT

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$$\int_{nT-T}^{nT} \frac{dy(t)}{dt} + a\int_{nT-T}^{nT} y(t) = b\int_{nT-T}^{nT} x(t) \quad \text{---} (3)$$

Trapezoidal rule for integration

$$\int_{nT-T}^{nT} y(t) = \frac{T}{2}\left[y(nT) + y(nT-T)\right]$$

sub this rule in (3)

$$y(nT) - y(nT-T) + a\left[T/_2\,(y(nT) + y(nT-T))\right]$$
$$= b\frac{T}{2}\left[x(nT) + x(nT-T)\right]$$

$$y(nT) - y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T)$$
$$= \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

$$\left[1+\frac{aT}{2}\right]y(nT) - \left[1-\frac{aT}{2}\right]y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$
$$\text{---} (4)$$

sub $y(nT) = y(n)$

$y(nT-T) = y(n-1)$

$$\left[1+\frac{aT}{2}\right]y(n) - \left[1-\frac{aT}{2}\right]y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$
$$\text{---} (5)$$

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Take z-transform on both sides.

$$\left[1+\frac{aT}{2}\right]Y(z) - \left[1-\frac{aT}{2}\right]z^{-1}Y(z) = \frac{bT}{2}\left[X(z)+z^{-1}X(z)\right] \quad \text{(6)}$$

$$\left[(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})\right]Y(z) = \frac{bT}{2}\left[1+z^{-1}\right]X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})}$$

divide by $\frac{T}{2}(1+z^{-1})$

$$\frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})} + a} \quad \text{——— (7)}$$

comparing (1) and (7)

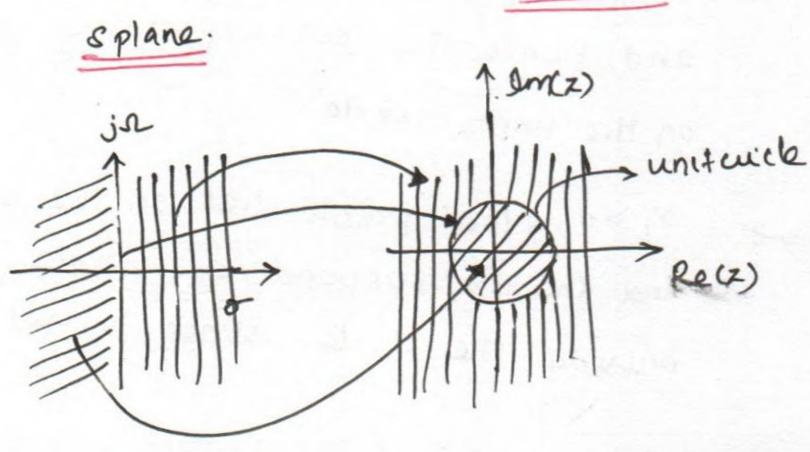
$$s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$$

Relation between Analog and Digital filter poles :-

$$s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$$

$$\frac{Ts}{2} = \frac{1-1/z}{1+1/z}$$

$$\frac{Ts}{2} = \frac{z-1}{z+1}$$



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$$\frac{Ts}{2}(z+1) = z-1$$

simplifying $\quad z = \dfrac{1+\frac{Ts}{2}}{1-Ts/2}.$

Let $s = \sigma_1 + j\Omega_1$

$$z = \frac{1 + T/2(\sigma_1 + j\Omega_1)}{1 - T/2(\sigma_1 + j\Omega_1)}$$

$$z = \frac{(1 + T/2\,\sigma_1) + j\Omega_1 . T/2}{(1 - T/2\,\sigma_1) + jT/2\,\Omega_1}.$$

$$|z| = \sqrt{\frac{(1 + T/2\,\sigma_1)^2 + (T/2\,\Omega_1)^2}{(1 - T/2\,\sigma_1)^2 + (T/2\,\Omega_1)^2}}$$

1. If $\sigma_1 < 0$, then the points $s = \sigma_1 + j\Omega_1$ lies on the left half of s plane. In this case $|z| < 1$, and hence the corresponding point in z-plane will lie inside the unit circle in z-plane

2. $\sigma_1 = 0$, (real part is zero), so the points lies on the imaginary axis in the s-plane. In this case $|z| = 1$ and hence the corresponding point in z-plane will lie on the unit circle.

3. $\sigma_1 > 0$, the points lies on the right half of the s-plane and the corresponding points in z-plane will lie outside the circle since $|z| > 1$.

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Relation b/w Analog and Digital frequency :-

Let $s = j\Omega$ be points on imaginary axis.

& $z = e^{j\omega}$ where ω is the digital frequency

Ω is the analog frequency.

$$s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$$

$$j\Omega = \frac{2}{T}\frac{(1-e^{-j\omega})}{1+e^{-j\omega}}$$

$$= \frac{2}{T}\left(\frac{e^{j\omega/2}e^{-j\omega/2} - e^{-j\omega}}{e^{j\omega/2}e^{-j\omega/2} + e^{-j\omega}}\right)$$

Taking $e^{-j\omega/2}$ outside

$$j\Omega = \frac{2}{T}\frac{e^{-j\omega/2}}{e^{-j\omega/2}}\left[\frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}\right]$$

$$j\Omega = \frac{2}{T} \times \frac{2j\sin(\omega/2)}{2\cos\omega/2}$$

$$\Omega = \frac{2}{T}\frac{\sin(\omega/2)}{\cos(\omega/2)}$$

$$\Omega = \frac{2}{T}\tan\omega/2$$

Digital freq.

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}.$$

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Warping Effect :-

For small value of ω

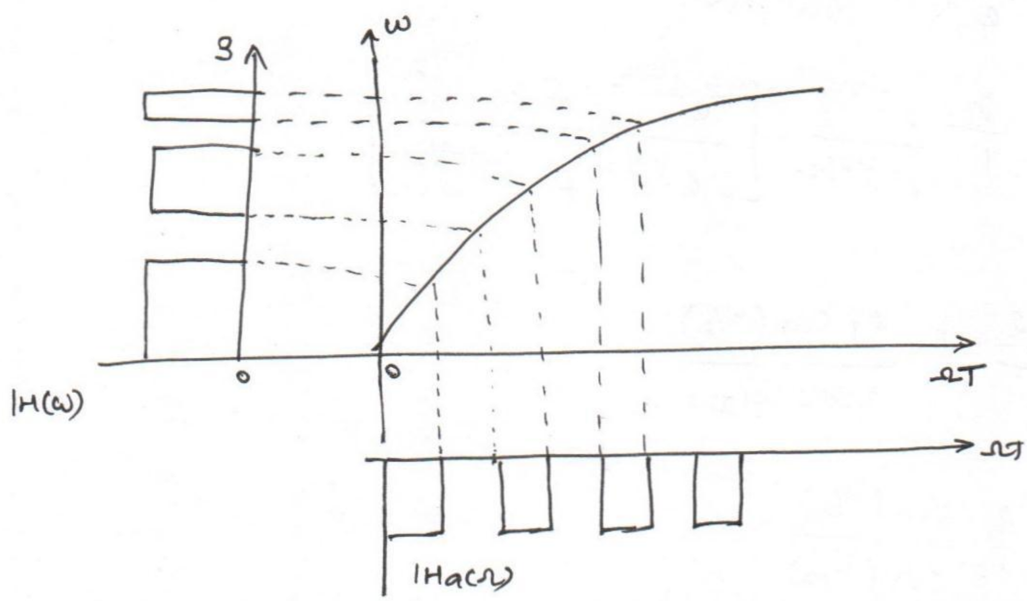
$$\Omega = \frac{2}{T}\,\omega/2$$

$$\Omega = \omega/T$$

$\omega = \Omega T$ for low frequencies the relationship b/w Ω and ω are linear, as a result the digital filter have the same amplitude response.

But for higher frequencies, they have non-linear relationship and it introduces a distortion in the frequency axis which is called frequency warping.

Effect of Warping on the Magnitude Response :-



Consider an analog filter with a number of passbands centered at regular intervals. The desired digital filter will have same number of passbands. But the center frequencies and bandwidth of higher frequency passband will tend to reduce disproportionately.

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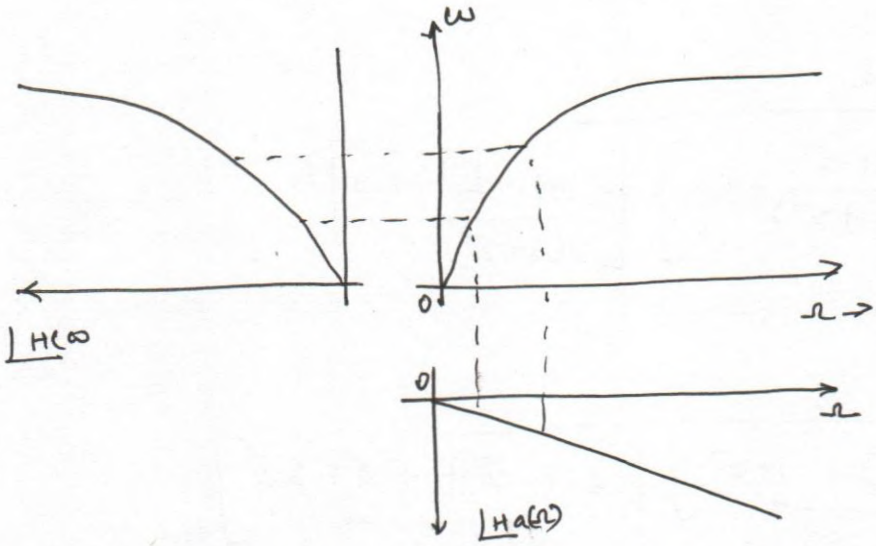
Prewarping :-

The effect of warping on amplitude responses can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

Effect of Warping on the Phase Response :-

Consider an analog filter with linear phase.



The phase response of the derived digital filter will be non-linear. So the bilinear transformation can be used only to design digital filter with prescribed magnitude response with piecewise constant values.

Disadvantage :-

A linear phase analog filter cannot be transformed to a linear phase digital filter using bilinear transformation.

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Problem No 1 :-

Apply bilinear transformation to

$H(s) = 2/(s+1)(s+2)$ with $T = 1$ sec & find $H(z)$.

Solution :-

sub $s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$

$$s = 2\frac{(1-z^{-1})}{(1+z^{-1})}$$

$$H(z) = \frac{2}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}+1\right]\left[\frac{2(1-z^{-1})}{(1+z^{-1})}+2\right]}$$

$$= \frac{2}{\left[\frac{2(1-z^{-1})+1+z^{-1}}{(1+z^{-1})}\right]\left[\frac{2(1-z^{-1})+2+2z^{-1}}{(1+z^{-1})}\right]}$$

$$= \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})(2-2z^{-1}+2+2z^{-1})}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}} = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

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Problem No 2 :

A digital filter with a 3 dB bandwidth of 0.25π is to be designed from the analog filter whose freq response is

$$H(s) = \frac{\Omega_c}{s+\Omega_c}.$$ use bilinear transformation.

Solution :

given data : $\omega_c = 0.25\pi$ (ie cut off frequency,

sometimes it may be given as resonant frequency)

$T = 1$ sec.

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2}$$

$$= 2\tan\frac{0.25\pi}{2}$$

$\Omega_c = 0.828$ rad/sec.

$$H(s) = \frac{0.828}{s+0.828}.$$

for bilinear sub $s = 2\frac{(1-z^{-1})}{(1+z^{-1})}$

$$H(z) = \frac{0.828}{2\frac{(1-z^{-1})}{(1+z^{-1})} + 0.828}$$

$$= \frac{0.828(1+z^{-1})}{2-2z^{-1}+0.828(1+z^{-1})} = \frac{0.828\ (1+z^{-1})}{2.828 - 2z^{-1} + 0.828z^{-1}}$$

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$$H(z) = \frac{0.828\,(1+z^{-1})}{2.828 - 1.172\,z^{-1}}$$ //.

Problem NO 3 :

convert the analog filter $H(s) = \frac{s+0.1}{(s+0.1)^2+\underbrace{9}_{\Omega_c^2}}$ into digital filter. $\omega_r = \pi/4$. is the resonant frequency.

Solution :

from the system function $\Omega_c = 3$

$$\omega_r = \omega_c = \pi/4$$

$$\Omega_c = 2/T \tan\frac{\omega_c}{2}$$

$$T = \frac{2}{3}\tan\frac{\pi/4}{2}$$

$$T = 0.276 \text{ sec}.$$

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

$$s = 2/T\,\frac{(1-z^{-1})}{(1+z^{-1})} = \frac{2}{0.276} \times \frac{(1-z^{-1})}{(1+z^{-1})}$$

$$s = 7.246\,\frac{(1-z^{-1})}{(1+z^{-1})}$$

$$H(z) = \frac{7.246\,\frac{(1-z^{-1})}{(1+z^{-1})} + 0.1}{\left[7.246\,\frac{(1-z^{-1})}{(1+z^{-1})} + 0.1\right]^2 + 9}$$

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$$H(z) = \frac{\dfrac{7.246(1-z^{-1}) + 0.1(1+z^{-1})}{1+z^{-1}}}{\left[7.246\dfrac{(1-z^{-1})}{(1+z^{-1})} + 0.1(1+z^{-1})\right]^2 + 9}$$

$$H(z) = \frac{\left[7.246(1-z^{-1}) + 0.1(1+z^{-1})\right](1+z^{-1})}{\left[7.246(1-z^{-1}) + 0.1(1+z^{-1})\right]^2 + 9(1+z^{-1})^2}$$

Sumplifying.

$$H(z) = \frac{1 + 0.027z^{-1} - 0.973z^{-2}}{8.572 - 11.84z^{-1} + 8.177z^{-2}}$$

Impulse Invariant Technique :-

The impulse response of the digital filter is obtained by uniformly sampling the impulse response of the analog filter.

Let $H_a(s)$ be the transfer function of the analog filter and $h_a(t)$ is the impulse response of the filter.

Let $H(z)$ be the transfer function of the digital filter and $h(n)$ be the impulse response,

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The $H_a(s)$ can be expressed in the form of summation of poles.

$$H_a(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s-p_3} \cdots\cdots \frac{c_N}{s-p_N}$$

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k} \quad \text{———} \quad (1)$$

$$h_a(t) = L^{-1}[H_a(s)$$

$$= L^{-1}\left[\sum_{k=1}^{N} \frac{c_k}{s-p_k}\right]$$

$$= \sum_{k=1}^{N} c_k \, L^{-1}\left[\frac{1}{s-p_k}\right]$$

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t}$$

$$h(n) = h_a(t)\Big|_{t=nT}$$

$$h(n) = \sum_{k=1}^{N} c_k e^{p_k nT}$$

$$= \sum_{k=1}^{N} c_k \left(e^{p_k T}\right)^n.$$ → take z transform of $h(n)$.

$$H(z) = \sum_{k=1}^{N} c_k \, z\left[e^{p_k T}\right]^n$$

$$a^n \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$$

$$H(z) = \sum_{k=1}^{N} c_k \frac{1}{1-e^{p_k T} z^{-1}} \quad \text{———} \quad (2)$$

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Comparing (i) & (2)

$$\frac{1}{s-P_k} \xrightarrow{\text{is transformed}} \frac{1}{1-e^{P_k T} z^{-1}}$$

Relation b/w Analog and Digital poles:-

Let $z = e^{P_i T}$

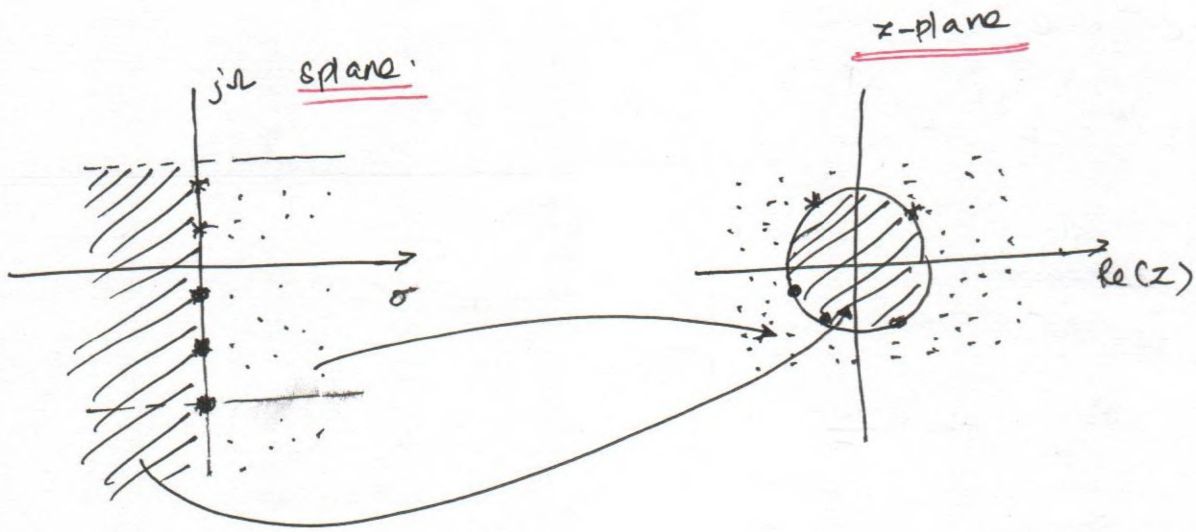
$z = e^{ST}$

$S = \sigma_1 + j\Omega_1$.

$z = e^{(\sigma_1 + j\Omega_1)T}$

$z = e^{\sigma_1 T} e^{j\Omega_1 T}$

$|z| = e^{\sigma_1 T} \qquad \angle z = \Omega_1 T$



(i) if $\sigma_1 < 0$, then the analog poles lies on left half of s plane. then $|z| < 1$, and hence the corresponding digital pole z will lie inside the unit circle in the z-plane.

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(ii) If $\sigma_1 = 0$, then the analog pole s lies on imaginary axis of s plane. then $|z| = 1$, and hence the corresponding digital pole will lie on the unit circle.

(iii) If $\sigma_1 > 0$, then the analog pole lies on right half of plane, then $|z| > 1$, so the digital poles lie outside the unit circle.

Relation b/w Analog & Digital Frequency :-

Let Ω be the analog freq in rad/sec

ω is digital freq.

$$z = re^{j\omega}$$

$$s = \sigma + j\Omega$$

$$z = e^{sT}$$

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

$$r = e^{\sigma T}$$

$$\omega = \Omega T$$

or $$\Omega = \frac{\omega}{T}$$

Disadvantage :-

The mapping of analog to digital is not one-to-one mapping rather it is many to one mapping. i.e many points in s plane are mapped into single point in the z-plane.

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For eg.

Let $s_1 = \sigma + j\Omega$

$$s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right)$$

The imaginary part is different by $2\pi/T$.

$$z_1 = e^{s_1 T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$$

$$z_2 = e^{s_2 T} = e^{\sigma T} * e^{j\Omega T} * e^{j2\pi} \qquad \text{Since } e^{j2\pi} = 1$$

$$= e^{\sigma T} e^{j\Omega T}$$

$$= z_1$$

Thus the two poles differing by $2\pi/T$ mapped into single pole

The analog poles will not be aliased by the impulse invariant mapping if they are confined to the s-plane's "primary strip" i.e within π/T of the real axis

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Impulse Invariant Method :-

1. $$\frac{1}{s-p_i} \longrightarrow \frac{1}{1-e^{p_i T} z^{-1}}$$

2. $$\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dp_i^{m-1}} \left(\frac{1}{1-e^{-p_i T} z^{-1}} \right)$$

3. For complex poles

$$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

Problem No 1 :

Convert the analog filter into a digital filter whose system function is

$$H(s) = \frac{s+0.2}{(s+0.2)^2+9}$$ use impulse invariant technique.

Assume $T = 1$ sec.

Solution :

The system response of the filter is of the standard form.

$$H(s) = \frac{s+a}{(s+a)^2+b^2}$$

so $a = 0.2$ $\quad b = 3$

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$$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$H(z) = \frac{1-e^{-0.2T}\cos 3T \; z^{-1}}{1-2e^{-0.2T}(\cos 3T)z^{-1}+e^{-0.4T}z^{-2}}$$

sub $T = 1$ sec

$$H(z) = \frac{1-(0.818)(-0.989)z^{-1}}{1-2(0.8187)(-0.989)z^{-1}+0.6703z^{-2}}$$

$$H(z) = \frac{1+0.809z^{-1}}{1+1.6193z^{-1}+0.6703z^{-2}}$$

Problem No 2:

For the analog transfer function

$H(s) = \frac{1}{(s+1)(s+2)}$. Determine $H(z)$ using impulse invariant technique.

$$H(s) = \frac{1}{(s+1)(s+2)}.$$

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

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where $A = 1$ and $B = -1$.

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}}$$

$T = 1$ sec

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} - \frac{1}{1-e^{-2}z^{-1}}$$

$$H(z) = \frac{1}{1-0.3678\,z^{-1}} - \frac{1}{1-0.135\,z^{-1}}$$

$$H(z) = \frac{1-0.135z^{-1} - 1 + 0.3678z^{-1}}{(1-0.3678z^{-1})(1-0.135z^{-1})}.$$

$$H(z) = \frac{0.2328}{1-0.5032z^{-1} + 0.0496z^{-2}}$$

Problem No 3:

$H(s) = \dfrac{1}{(s+1)(s+2)}$ find $H(z)$ using impulse invariance method for sampling frequency of 5 samples/sec.

Solution:

$$H(s) = \frac{1}{(s+1)(s+2)}$$

After partial fraction

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

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Sampling frequency is given as $F_s = 5$

$$T = 1/F_s = 1/5 = 0.2 \text{ sec.}$$

Using impulse invariance

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{2T}z^{-1}}$$

$$= \frac{1}{1-e^{-0.2}z^{-1}} - \frac{1}{1-e^{-0.4}z^{-1}}$$

$$H(z) = \frac{1}{1-0.8187z^{-1}} - \frac{1}{1-0.6703z^{-1}}$$

$$H(z) = \frac{0.1484z^{-1}}{(1-1.489z^{-1}+0.5488z^{-2})}$$

Problem No 4:

Apply impulse invariant method and find $H(z)$ for

$$H(s) = \frac{s+a}{(s+a)^2+b^2}$$

Solution:

The inverse Laplace transform.

$$h(t) = \begin{cases} e^{-at}\cos bt & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$h(n) = h(t)\Big|_{t=nT}$$

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$$h(n) = \begin{cases} e^{-anT}\cos(bnT) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The digital transfer function

$$H(z) = \sum_{n=0}^{\infty} e^{-anT}\cos(bnT)\, z^{-n}$$

Infinite series formula.
$[a^n] = \frac{1}{1-a}$

$$= \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{jbnT} + e^{-jbnT}}{2}\right) z^{-n}$$

$$H(z) = \frac{1}{2}\sum_{n=0}^{\infty} \left(e^{-(a-jb)T} z^{-1}\right)^n + \left(e^{-(a+jb)T} z^{-1}\right)^n \Big]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-(a+jb)T}z^{-1} + 1 - e^{-(a-jb)T}z^{-1}}{\left(1-e^{-(a-jb)T}z^{-1}\right)\left(1-e^{-(a+jb)T}z^{-1}\right)}\right]$$

After multiplication & simplifying,

$$H(z) = \frac{\frac{1}{2}\left[2 - e^{-aT}\left(e^{-jbT} + e^{jbT}\right) z^{-1}\right]}{1 - 2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$H(z) = \frac{1 - e^{-aT}(\cos bT) z^{-1}}{1 - 2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

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Realization of Digital filters:-

A digital filter transfer function can be realized in a variety of ways.

There are two types of realization

(i) Recursive

(ii) Non-recursive.

1. For recursive realization the current o/p y(n) is a function of past o/p, past and present i/p's. This form corresponds to an Infinite Impulse Response (IIR) digital filter.

2. For Non-recursive realization the current o/p sample y(n) is a function of only past and present i/p's. This form corresponds to a Finite Impulse Response digital filter.

IIR filter can be realized in many forms.

1. Direct form I realization
2. Direct form II realization
3. Transposed direct form realization
4. Cascade form realization
5. Parallel form realization
6. Lattice - Ladder structure.

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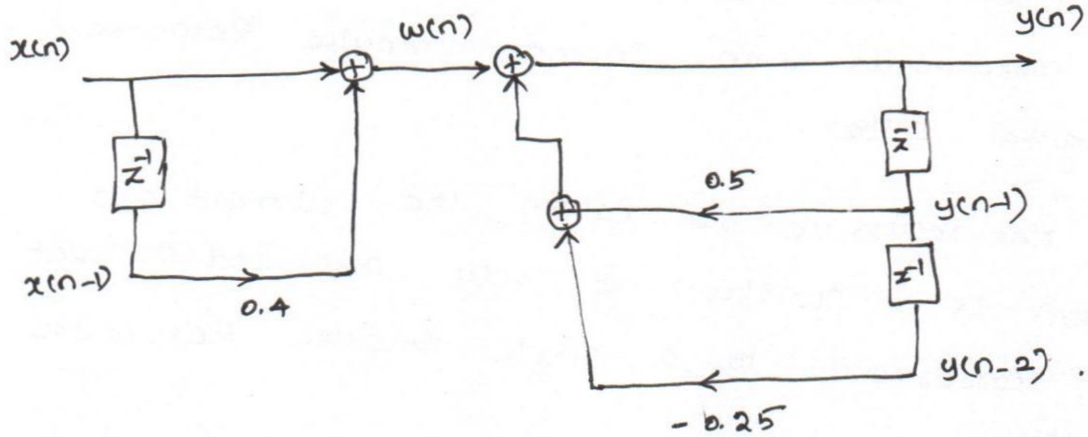
Problem No 1:

obtain the direct form I realization for the system described by difference equation

$$y(n) = 0.5\, y(n-1) - 0.25\, y(n-2) + x(n) + 0.4\, x(n-1)$$

Let $x(n) + 0.4\,x(n-1) = w(n)$

so $y(n) = 0.5\,y(n-1) - 0.25\,y(n-2) + w(n)$



Problem No: 2

Determine the direct form II realization for the following system $y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7\,x(n) - 0.252\,x(n-2)$.

The system function is given by taking z-transform on both sides.

$$Y(z) = -0.1\,z^{-1}Y(z) + 0.72\,z^{-2}Y(z) + 0.7X(z) - 0.252\,z^{-2}X(z)$$

$$Y(z) + 0.1\,z^{-1}Y(z) - 0.72\,z^{-2}Y(z) = \left[0.7 - 0.252\,z^{-2}\right]X(z).$$

$$Y(z)\left[1 + 0.1\,z^{-1} - 0.72\,z^{-2}\right] = \left[0.7 - 0.252\,z^{-2}\right]X(z)$$

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$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\frac{Y(z)}{X(z)} \cdot \frac{W(z)}{W(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-2}$$

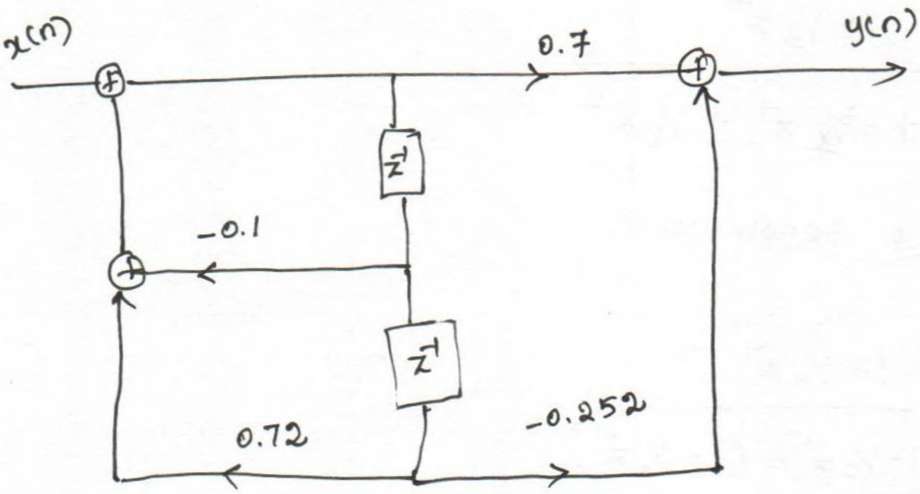
$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$Y(z) = (0.7 - 0.252z^{-2})W(z)$$

$$W(z)(1 + 0.1z^{-1} - 0.72z^{-2}) = X(z)$$

$$Y(z) = 0.7W(z) - 0.252z^{-2}W(z)$$

$$W(z) = X(z) - 0.1z^{-1}W(z) - 0.72z^{-2}W(z).$$



(i) Direct form II realization

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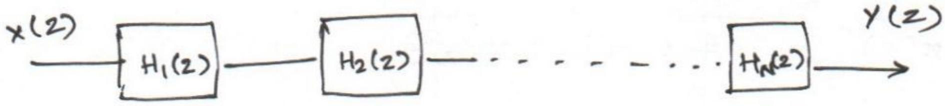
Cascade form:

Let us consider an IIR system with function

$$H(z) = H_1(z)\, H_2(z)\, H_3(z) \dots H_N(z)$$



* Realize each $H_i(z)$ in direct form II
* Connect the structure in cascade

Problem No 3:

Realize the system with difference equation

$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{3} x(n-1)$ in cascade form

Take z transform on both sides

$$Y(z) = \frac{3}{4} z^{-1} Y(z) - \frac{1}{8} z^{-2} Y(z) + X(z) + \frac{1}{3} z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

Factorize the denominator

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3} z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)}$$

This can be written in form of

$$H(z) = H_1(z)\, H_2(z).$$

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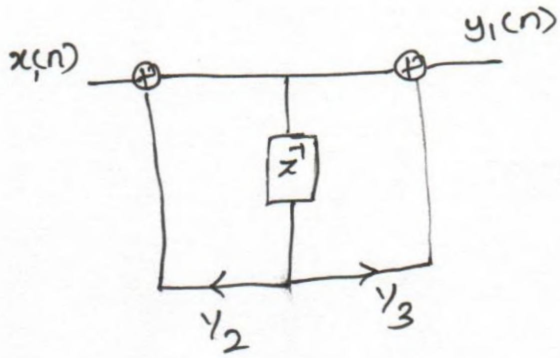
where

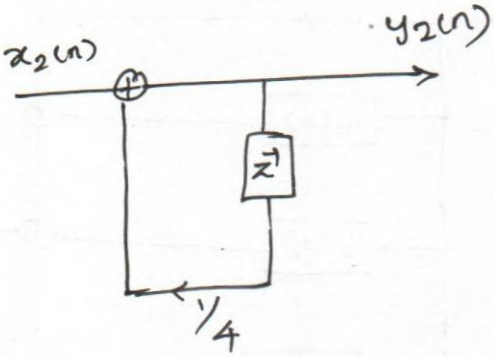
$$H_1(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$H_2(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$H_1(z)$ and $H_2(z)$ can be realized in direct form II.

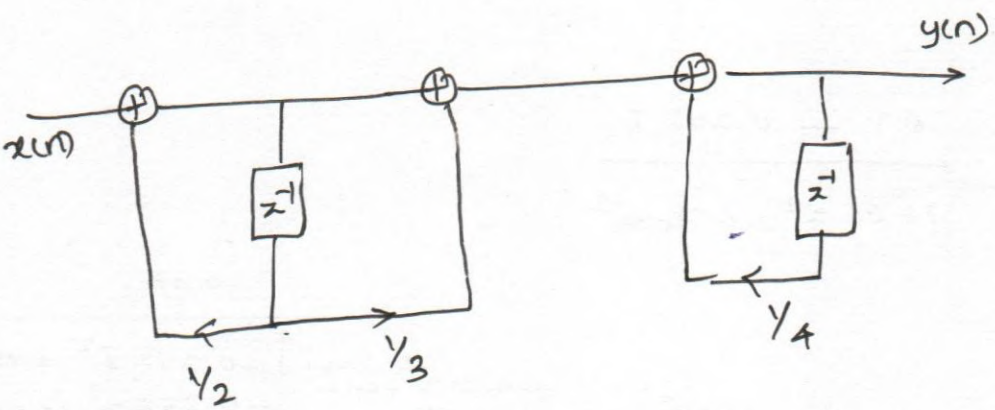


$H_1(z)$



$H_2(z)$

cascading $H_1(z)$ and $H_2(z)$



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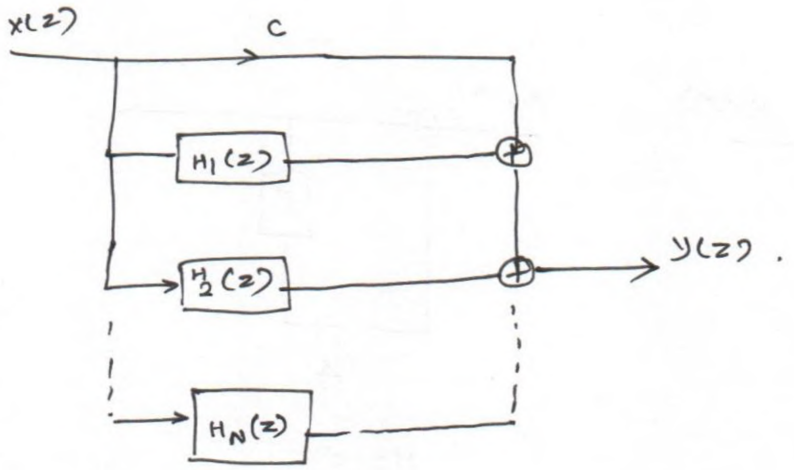
Parallel form :-

A parallel form realization of an IIR system can be obtained by performing a partial expansion of

$$H(z) = C + \frac{C_1}{1-P_1 z^{-1}} + \frac{C_2}{1-P_2 z^{-1}} \cdots \cdots \frac{C_N}{1-P_N z^{-1}}$$

$$H(z) = C + H_1(z) + H_2(z) \cdots \cdots H_N(z)$$



Problem No 4 :

Realize the system given by difference equation

$$y(n) = -0.1\, y(n-1) + 0.72\, y(n-2) + 0.7\, x(n) - 0.252\, x(n-2).$$

$$H(z) = \frac{0.7 - 0.252 z^{-2}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$H(z) = 0.35 + \frac{0.35 - 0.035 z^{-1}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$\begin{array}{r|l} & 0.35 \\ \hline -0.72 z^{-2} + 0.1 z^{-1} + 1 & -0.252 z^{-2} + 0 + 0.7 \\ & -0.252 z^{-2} + 0.035 z^{-1} + 0.35 \\ \hline & -0.035 z^{-1} + 0.35 \end{array}$$

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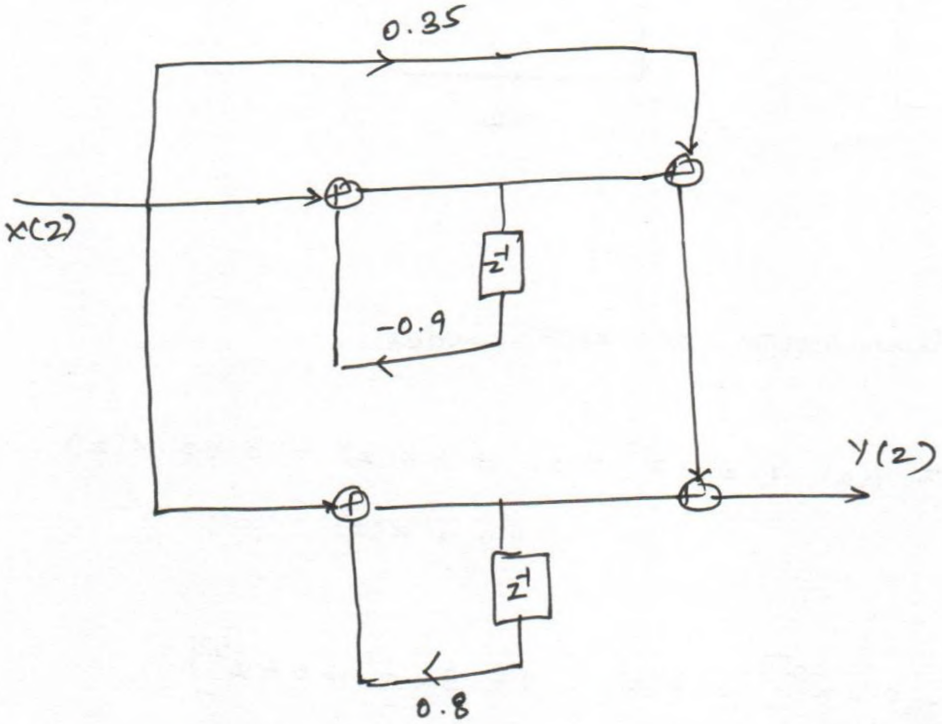
$$H(z) = 0.35 + \frac{0.206}{1+0.9z^{-1}} + \frac{0.144}{1-0.8z^{-1}}$$

$$H(z) = c + H_1(z) + H_2(z)$$



Problem No 5:

Obtain the directform I, directform II, cascade and parallel form.

$$y(n) = -0.1\,y(n-1) + 0.2y(n-2) + 3x(n) + 3.6\,x(n-1) + 0.6x(n-2)$$

Direct form I:

$$y(n) = -0.1\,y(n-1) + 0.2y(n-2) + 3x(n) + 3.6\,x(n-1) + 0.6x(n-2)$$

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + w(n)$$

where $w(n) = 3x(n) + 3.6x(n-1) + 0.6x(n-2)$

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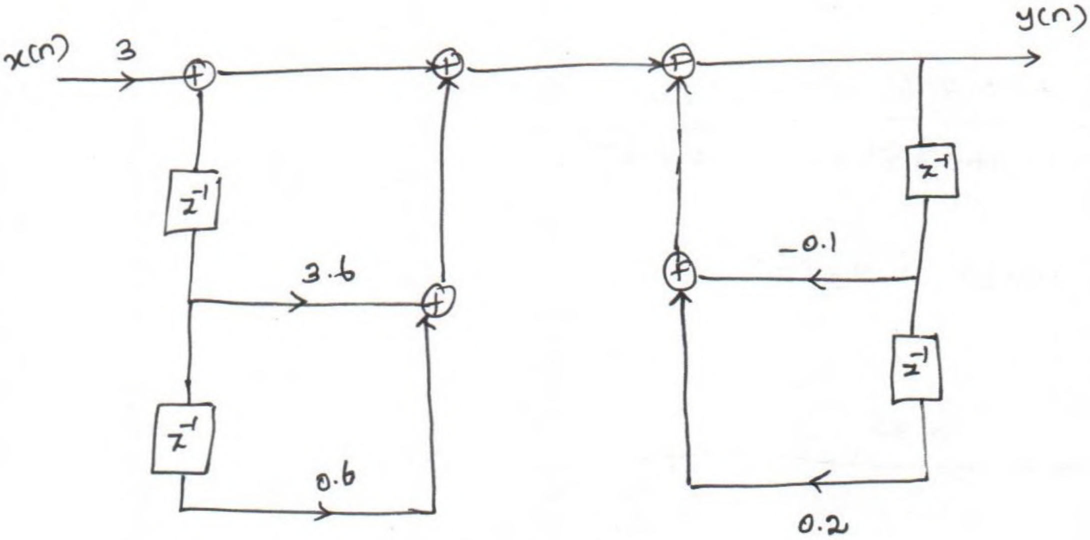


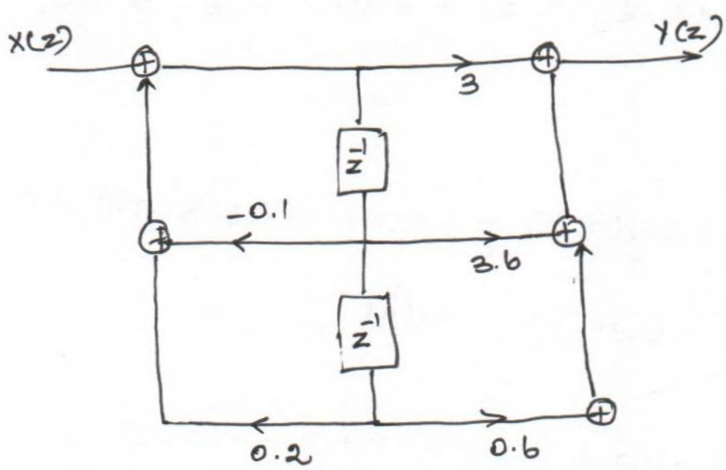
Directform II :

Taking z transform on both sides.

$$Y(z) = -0.1\, z^{-1}Y(z) + 0.2z^{-2}\, Y(z) + 3X(z) + 3.6z^{-1}X(z) + 0.6\, z^{-2}X(z)$$

$$Y(z)\left[1 + 0.1z^{-1} - 0.2z^{-2}\right] = X(z)\left[3 + 3.6z^{-1} + 0.6z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$



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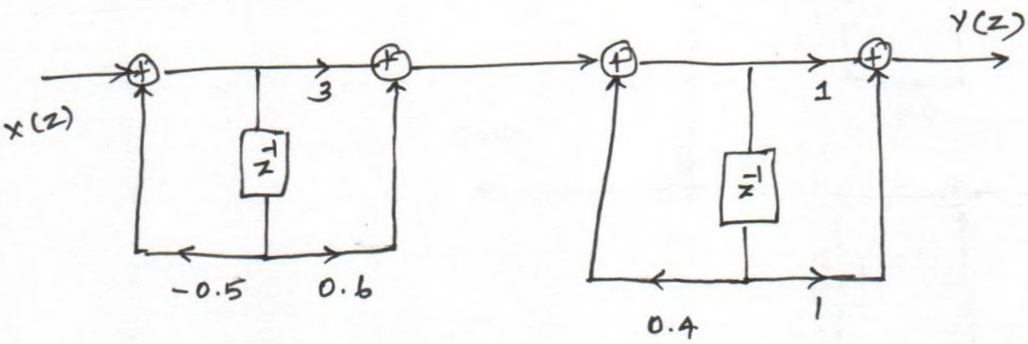
Cascade Form :

$$\frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} + 0.2z^{-2}}$$

$$= \frac{(3+0.6z^{-1})(1+z^{-1})}{(1+0.5z^{-1})(1-0.4z^{-1})}$$

$$H(z) = H_1(z)\,H_2(z)$$

$$H_1(z) = \frac{3+0.6z^{-1}}{1+0.5z^{-1}} \qquad H_2(z) = \frac{1+z^{-1}}{1-0.4z^{-1}}$$



Parallel form :-

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$= -3 + \frac{3.9z^{-1} + 6}{1+0.1z^{-1} - 0.2z^{-2}}$$

$$\begin{array}{r|l} & -3 \\ \hline -0.2z^{-2}+0.1z^{-1}+1 & 0.6z^{-2} + 3.6z^{-1} + 3 \\ & 0.6z^{-2} - 0.3z^{-1} - 3 \\ \hline & 3.9z^{-1} + 6. \end{array}$$

Consider $\dfrac{3.9z^{-1}+6}{1+0.1z^{-1}-0.2z^{-2}} = \dfrac{3.9z^{-1}+6}{(1-0.4z^{-1})(1+0.5z^{-1})}$

$$= \frac{A}{1-0.4z^{-1}} + \frac{B}{1+0.5z^{-1}}$$

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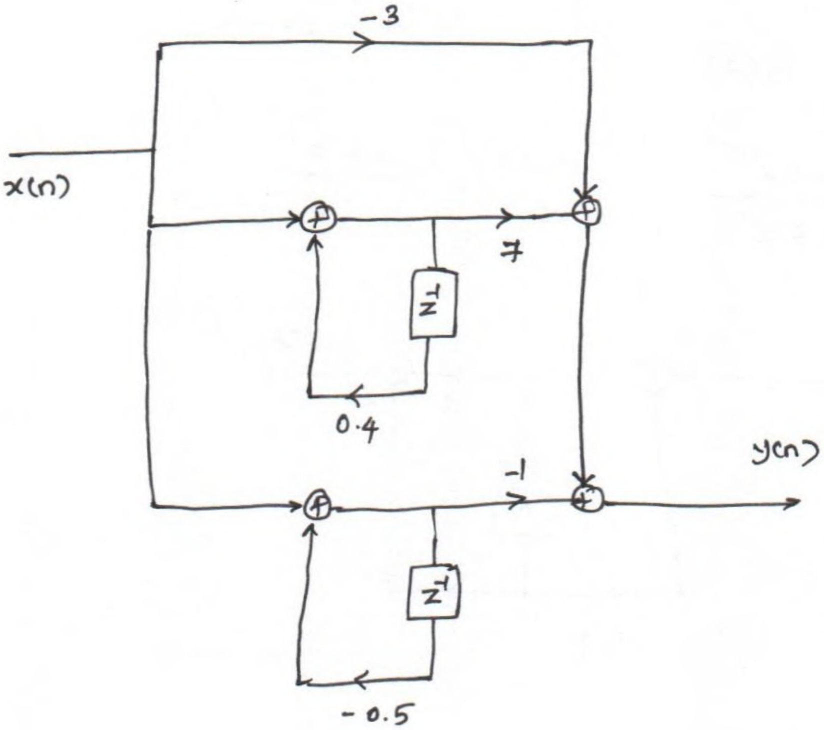
A = 7 , B = -1

$$H(z) = -3 + \frac{7}{1-0.4z^{-1}} - \frac{1}{1+0.5z^{-1}}$$

$$H(z) = c + H_1(z) + H_2(z)$$



Problem NO 6 :

Obtain the cascade realization for the following systems.

$$H(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2})}{(1+\frac{1}{4}z^{-1})(1+z^{-1}+\frac{1}{2}z^{-2})(1-\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2})}$$

$$H_1(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}} \qquad H_2(z) = \frac{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}{1+z^{-1}+\frac{1}{2}z^{-2}} \qquad H_3(z) = \frac{1}{1-\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2}}$$

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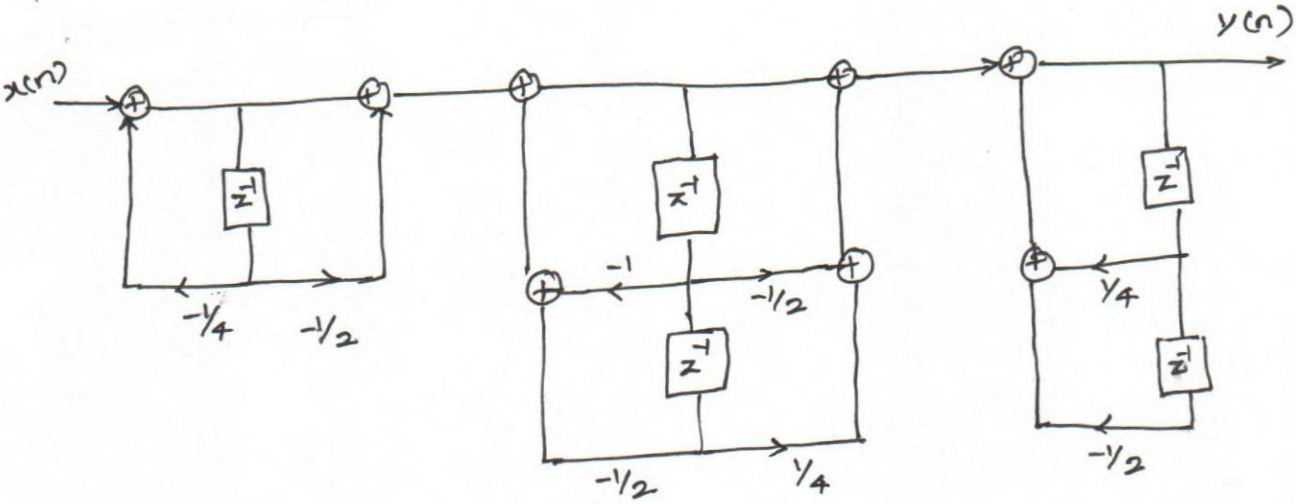

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Butterworth Filters:

The butterworth low-pass filter has a magnitude response given by

$$|H(j\Omega)| = \frac{A}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{0.5}}$$

where A is the filter gain.

Ω_c is the dB cut-off frequency

N is the order of the filter.

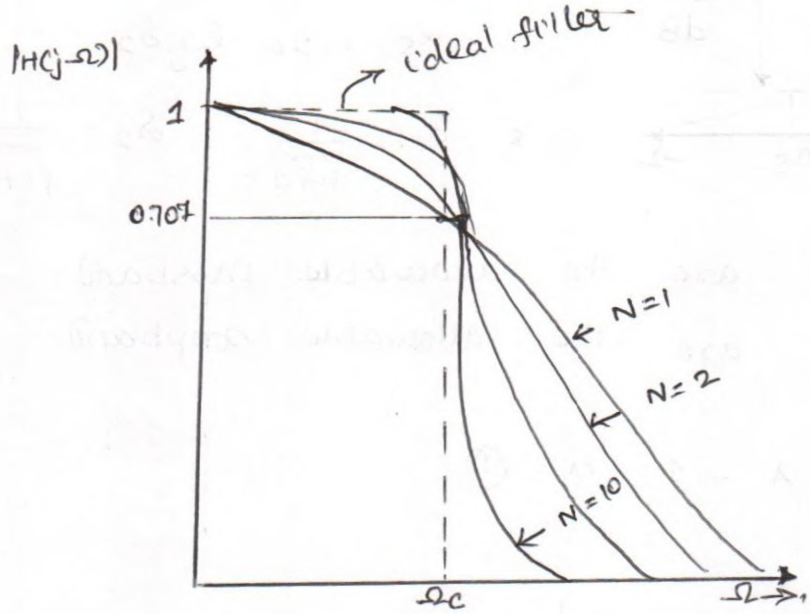

fig LPF Butterworth Magnitude Response:

* The magnitude response has a maximally flat passband and stopband.

* The magnitude response approaches the ideal response as the order of the filter increases.

* The phase response of the butterworth filter becomes more non-linear with increasing 'N'.

* At $\Omega = \Omega_c$, the curve passes through 0.707 A which corresponds to -3 dB

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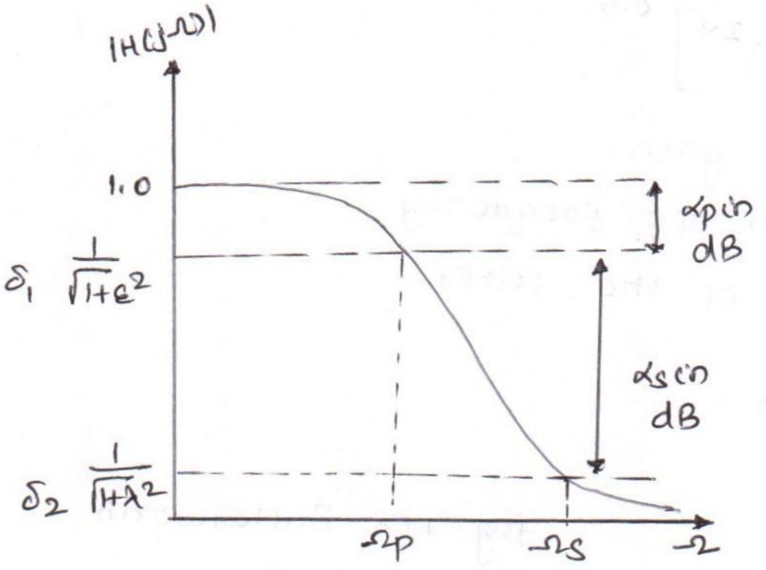
The design parameters of the butterworth filter are obtained by considering the low-pass filter with the desired specifications



$$\delta_1 \le |H(j\Omega)| \le 1 \qquad 0 \le \Omega \le \Omega_p$$

$$|H(j\Omega)| \le \delta_2 \qquad \Omega_s \le \Omega \le \pi$$

— ②

From the fig:

$$\alpha_p = -20 \log \delta_1$$

$$\alpha_s = -20 \log \delta_2$$

& $\delta_1 = \dfrac{1}{\sqrt{1+\epsilon^2}}$ $\quad \delta_2 = \dfrac{1}{\sqrt{1+\lambda^2}}$

where ϵ and δ_1 are the allowable passband

δ_2 and λ are the allowable stopband.

Sub ① and $A = 1$ in ②

$$\delta_1^2 \le \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} \le 1$$

$$\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \le \delta_2^2 \quad \text{——— ③}$$

This can be written as

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \le \frac{1}{\delta_1^2} - 1$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \le \frac{1}{\delta_2^2} - 1 \quad \text{——— ④}$$

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Dividing the above two expression

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{\left(1/\delta_2^2 - 1\right)}{1/\delta_1^2 - 1}$$

So the order of the filter 'N' is

$$N \geq \frac{\log\left(\sqrt{1/\delta_2^2 - 1} \Big/ \sqrt{1/\delta_1^2 - 1}\right)}{\log \frac{\Omega_s}{\Omega_p}}$$

$$\boxed{N > \frac{\log\left(\lambda/\varepsilon\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

The cut off frequency. 'Ω_c'

$$\Omega_c = \frac{\Omega_p}{\left(\sqrt{1/\delta_1^2 - 1}\right)^{1/N}} = \boxed{\frac{\Omega_p}{(\varepsilon)^{1/N}}}$$

The transfer function of the Butterworth filter is.

$$H(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

(or)

$$H(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}.$$

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where $b_K = 2 \sin\left(\frac{(2K-1)\pi}{2N}\right)$

Poles of a Normalised Butterworth Filter :-

The magnitude response of the butterworth filter

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}.$$

For a Normalised Filter $\Omega_c = 1$

So $$|H(j\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$$

The normalised poles in the s-domain can be obtained by substituting $\Omega = s/j$ and equating the denominator polynomial to zero.

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}}$$

$$1 + \left(\frac{s}{j}\right)^{2N} = 0$$

$$1 + \frac{s^{2N}}{j^{2N}} = 0$$

$$1 + (-s)^{2N} = 0$$

$$(-1)^N s^{2N} = -1$$

For N odd ; $s^{2N} = 1 \Rightarrow s^{2N} = e^{j2\pi k}$

$$s_k = e^{j\pi k/N} \qquad k = 1, 2 \ldots 2N$$

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For <u>N even</u>; $s^{2N} = -1 \Rightarrow s^{2N} = e^{j(2K-1)\pi}$

$$s_K = e^{j(2K-1)\pi/2N} \quad \text{for } K = 1,2 \ldots 2N.$$

To ensure stability and considering only the poles that lie on the left half of the plane.

So $\quad s_K = e^{j\phi_K}$

$$\text{where} \quad \phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N} \qquad K = 1,2,\ldots N.$$

In general, the unnormalized poles are given by

$$s_K' = \Omega_c s_K$$

$$\boxed{s_K' = \Omega_c e^{j\phi_K}}$$

Chebyshev Filters;

There are two types of chebyshev filter.

Type I

* All pole filters
* Exhibit equiripple behaviour in the passband and a monotonic characteristics in stopband

Type II

* It contains both poles and zeros
* Exhibit monotonic behaviour in the passband and an equiripple behaviour in stopband.

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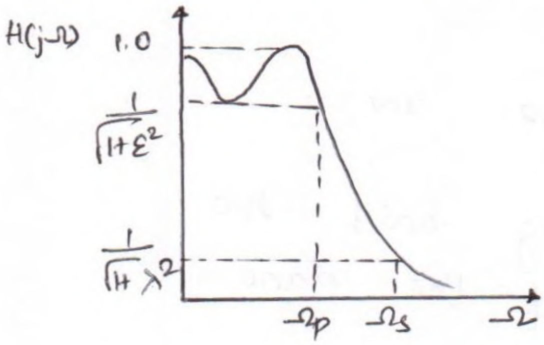
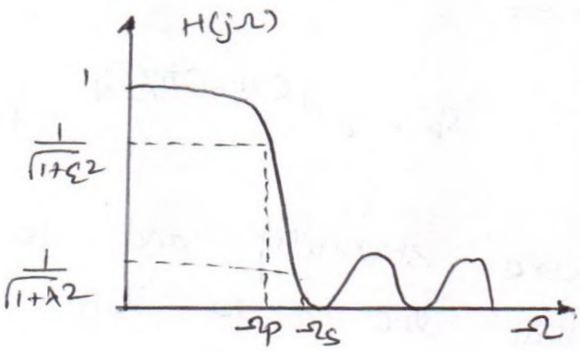
Type I



Type II



The magnitude square response of the filter (type I) is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega}\right)} \qquad N = 1, 2 \ldots$$

where ε is a constant and $C_N(x)$ is the N^{th} order chebyshev polynomial.

$$C_N(x) = \begin{cases} \cos(N\cos^{-1}x) & , |x| \le 1 \quad \text{(passband)} \\ \cosh(N\cosh^{-1}x) & , |x| > 1 \quad \text{(stopband)} \end{cases}$$

Properties of chebyshev polynomial:

(i) $C_N(1) = 1$ for all N.

$C_N(x) = C_N(-x)$ for N even

$C_N(x) = -C_N(-x)$ for N odd

$C_N(0) = 0$ for N odd

$C_N(-1) = 1$ for N even

$C_N(-1) = -1$ for N odd.

(ii) $C_N(x) \le 1$ for all $|x| \le 1$

(iii) The roots of the polynomials $C_N(x)$ occur in the interval $-1 \le x \le 1$

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$$C_N\left(\frac{\Omega_s}{\Omega_p}\right) \geq \frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2}-1\right]^{0.5}$$

so $\cosh\left[N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right] \geq \frac{\lambda}{\varepsilon}$

$$N \geq \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Pole Locations for Chebyshev filter.

The magnitude square response of the chebyshev filter is

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

The normalised poles in the s-domain can be obtained by substituting $\Omega = s/j$ and $\Omega_c = 1$ equating the denominator polynomial to zero.

$$1+\varepsilon^2 C_N^2(-js) = 0.$$

By solving & simplifying.

$$s_K = a\cos\phi_K + jb\sin\phi_K$$

$$s_K = \sigma_K + j\Omega_K \qquad K = 1, 2 \ldots N$$

The poles of a chebyshev filter can be determined by

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(ii) $C_N(x)$ is monotonically increasing for $|x| > 1$

The design parameters of the chebyshev filter.

$$\delta_1 \le H(j\omega) \le 1 \qquad 0 \le \omega \le \omega_p$$

$$H(j\omega) \le \delta_2 \qquad \omega_s \le \omega \le \pi$$

Using the expression.

$$\delta_1^2 \le \frac{1}{1 + \varepsilon^2 C_N^2\left(\frac{\Omega_p}{\Omega_c}\right)} \le 1 \qquad \text{——(1)}$$

$$\frac{1}{1 + \varepsilon^2 C_N^2\left(\frac{\Omega_s}{\Omega_c}\right)} \le \delta_2^2 \qquad \text{——(2)}$$

Assuming $\Omega_p = \Omega_c$

$$C_N\left(\frac{\Omega_c}{\Omega_c}\right) = C_N(1) = 1$$

So $\delta_1^2 \le \frac{1}{1+\varepsilon^2} \le 1$

$$\varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1}$$

using $\Omega_c = \Omega_p$ in (2).

$$\frac{1}{1 + \varepsilon^2 C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)} \le \delta_2^2$$

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$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$$

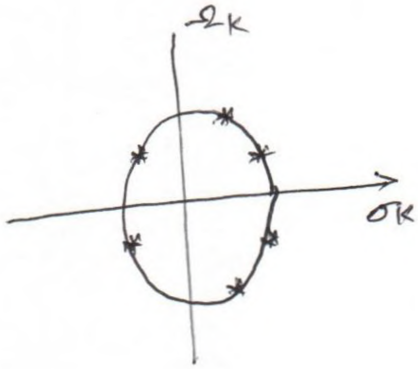
$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

$$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N}$$

where $\mu = \frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}$

The poles of the chebyshev transfer function are located on an ellipse.

$$\frac{\sigma_K^2}{a^2} + \frac{\Omega_K^2}{b^2} = 1$$ where a and b are minor and major axes of the ellipse.



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Design procedure for digital Butterworth Lowpass Filter:

step 1:

From the given specifications, find out the analog frequency.

For impulse invariant method

$$\Omega_s = \frac{\omega_s}{T} \qquad \Omega_p = \frac{\omega_p}{T}$$

For Bilinear transformation

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} \qquad \Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2}$$

Note: If 'T' value is not given, assume T = 1 sec.

Step 2:

Find the value of λ and ε

If δ_1 and δ_2 (values) are given

$$\varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1} \qquad \lambda = \sqrt{\frac{1}{\delta_2^2} - 1}$$

If α_p and α_s (in dB) are given

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} \quad \text{and} \quad \lambda = \sqrt{10^{0.1\alpha_s} - 1}$$

step 3: Find the order of the filter 'N'

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

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Round it off to the next higher integer value.

Step 4: Find the cut off frequency 'Ω_c'

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} \text{ rad/sec.}$$

Step 5: Find the transfer function of the analog filter

(i) when N is odd

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

(ii) when N is even

$$H(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

where $b_k = 2 \sin\left[\left(\frac{2k-1}{2N}\right)\pi\right]$

Step 6: Convert H(s) into H(z) using impulse invariant or bilinear transformation.

Impulse Invariant:

1. $$\frac{1}{s-p_i} \Rightarrow \frac{1}{1-e^{p_i T} z^{-1}}$$

2. $$\frac{s+a}{(s+a)^2+b^2} \Rightarrow \frac{1 - e^{-aT}(\cos bT) z^{-1}}{1 - 2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

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3. $$\frac{b}{(S+a)^2+b^2} \Rightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

Bilinear Transformation:

$$s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$$

Problem No 1:

1. Design a digital butterworth filter that satisfies the following constraints

$$\sqrt{0.5} \le H(e^{j\omega}) \le 1 \qquad 0 \le \omega \le \pi/2$$

$$|H(e^{j\omega})| \le 0.2 \qquad 3\pi/4 \le \omega \le \pi$$. using

impulse invariant transformation.

Given data:

$\delta_1 = \sqrt{0.5}$ $\quad \delta_2 = 0.2$ $\quad T = 1$ sec

$\omega_p = \pi/2$ $\quad \omega_s = 3\pi/4$

Step 1:

convert digital to analog frequency.

$$\Omega_p = \frac{\omega_p}{T} = \frac{\pi}{2}$$

$$\Omega_s = \frac{\omega_s}{T} = \frac{3\pi}{4}$$

Step 2:

Find the value of λ and ε

$$\varepsilon = \sqrt{\frac{1}{(\sqrt{0.5})^2}-1} = \boxed{1}$$

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$$\lambda = \sqrt{\frac{1}{(0.2)^2} - 1}$$

$$\boxed{\lambda = 4.89}$$

Step 3:

Find the order of the filter

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\geq \frac{\log\left(\frac{4.89}{1}\right)}{\log\left(\frac{3\pi/4}{\pi/2}\right)} \geq \frac{0.689}{0.176}$$

$$N \geq 3.9$$

$$\boxed{N = 4}$$

Step 4:

Find the cut off frequency Ω_c

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{\pi}{2(1)^{1/4}} = 1.57 \text{ rad/sec.}$$

Step 5:

Find the analog transfer function

N is even.

$$H(S) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \qquad N = 4.$$

$$H(S) = \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2} \quad \frac{\Omega_c^2}{s^2 + b_2 \Omega_c s + \Omega_c^2}$$

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$$b_k = 2\sin\left[\left(\frac{2K-1}{2N}\right)\pi\right]$$

$$b_1 = 2\sin\left[\left(\frac{2-1}{8}\right)\pi\right] = \boxed{0.765}$$

$$b_2 = 2\sin\left[\frac{(4-1)\pi}{8}\right] = \boxed{1.85}$$

$$H(s) = \frac{2.46}{s^2 + 0.765 \times 1.57\, s + 2.46} * \frac{2.46}{s^2 + 1.85 \times 1.57\, s + 2.46}$$

$$H(s) = \frac{6.075}{(s^2 + 1.2s + 2.46)(s^2 + 2.9s + 2.46)}$$

(Do the partial fraction)

$$H(s) = \frac{As + B}{s^2 + 1.2s + 2.46} + \frac{Cs + D}{s^2 + 2.9s + 2.46}$$

$$(As+B)(s^2 + 2.9s + 2.46) + (Cs+D)(s^2 + 1.2s + 2.46) = 6.075$$

$$As^3 + 2.9As^2 + 2.46As + Bs^2 + 2.9Bs + 2.46B$$
$$+ Cs^3 + 1.2Cs^2 + 2.46Cs + Ds^2 + 1.2Ds + 2.46D = 6.075$$

Equating s^3 : $A + C = 0$ —— ①

Equating s^2 : $2.9A + B + 1.2C + D = 0$ —— ②

Equating s : $2.46A + 2.9B + 2.46C + 1.2D = 0$ —— ③

Equating constant: $2.46B + 2.46D = 6.075$ —— ④

$$2.46(B + D) = 6.075$$

$$B + D = 2.46 \quad \text{—— ⑤}$$

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sub this in ②

$2.9A + 1.2C + B + D = 0$

$2.9A - 1.2A + 2.46 = 0$

$1.7A = -2.46$

$\boxed{A = -1.4509}$

$\boxed{C = 1.409}$

from ③

$2.9B + 1.2D = 0$

$B + D = 2.46$

solving above equation.

$\boxed{B = -1.7443}$ $\boxed{D = 4.2113}$

Substitute those values in H(S)

$$H(S) = \frac{-1.4509S - 1.7443}{S^2 + 1.202S + 2.46} + \frac{1.4509S + 4.2113}{S^2 + 2.9S + 2.46}$$

$$H(S) = H_1(S) + H_2(S)$$

convert $H_1(S)$ into $H_1(z)$

$$H_1(S) = \frac{-1.4509S - 1.7443}{S^2 + 1.202S + 2.46}$$

$$H_1(S) \; \frac{-1.4509\left(S + 1.202\right)}{(S + 0.601)^2 + (1.451)^2}$$

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In order to bring it to the format of $\frac{s+a}{(s+a)^2+b^2}$

substitute $+a$ and $-a$.

$$= \frac{-1.4509\,(s+1.202+0.601-0.601)}{(s+0.601)^2+(1.451)^2}$$

$$= \frac{-1.4509\,(s+0.601+0.601)}{(s+0.601)^2+(1.451)^2}$$

$$= -1.4509\,\frac{(s+0.601)}{(s+0.601)^2+(1.451)^2} + \frac{(-1.4509)\times 0.601}{(s+0.601)^2+(1.451)^2}$$

$$= -1.4509\,\frac{(s+0.601)}{(s+0.601)^2+(1.451)^2} + \frac{(-1.4509)\times 0.601 \times \frac{1.451}{1.451}}{(s+0.601)^2+(1.451)^2}$$

$$= \frac{-1.4509\,(s+0.601)}{(s+0.601)^2+(1.451)^2} + 0.6 \times \frac{1.451}{(s+0.601)^2+(1.451)^2}$$

Apply the formula.

$$= \frac{-1.4509\,(1-e^{-0.601}\cos(1.451)z^{-1})}{(1-2e^{-0.601}\cos(1.451)z^{-1}+e^{-2\times 0.601}z^{-2}} + \frac{0.6\,(e^{-0.601}\sin(1.451)z^{-1})}{1-2e^{-0.601}\cos(1.451)z^{-1}+e^{-2\times 0.601}z^{-2}}$$

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$$= \frac{-1.4509\left(1-0.065z^{-1}\right)}{1-0.131z^{-1}+0.3036z^{-2}} - \frac{0.326z^{-1}}{1-0.131z^{-1}+0.3036z^{-2}}$$

$$H_1(z) = \frac{-1.459 - 0.2316z^{-1}}{1-0.131z^{-1}+0.3036z^{-2}}$$

Illy.

Convert $H_2(s)$ into $H_2(z)$

$$H_2(s) = \frac{1.4509s + 4.2113}{s^2 + 2.9025s + 2.46}$$

$$H_2(s) = \frac{1.4509(s+2.9)}{(s+1.451)^2 + (0.6)^2}$$

$$= \frac{1.4509(s+2.9+1.451-1.451)}{(s+1.451)^2 + (0.6)^2}$$

$$= \frac{1.4509(s+1.451)}{(s+1.451)^2+(0.6)^2} + \frac{1.449 \times \frac{0.6}{0.6} \times 1.4509}{(s+1.451)^2+(0.6)^2}$$

Apply the formula.

$$H_2(z) = \frac{1.4509 - 0.185z^{-1}}{1-0.386z^{-1}+0.05z^{-2}}$$

Adding $H_1(z)$ & $H_2(z)$

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$$H(z) = \frac{-1.4509 - 0.235 z^{-1}}{1 - 0.131 z^{-1} + 0.3036 z^{-2}} + \frac{1.4509 + 0.185 z^{-1}}{1 - 0.386 z^{-1} + 0.05 z^{-2}}$$

Problem No 2 :

1. Design a digital butterworth filter having the following specifications when $T = 1 sec$ using Bilinear Transformation.

$$0.9 \le |H(j\omega)| \le 1 \qquad 0 \le \omega \le \pi/2$$

$$|H(j\omega)| \le 0.2 \qquad 3\pi/4 \le \omega \le \pi$$

step 1 :

Convert digital freq to analog freq.

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} = 2 \tan \pi/4 = 2$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} = 2 \tan \frac{3\pi}{8} = 4.828$$

step 2 :

Calculation of ε & λ value.

$$\varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1} = \sqrt{\frac{1}{(0.9)^2} - 1} = 0.4843$$

$$\lambda = \sqrt{\frac{1}{\delta_2^2} - 1} = \sqrt{\frac{1}{(0.2)^2} - 1} = 4.8989$$

step 3 :

Calculate the order of the filter

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$$N \geq \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\left(\frac{4.8989}{0.4843}\right)}{\log\left(\frac{4.828}{2}\right)} \qquad N \geq 2.626$$

Round it to the next highest integer value

$N = 3$

Step 4: Determination of -3dB cut off frequency.

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{2}{(0.4843)^{1/3}} = 2.546$$

$\Omega_c = 2.546$ r/sec

Step 5: Determination of H(s)

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \cdot \frac{\Omega_c^2}{s^2 + b_1 \Omega_c s + \Omega_c^2}$$

$$b_k = 2\sin\left(\frac{2k-1}{2N}\right)\pi = 2\sin\left(\frac{2-1}{6}\right)\pi = 1$$

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$$H(S) = \frac{2.546}{S+2.546} * \frac{(2.546)^2}{S^2 + 2.546S + 2.546^2}$$

$$H(S) = \frac{16.5}{(S+2.546)(S^2 + 2.546S + 6.482)}$$

Step 6: Determination of $H(z)$.

$$H(z) = H(S)\Big|_{S = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}}$$

$$H(z) = \frac{16.5}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})} + 2.546\right]\left[\frac{4(1-z^{-1})^2}{(1+z^{-1})^2} + 2.546 \times 2\frac{(1-z^{-1})}{(1+z^{-1})} + 6.482\right]}$$

$$H(z) = \frac{16.5}{\left[\frac{2(1-z^{-1}) + 2.546(1+z^{-1})}{(1+z^{-1})}\right]\left[\frac{4(1-z^{-1})^2 + 5.092(1-z^{-1})(1+z^{-1}) + 6.482(1+z^{-1})^2}{(1+z^{-1})^2}\right]}$$

$$H(z) = \frac{16.5\,(1+z^{-1})^3}{\left[4.546 + 0.546z^{-1}\right]\left[15.574 + 4.96z^{-1} + 5.39z^{-2}\right]}$$

Simplifying.

$$H(z) = \frac{16.5\,(1+z^{-1})^3}{70.79 + 31.063z^{-1} + 27.21z^{-2} + 2.94z^{-3}}$$

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$\div$ 16.5 both num & denom

$$\boxed{H(z) = \frac{0.233\,(1+z^{-1})^3}{1 + 0.438 z^{-1} + 0.384 z^{-2} + 0.041 z^{-3}}}$$

Problem No 3:

Design a Butterworth filter using impulse invariant method for the following specifications

$$0.8 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$$

given data:

$\delta_1 = 0.8$ $\quad \delta_2 = 0.2$ $\quad \omega_p = 0.2\pi$ $\quad \omega_s = 0.6\pi$ $\quad T = 1\,\text{sec}$

step 1:

$$\Omega_p = \frac{\omega_p}{T} = 0.2\pi$$

$$\Omega_s = \frac{\omega_s}{T} = 0.6\pi$$

step 2:

$$\varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1} = \sqrt{\frac{1}{0.8^2} - 1} = 0.75$$

$$\lambda = \sqrt{\frac{1}{\delta_2^2} - 1} = \sqrt{\frac{1}{0.2^2} - 1} = 4.899$$

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step 3 :

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\left(\frac{4.899}{0.75}\right)}{\log\left(\frac{0.6\pi}{0.2\pi}\right)} \geq 1.71$$

$$\boxed{N = 2}$$

step 4 :-

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{0.2\pi}{(0.75)^{1/2}} = 0.7257.$$

step 5 :

To find the transfer function N is even

$$H(s) = \prod_{1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

$$b_1 = 2 \sin\left(\frac{2k-1}{4}\right)\pi = \boxed{1.414}$$

$$H(s) = \frac{0.5266}{s^2 + 1.026\, s + 0.5266}$$

step 6 : convert to H(z)

$$H(s) = \frac{0.5266}{(s+0.513)^2 + (0.513)^2}.$$

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$$H(S) = \frac{0.5266}{0.513} \times \frac{0.513}{(S+0.513)^2 + 0.513}$$

$$H(S) = 1.026 \times \frac{0.513}{(S+0.513)^2 + 0.513}$$

$$\frac{b}{(S+a)^2+b^2} \Rightarrow \frac{e^{-aT} \sin bT z^{-1}}{1 - 2e^{-aT} \cos bT z^{-1} + e^{-2aT} z^{-2}}$$

$$H(z) = \frac{1.026 \times e^{-0.513} \sin 0.513 z^{-1}}{1 - 2e^{-0.513} \cos 0.513 z^{-1} + e^{-2\times 0.513} z^{-2}}$$

$$H(z) = \frac{1.026 \times 0.598 \times 0.4907 z^{-1}}{1 - 2 \times 0.598 \times 0.871 z^{-1} + 0.358 z^{-2}}$$

$$\boxed{H(z) = \frac{0.301 z^{-1}}{1 - 1.041 z^{-1} + 0.358 z^{-2}}}$$

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Problem No 4:

Design a third order Butterworth filter using impulse invariant technique. Assume sampling period T = 1 sec.

Given

$N = 3$

$\Omega_c = 1$ rad/sec.

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2+b_k\Omega_c s+\Omega_c^2}$$

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \frac{\Omega_c^2}{s^2+b_1\Omega_c s+\Omega_c^2}$$

$\therefore b_1 = 2\sin\left(\frac{2-1}{6}\right)\pi$

$b_1 = 1$

$$H(s) = \frac{1}{(s+1)} \frac{1}{(s^2+s+1)}$$

$$H(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}$$

$$A(s^2+s+1) + Bs(s+1) + c(s+1) = 1$$

Equating s^2

$A + B = 0$

equating s $\quad A + B + C = 0$

Equating constant $\quad A + C = 1$

Solving $\quad C = 0$

$A = 1$

$B = -1$

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$$H(s) = \frac{1}{s+1} + \frac{(-s)}{s^2+s+1}.$$

$H_1(s) \qquad H_2(s)$

$$H_1(z) = \frac{1}{1-e^{-1}z^{-1}} = \frac{1}{1-0.3678z^{-1}}$$

$$H_2(s) = \frac{-s}{s^2+s+1} = \frac{-s}{(s+0.5)^2+(0.866)^2}$$

Inorder to bring to the format of $\frac{s+a}{(s+a)^2+b^2}$

add & subtract 0.5

$$= -\left[\frac{s+0.5 \quad -0.5}{(s+0.5)^2+(0.866)^2}\right]$$

$$= \frac{-(s+0.5)}{(s+0.5)^2+(0.866)^2} + \frac{0.5}{(s+0.5)^2+(0.866)^2}$$

Inorder to bring to the format of $\frac{b}{(s+a)^2+b^2}$

divide & multiply by 0.866

$$= \frac{-s+0.5}{(s+0.5)^2+(0.866)^2} + \frac{0.5}{0.866} \times \frac{0.866}{(s+0.5)^2+(0.866)^2}$$

$$H(z) = \frac{-\left[1-e^{-0.5}\cos 0.866\, z^{-1}\right]}{1-2e^{-0.5}\cos 0.86\, z^{-1}+e^{-2\times 0.5}z^{-2}} + 0.581\, \frac{e^{-0.5}\sin 0.866}{1-2e^{-0.5}\cos 0.86\, z^{-1} + e^{-2\times 0.5}z^{-2}}$$

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$$H_2(z) \Rightarrow \frac{-1 + 0.66z^{-1}}{1 - 0.791z^{-1} + 0.3678z^{-2}}$$

$$H(z) = H_1(z) + H_2(z)$$

$$= \frac{1}{1 - 0.3678} + \frac{-1 + 0.66z^{-1}}{1 - 0.791z^{-1} + 0.3678z^{-2}}$$

Problem No 5:

For the given specification design an analog Butterworth filter:

$\alpha_p = 3\,dB$ $\quad \alpha_s = 18\,dB$ $\quad f_p = 1\,kHz$ $\quad f_s = 2\,kHz$.

Step 1:

$$\Omega_s = 2\pi f_s = 2 \times \pi \times 2 \times 10^3 = 4000\pi$$

$$\Omega_p = 2\pi f_p = 2 \times \pi \times 1 \times 10^3 = 2000\pi$$

Step 2:

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = \sqrt{10^{0.1 \times 3} - 1} =$$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = \sqrt{10^{0.1 \times 18} - 1} =$$

Step 3:

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{4000\pi}{2000\pi}\right)}$$

$$N \geq \frac{\log\left(\frac{7.88}{0.9952}\right)}{\log(2)} \qquad N \geq 2.98$$

$$N = 3.$$

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Step 4:

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \frac{2000\pi}{(0.9952)^{1/3}} = 2001.6\,\pi$$

Step 5:

$$H(S) = \frac{\Omega_c}{S+\Omega_c} \prod_{K=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{S^2+b_K\Omega_c S+\Omega_c^2}$$

$b_1 = 1$

$$H(S) = \frac{\Omega_c}{S+\Omega_c} \cdot \frac{\Omega_c^2}{S^2+1\cdot\Omega_c S+\Omega_c^2}$$

$$\boxed{H(s) = \frac{(2001.6)^3\,\pi^3}{(s+2001.6\pi)\left(s^2+2001.6\pi\, s+(2001.6)^2\right)}}$$

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Design procedure for digital chebyshev Lowpass filter :

Step 1 :

From the given specifications, find out the analog frequency.

For impulse invariant method :

$$\Omega_s = \frac{\omega_s}{T} \qquad \Omega_p = \frac{\omega_p}{T}$$

For Bilinear transformation :

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} \qquad \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$$

Step 2 :

Find the order of the filter 'N'.

If δ_1 and δ_2 are given

$$\lambda = \sqrt{\frac{1}{\delta_2^2} - 1} \qquad \varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1}$$

If α_p and α_s (in dB) are given

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} \qquad \varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Step 3 :

Round it off to the next higher integer

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Step 4 :

Using the following formulas find the values of a and b which are minor and major axis of the ellipse.

$$a = \Omega_p \frac{\left[\mu^{1/N} - \mu^{-1/N}\right]}{2} \qquad b = \Omega_p \frac{\left[\mu^{1/N} + \mu^{-1/N}\right]}{2}$$

where

$$\mu = \frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}$$

Step 5 :

Find the poles of chebyshev filter

$$s_K = a\cos\phi_K + j\, b\sin\phi_K$$

where $K = 1, 2, \ldots N$

$$\phi_K = \frac{\pi}{2} + \left(\frac{2K-1}{2N}\right)\pi \qquad K = 1, 2 \ldots N.$$

Step 6 :

Find the denominator polynomial of the transfer function using the poles

$$\text{Denominator} = (s-s_1)(s-s_2)\ldots\ldots(s-s_N)$$

Step 7 :

To find the numerator

(i) For N odd substitute $s=0$ in denominator and find the value and that is equal to numerator

(ii) For N even substitute $s=0$ in denom and divide the result by $\sqrt{1+\varepsilon^2}$ and this value is equal to numerator.

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Step 8 :

Convert $H(s)$ to $H(z)$ using impulse invariant method or Bilinear transformation.

Problem No 1 :

Design a chebyshev filter for the following specification using bilinear transformation.

$$\underset{\delta_1}{0.8} \leq |H(j\omega)| \leq 1 \qquad 0 \leq \omega \leq \underset{\omega_p}{0.2\pi}$$

$$|H(j\omega)| \leq \underset{\delta_2}{0.2} \qquad \underset{\omega_s}{0.6\pi} \leq \omega \leq \pi$$

given data:

$\delta_1 = 0.8 \quad \delta_2 = 0.2 \quad \omega_p = 0.2\pi \quad \omega_s = 0.6\pi \quad T = 1\,sec$

Step 1 :

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} = 0.6498$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} = 2.752$$

Step 2 :

$$\lambda = \sqrt{\frac{1}{\delta_2^2} - 1} = \sqrt{\frac{1}{(0.2)^2} - 1} = 4.8999$$

$$\varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1} = \sqrt{\frac{1}{(0.8)^2} - 1} = 0.75$$

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\left(\frac{4.899}{0.75}\right)}{\cosh^{-1}\left(\frac{2.752}{0.6498}\right)}$$

$N \geq 1.208$

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Step 3 :

$N = 2$

Step 4 :

$$\mu = \frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}$$

$$= \frac{1}{0.75} + \sqrt{\frac{1}{(0.75)^2} + 1} = 3$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.6498 \left[\frac{3^{1/2} - 3^{-1/2}}{2}\right]$$

$a = 0.3752$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.6498 \left[\frac{3^{1/2} + 3^{-1/2}}{2}\right]$$

$b = 0.75$

Step 5 : To find the poles .

$$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N} \qquad K = 1, 2$$

when $K = 1$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 2.356 = (135^0)$$

when $K = 2$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 3.926 = (225^0).$$

Step 6 : (Do in Degree mode) to get the accurate result

$$s_K = a\cos\phi_K + j\, b\sin\phi_K$$

$$s_1 = 0.3752 \cos 2.356 + j\, 0.75 \sin 2.356$$

$$s_1 = -0.2653 + j\, 0.53$$

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$S_2 = 0.3752 \cos 3.926 + j\, 0.75 \sin 3.926$

$S_2 = -0.2653 - j0.53$

Step 7:

$$Denom = (S-S_1)(S-S_2)$$

$$= (s+0.2653 - j0.53)(s+0.2653 - j0.53)$$

$$Denom = (s+0.2653)^2 + (0.53)^2$$

$$= s^2 + 0.5306\, s + 0.3516$$

Step 8:

Numerator of H(s) ; sub $s=0$ in denom

Since N is even ; $\dfrac{0.3516}{\sqrt{1+0.75^2}} = 0.28$

$$\text{so } H(s) = \frac{0.28}{s^2 + 0.5306 s + 0.3516}$$

Using Bilinear transformation.

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.28}{\left[\dfrac{2(1-z^{-1})}{(1+z^{-1})}\right]^2 + 0.5306 \times 2\dfrac{(1-z^{-1})}{(1+z^{-1})} + 0.3516}$$

$$= \frac{0.28\,(1+z^{-1})^2}{4(1-z^{-1})^2 + 1.0612(1-z^{-1})(1+z^{-1}) + 0.3516(1+z^{-1})^2}$$

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$$= \frac{0.28(1+z^{-1})^2}{4(1-2z^{-1}+z^{-2}) + 1.0612(1-z^{-2}) + 0.3516(1-2z^{-1}+z^{-2})}$$

$$H(z) = \frac{0.28(1+z^{-1})^2}{5.4128 - 7.298z^{-1} + 3.29z^{-2}}$$

$\div 5.4128$

$$H(z) = \frac{0.052(1+z^{-1})^2}{1 - 1.348z^{-1} + 0.608z^{-2}}.$$

Problem No 2:

Design a chebyshev filter for the following specification using impulse invariant method with T = 1 sec.

$$0.8 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \qquad 0.6\pi \le \omega \le \pi$$

given:

$\delta_1 = 0.8$ $\delta_2 = 0.2$ $\omega_p = 0.2\pi$ $\omega_s = 0.6\pi$ $T = 1$ sec.

Step 1:

$$\Omega_s = \frac{\omega_s}{T} = 0.6\pi \qquad \Omega_p = \frac{\omega_p}{T} = 0.2\pi$$

step 2:

$$\lambda = \sqrt{\frac{1}{\delta_2^2} - 1} = \sqrt{\frac{1}{(0.2)^2} - 1} = 4.899$$

$$\varepsilon = \sqrt{\frac{1}{\delta_1^2} - 1} = \sqrt{\frac{1}{0.8^2} - 1} = 0.75$$

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$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\left(\frac{4.899}{0.75}\right)}{\cosh^{-1}\left(\frac{0.6\pi}{0.2\pi}\right)}$$

$N \geq 1.45$

step 3:

$N = 2$

step 4:

$$\mu = \frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}$$

$$= \frac{1}{0.75} + \sqrt{\frac{1}{0.75^2} + 1}$$

$\mu = 3$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.2\pi \left[\frac{3^{1/2} - 3^{-1/2}}{2}\right]$$

$a = 0.3627$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.2\pi \left[\frac{3^{1/2} + 3^{-1/2}}{2}\right]$$

$b = 0.7255$

step 5:

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 2.356 \quad (135^0)$$

$$\phi_2 = \pi/2 + 3\pi/4 = 3.926 \quad (225^0)$$

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$s_K = a\cos\phi_K + jb\sin\phi_K$

$s_1 = 0.3627 \cos 2.356 + j\,0.7255 \sin 2.356$

$s_1 = -0.2564 + j0.513$

$s_2 = 0.3627 \cos 3.926 + j\,0.7255 \sin 3.926$

$s_2 = -0.2564 - j0.513$

step 6 :

Denom : $(s-s_1)(s-s_2)$

$\Rightarrow (s+0.2564)^2 + (0.513)^2$.

Num : sub $s=0$ in denominator

$\Rightarrow \dfrac{0.33}{\sqrt{1+\varepsilon^2}}$ since N is even

$\Rightarrow 0.264$

$$H(s) = \frac{0.264}{(s+0.2564)^2 + (0.513)^2}$$

To bring it to the form of $\dfrac{b}{(s+a)^2+b^2}$

$\times$ and $\div$ by b

$$H(s) = \frac{0.264}{0.513} \times \frac{0.513}{(s+0.2564)^2 + (0.513)^2}$$

$$H(s) = 0.5146 \times \frac{0.513}{(s+0.2564)^2 + (0.513)^2}$$

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$$\frac{b}{(s+a)^2+b^2} \Rightarrow \frac{e^{-aT}\sin bT\, z^{-1}}{1-2e^{-aT}\cos bT\, z^{-1}+e^{-2aT}z^{-2}}$$

$$H(z) = \frac{0.5146\; e^{-0.2564}\sin(0.513)\, z^{-1}}{1-2e^{-0.2564}\cos(0.513)z^{-1}+e^{-0.513}z^{-2}}$$

$$H(z) = \frac{0.1954\, z^{-1}}{1-1.3483z^{-1}+0.5987z^{-2}}$$

Problem No 3:

Determine the system function $H(z)$ of the lowest order chebyshev filter with the following specification.

(a) 3 dB ripple in passband $0 \le \omega \le 0.2\pi$

(b) 25 dB attenuation in stopband $0.45\pi \le \omega \le \pi$

Given data:

$\alpha_p = 3\,dB \quad \omega_p = 0.2\pi$

$\alpha_s = 25\,dB \quad \omega_s = 0.45\pi$

Since the conversion technique is not given choose Bilinear transformation.

Step 1:

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 0.6498$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 1.708$$

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step 2 :

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = \sqrt{10^{2.5} - 1} = 17.7546$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = \sqrt{10^{0.3} - 1} = 0.997 \simeq 1$$

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\left(\frac{17.7546}{0.997}\right)}{\cosh^{-1}\left(\frac{0.6498}{1.708}\right)}$$

$$N \geq 2.203$$

step 3 :

$$N = 3$$

step 4 :

$$\mu = \frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}$$

$$\mu = 2.414$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.6498 \left[\frac{2.414^{1/3} - 2.414^{-1/3}}{2}\right]$$

$$a = 0.1935$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.6498 \left[\frac{2.414^{1/3} + 2.414^{-1/3}}{2}\right]$$

$$b = 0.678$$

step 5 :

$$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N}$$

$\phi_1 = 2.094$ $\quad$ $\phi_2 = 3.1415$ $\quad$ $\phi_3 = 4.1887$

$\phi_1 = 120^\circ$ $\quad$ $\phi_2 = 180^\circ$ $\quad$ $\phi_3 = 240^\circ$

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$s_K = a\cos\phi_K + jb\sin\phi_K$

$s_1 = a\cos\phi_1 + jb\sin\phi_1$

$= 0.1935 \cos 2.094 + j\, b \sin 2.094$

$s_1 = -0.09675 + j\,0.587$

$s_2 = a\cos\phi_2 + jb\sin\phi_2$

$s_2 = -0.1935$

$s_3 = -0.09675 - j0.587.$

Step 6:

Denominator of $H(s) = (s-s_1)(s-s_2)(s-s_3)$

$= (s+0.09675-j0.587)(s+0.1935)(s+0.09675+j0.587)$

$= [(s+0.09675)^2 + (0.587)^2][s+0.1935]$

sub $s=0$ in denom.

since N is odd, the value is numerator

$\Rightarrow 0.0666$ //

Step 7:

$$H(s) = \frac{0.0666}{(s+0.1935)(s^2+0.1935s+0.354)}$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}}$$

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$$= \frac{0.0666}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})} + 0.1935\right]\left[\frac{4(1-z^{-1})^2}{(1+z^{-1})^2} + 0.1935 \times 2\frac{(1-z^{-1})}{(1+z^{-1})} + 0.354\right]}$$

$$= \frac{0.0666}{\left[\frac{2(1-z^{-1}) + 0.1935(1+z^{-1})}{(1+z^{-1})}\right]\left[\frac{4(1-z^{-1})^2 + 0.1935 \times 2(1-z^{-1})(1+z^{-1}) + 0.354(1+z^{-1})^2}{(1+z^{-1})^2}\right]}$$

$$= \frac{0.0666\,(1+z^{-1})^2(1+z^{-1})}{(2.1935 - 1.8065z^{-1})(4.5475 - 7.29z^{-1} + 4.1605z^{-2})}$$

$$\boxed{H(z) = \frac{0.00667\,(1+z^{-1})^3}{(1-0.823z^{-1})(1-1.6z^{-1}+0.915z^{-2})}}$$

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Frequency Transformation in Analog Domain :-

LPF is a prototype filter, then if a highpass or bandpass or bandstop is to be designed, frequency transformation is done.

These formulae are used to convert a LPF ($\Omega_c = 1$) into a lowpass (with different Ω_c), highpass, bandpass or bandstop

1. Normalized LPF to another LPF

$$s \rightarrow \frac{s}{\Omega_c}$$

2. Normalized LPF to HPF

$$s \rightarrow \frac{\Omega_c}{s}$$

3. Normalized LPF to BPF

$$s \rightarrow \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)}$$

If Q & Ω_o is given

$$s \rightarrow \frac{s^2 + \Omega_o^2}{s\left(\frac{\Omega_o}{Q}\right)}$$

where $\Omega_o = \sqrt{\Omega_u \Omega_l}$ $\Rightarrow$ centre frequency

$Q = \frac{\Omega_o}{\Omega_u - \Omega_l}$ $\Rightarrow$ Quality factor.

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4. Normalized LPF to BSF :

$$S \rightarrow \frac{s(\Omega u - \Omega l)}{s^2 + \Omega u \Omega l}$$

$$S \rightarrow \frac{s(\Omega o/Q)}{s^2 + \Omega o^2}$$

Problem No 1 :

1. A LPF has the transfer function $H(s) = \frac{1}{s^2+2s+1}$ convert this into Bandpass filter when $Q = 10$ and $\Omega o = 2$.

solution :

$$H(s) = \frac{1}{s^2 + 2s + 1}$$

$\Omega c^2 = 1 \Rightarrow \Omega c = 1$

$$S \Rightarrow \frac{s^2 + \Omega o^2}{s(\Omega o/Q)} \Rightarrow \frac{s^2+4}{s(2/10)} \Rightarrow \frac{5(s^2+4)}{s}$$

$$H(s) = \frac{1}{\left[\frac{5(s^2+4)}{s}\right]^2 + 10\frac{(s^2+4)}{s} + 1}$$

$$H(s) = \frac{1}{\frac{25(s^4 + 8s^2 + 16) + 10(s^2+4)s + s^2}{s^2}}$$

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$$H(s) = \frac{s^2}{25(s^4+8s^2+16) + 10s^3 + 40s + s^2}$$

$$\boxed{H(s) = \frac{s^2}{25s^4 + 10s^3 + 201s^2 + 40s + 400}}.$$

Problem No 2:

A LPF has the following transfer function $H(s) = \frac{\Omega_c}{s+\Omega_c}$ convert this into HPF having cut off freq $\Omega_c = 5$ rad/sec.

Normalized LPF Transfer function $H(s) = \frac{1}{s+1}$

The formula for converting to HPF with $\Omega_c = 5$

$$s \to \frac{\Omega_c}{s}$$

$$s \to \frac{5}{s}$$

$$H(s) = \frac{1}{\frac{5}{s}+1} = \boxed{\frac{s}{s+5}}.$$

Problem No 3!

Design an analog high pass butterworth filter with $\alpha_p = 3$ dB $\alpha_s = 15$ dB $\Omega_p = 1000$ r/sec and $\Omega_s = 500$ r/sec.

The specifications given in the problem is for HPF. So that should be converted for LPF and then LPF is designed first. After that frequency

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transformation is done.

LPF specifications:

$$\alpha_p = 3\ dB$$
$$\alpha_s = 15\ dB$$
$$\left.\begin{array}{l}\Omega_s = 1000\ rad/sec \\ \Omega_p = 500\ rad/sec.\end{array}\right\}$$

Step 1:

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = \sqrt{31.62 - 1} = 5.53$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} \simeq 1$$

Step 2:

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left(\frac{5.53}{1}\right)}{\log\left(\frac{1000}{500}\right)}$$

$$N \geq 2.4$$

$$N = 3$$

Step 3:

Normalized freq. $\Omega_c = 1\ rad/sec.$

Step 4:

$$H(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

$$H(s) = \frac{1}{s+1} \cdot \frac{1}{s^2 + b_1 s + 1}$$

$$b_1 = 2\sin(\pi/6) = 1$$

$$H(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+s+1}$$

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Step 5 :

Convert this into HPF

$$s \rightarrow \frac{\Omega_c}{s} \qquad s \rightarrow \frac{1000}{s}$$

$$\Omega_c = \frac{\Omega_p}{(\varepsilon)^{1/N}}$$

$$\Omega_c = \frac{1000}{(1)^{1/3}}$$

$\Omega_c = 1000$ rad/sec.

$$H(s) = \frac{1}{\frac{1000}{s}+1} \cdot \frac{1}{\frac{(1000)^2}{s^2} + \frac{1000}{s} + 1}$$

$$\boxed{H(s) = \frac{s}{s+1000} \cdot \frac{s^2}{(1000)^2 + 1000s + s^2}}$$

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Design of IIR filters from analog filters:-

There are four methods for converting the analog filter into digital filter.

1. Approximation of derivatives
2. Impulse Invariant transformation
3. Bilinear transformation.
4. The matched - z transformation technique.

If the conversion technique is to be effective, it should possess the following desirable properties

* The $j\Omega$ axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship b/w the two freq variables in the two domains
* The left half of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted into a stable digital filter.

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Approximation of Derivatives :-

In this method, an analog filter is converted into digital filter by approximating the differential equation by an equivalent difference equation.

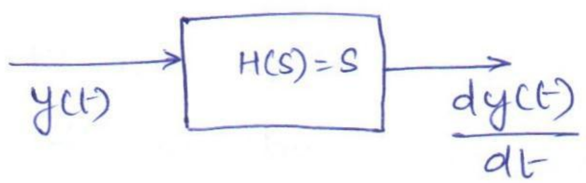
The backward difference formula is substituted for the derivative $\frac{dy(t)}{dt}$ at time $t = nT$. Thus

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T}$$

where T is the sampling interval.

We know that Laplace transform of $\frac{dy(t)}{dt} = sY(s)$ ——— ①



The z-transform of $\frac{y(n) - y(n-1)}{T}$

is $\frac{(1 - z^{-1})\, Y(z)}{T}$ ——— ②.

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Comparing ① and ②

$$\boxed{s = \frac{1 - z^{-1}}{T}} \quad \text{——} \quad ③$$

$$H(z) = H(s) \Big|_{s = \frac{1 - z^{-1}}{T}}$$

The equation ③ can be written as

$$s = \frac{1 - z^{-1}}{T}$$

$$sT = 1 - z^{-1}$$

$$z^{-1} = 1 - sT$$

$$z = \frac{1}{1 - sT} \qquad \text{sub } s = j\Omega$$

$$z = \frac{1}{1 - j\Omega T}$$

$$z = \frac{1}{1 - j\Omega T} \frac{1 + j\Omega T}{1 + j\Omega T}$$

$$z = \frac{1}{1 + \Omega^2 T^2} + j \frac{\Omega T}{1 + \Omega^2 T^2}.$$

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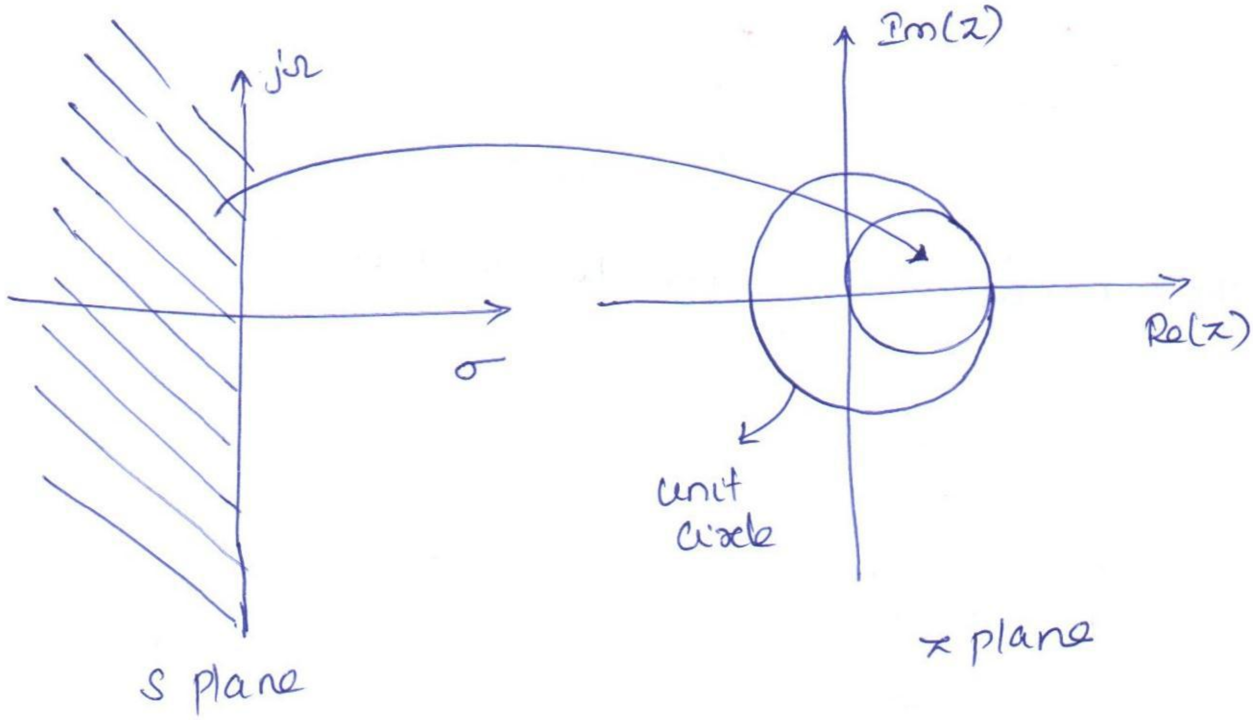
Varying Ω from $-\infty$ to ∞, the points will be in the z-plane and it is a circle with radius $1/2$ and centre $1/2$.



* The left half of the plane maps into the circle of radius $1/2$ and centre $1/2$

* The right half of the plane maps outside the circle

* The $j\Omega$ axis maps on to the perimeter of the circle of radius $1/2$.

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This type of mapping is restricted to the design of lowpass and bandpass filter only. High pass is not possible.

Problem No.1 :

Use the backward difference formula or approximation derivative to convert analog low pass filter with system function

$$H(s) = \frac{1}{s+2}.$$

Formula : $s = \frac{1-z^{-1}}{T}$

Assume $T=1$, so $s = 1-z^{-1}$

$$H(z) = \frac{1}{1-z^{-1}+2}$$

$$\boxed{H(z) = \frac{1}{3-z^{-1}}}$$

Problem No 2 :

An analog filter has the following system function. Use backward difference to convert

$$H(s) = \frac{1}{(s+0.1)^2+9}$$

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Formula:

$$S = \frac{1 - z^{-1}}{T} \quad \text{sub } T = 1$$

$$S = 1 - z^{-1})$$

$$H(s) = \frac{1}{(S+0.1)^2 + 9}$$

$$H(z) = \frac{1}{(1 - z^{-1} + 0.1)^2 + 9}$$

$$= \frac{1}{(1.1 - z^{-1})^2 + 9} = \frac{1}{(1.1^2 - 2.2z^{-1} + z^{-2}) + 9}$$

$$= \frac{1}{10.21 - 2.2z^{-1} + z^{2}}$$

$$\boxed{H(z) = \frac{0.0979}{1 - 0.2154z^{-1} + 0.0979z^{-2}}}$$

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FINITE IMPULSE RESPONSE FILTERS

Digital filters are classified as

(i) FIR filter

(ii) IIR filter

In the FIR system, the impulse response is of finite duration. i.e it has a finite number of non-zero terms.

for Eg. $h(n) = \{\underset{h(0)}{-4}, \underset{h(1)}{1}, \underset{h(2)}{0}, \underset{h(3)}{2}\}$.

It has only a finite number of non-zero terms. An FIR of length 'N' is described by the difference equation

$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$ where b_k is the set of filter coefficients, the response of the filter depends only on the present and past input samples.

FIR filters have the following advantages

1. Fir filters are always stable
2. Fir filter with exactly linear phase can easily be designed.

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3. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital system.

Disadvantages of FIR filter :-

1. The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.

2. Memory requirement and execution time are very high, if N value is increased.

Characteristics of Linear Phase FIR filter :-

Let h(n) be the finite impulse response and its fourier transform is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}.$$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) [\cos\omega n - j \sin \omega n]$$

$$= \sum_{n=0}^{N-1} h(n) \cos\omega n - j \sum_{n=0}^{N-1} h(n) \sin\omega n$$

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The magnitude and phase responses are

$$|H(e^{j\omega})| = \sqrt{Re(H(e^{j\omega}))^2 + Im(H(e^{j\omega}))^2}$$

$$\theta(\omega) = \tan^{-1} \frac{Im[H(e^{j\omega})]}{Re[H(e^{j\omega})]}$$

Filters can have a linear phase depending upon the delay function,

Phase delay

$$\alpha = \tau_p = -\frac{\theta(\omega)}{\omega}$$

Group delay

$$\tau_g = -\frac{d\theta(\omega)}{d\omega}$$

The group delay is defined as the delayed response of the filter as a function of ω

* Linear phase filters are those filters in which the phase delay and group delay are constant.

Let us obtain the conditions for having a linear phase. Now

$$\frac{\theta(\omega)}{\omega} = -\alpha$$

$$\theta(\omega) = -\alpha\omega = \tan^{-1} \frac{Im(H(e^{j\omega}))}{Re(H(e^{j\omega}))}$$

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$$\tan^{-1}\left(\frac{\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n}\right) = \alpha\omega$$

$$\tan\alpha\omega = \frac{\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n}$$

$$\frac{\sin\alpha\omega}{\cos\alpha\omega} = \frac{\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n}$$

$$\sum_{n=0}^{N-1} h(n)\sin\alpha\omega\cos\omega n - \sum_{n=0}^{N-1} h(n)\cos\alpha\omega\sin\omega n = 0$$

$$\sum_{n=0}^{N-1} h(n)\left[\sin(\alpha - n)\omega\right] = 0$$

The solution to this expression is

$$\boxed{\alpha = \frac{N-1}{2} \qquad h(n) = h(N-1-n)\ \{\text{symmetrical}).}$$

If these two expressions are satisfied then the FIR filter will have constant phase and group delay & thus the phase of the filter will be linear.

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Whenever Constant Group Delay alone is preferred

$$\theta(\omega) = \beta - \alpha\omega$$

The final expression is

$$\sum_{n=0}^{N-1} h(n) \sin\left[\beta - (\alpha - n)\omega\right] = 0$$

The solution to this

$$\beta = \pm \pi/2 \qquad \alpha = \frac{N-1}{2} \qquad h(n) = -h(N-1-n)$$

antisymmetric

FIR filters like wideband differentiator and Hilbert transformer use such antisymmetric impulse response sequences.

Frequency Response of Linear phase FIR filter

Case (i) When N is odd and impulse response is symmetrical i.e $h(n) = h(N-1-n)$

Let $h(n)$ be the impulse response and its fourier transform.

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \qquad \text{——— (1)}$$

When N is odd, the centre of symmetry is

$$\alpha = \frac{N-1}{2}$$

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Splitting the summation

For Eg if $N=7$, then 0 to 2, 3, 4 to 6 , $\alpha = \frac{N-1}{2} = 3$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n} \quad \text{--- (2)}$$

Let us change the index of third term

Let $m = N-1-n$

then $n = N-1-m$

when $n = \frac{N+1}{2}$, then $m = N-1-\left(\frac{N+1}{2}\right) = \frac{N-3}{2}$

when $n = N-1$, then $m = N-1-(N-1) = 0$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m)e^{-j\omega(N-1-m)} \quad \text{--- (3)}$$

Since $\boxed{h(N-1-n) = h(n)}$

Replace m by n in the third term

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega(N-1-n)} \quad \text{--- (4)}$$

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$$e^{j\omega\left(\frac{N-1}{2}\right)} \; e^{-j\omega\left(\frac{N-1}{2}\right)} = 1$$

Add above term in first & third summation

$$H(e^{j\omega}) = h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} e^{j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$+ \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)} e^{j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)} \quad \text{——— (5)}$$

Taking $e^{-j\omega\left(\frac{N-1}{2}\right)}$ outside

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} e^{j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)} e^{j\omega\left(\frac{N-1}{2}\right)} \right] \quad \text{——— (6)}$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{j\omega\left(\frac{N-1}{2}-n\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)} \right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{j\omega\left(\frac{N-1}{2}-n\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right]$$

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$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{n=0}^{\frac{N-3}{2}} h(n)\cos\left(\frac{N-1}{2}-n\right)\omega\right]$$

Let $k = \frac{N-1}{2} - n$

$n = \frac{N-1}{2} - k$

when $n=0$, $k = \frac{N-1}{2} - 0$, $k = \frac{N-1}{2}$

when $n = \frac{N-3}{2}$, $k = \frac{N-1}{2} - \left(\frac{N-3}{2}\right)$, $k = 1$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{k=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-k\right)\cos k\omega\right]$$

Replace k by n

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos n\omega\right]$$

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\theta(\omega)}$$

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega$$

$$\boxed{|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos n\omega}$$

It is observed that the frequency response is symmetric with $\omega = \pi$, when the impulse response is symmetric and N is odd.

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Problem No 1:

Determine the frequency response of FIR filter defined by $y(n) = 0.25\,x(n) + x(n-1) + 0.25\,x(n-2)$. Calculate phase delay and group delay.

$y(n) = 0.25\,x(n) + x(n-1) + 0.25\,x(n-2)$

Take z transform on both sides

$Y(z) = 0.25\,X(z) + z^{-1}X(z) + 0.25\,z^{-2}X(z)$

$$\frac{Y(z)}{X(z)} = 0.25 + z^{-1} + 0.25\,z^{-2}$$

$$\left.\begin{aligned} h(0) &= 0.25 \\ h(1) &= 1 \\ h(2) &= 0.25 \end{aligned}\right\} \quad h(n) \text{ is symmetric.}$$

$N-1 = 2 \qquad N = 3$

$$\alpha = \frac{N-1}{2} = 1$$

So when N is odd & h(n) is symmetric

$$H(j\omega) = e^{-j\alpha\omega}\,|H(j\omega)|$$

$$= e^{-j\omega}\left[h\frac{(N-1)}{2} + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos n\omega\right]$$

$$= e^{-j\omega}\left[1 + 2\times 0.25\cos\omega\right]$$

$$\boxed{H(j\omega) = e^{-j\omega}\left[1 + 0.5\cos\omega\right].}$$

$\theta(\omega) = -\alpha\omega$

$\theta(\omega) = -\omega$

$$\tau_p = -\frac{\theta(\omega)}{\omega} = \frac{\omega}{\omega} = 1 \longrightarrow \text{Phase delay}$$

$$\tau_g = -\frac{d\theta(\omega)}{d\omega} = \frac{-d}{d\omega}(-\omega) = 1 \longrightarrow \text{Group delay.}$$

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Problem NO : 2

Show that $h_n = \{1, 0, 1\}$ is a linear phase filter
$0, 1, 2$

$$N-1 = 2$$

$$N = 3$$

$$\alpha = \frac{N-1}{2} = 1.$$

$$h(n) = \{1, 0, 1\}$$

Centre

$$h(n) = h(N-1-n)$$

$$h(2) = h(2-2) = h(0)$$

Since the condition is satisfied, It is a linear phase filter.

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Design of FIR filters using Windows :-

The desired frequency response of filter is $H_d(e^{j\omega})$ and can be expanded in a Fourier series. The series is given by

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

where

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega$$

These fourier coefficients have infinite length. This is converted into finite filter by truncating the infinite fourier series at $n = \pm\left(\frac{N-1}{2}\right)$

* But abrupt truncation of the fourier series results in oscillation in the passband & stopband.

* These oscillations are due to slow convergence of the fourier series and this effect is known as gibbs phenomenon.

* To reduce these oscillations, the fourier coefficients are multiplied with a finite weighing sequence w(n) called window function.

The desirable characteristics of window are

1. The central lobe of the frequency response of the window should contain most of the energy and should be narrow.

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2. The highest side lobe level of the frequency response should be small.

3. The sidelobes of the frequency response should decrease in energy rapidly as ω tends to π

Types of window sequences for FIR filter design:

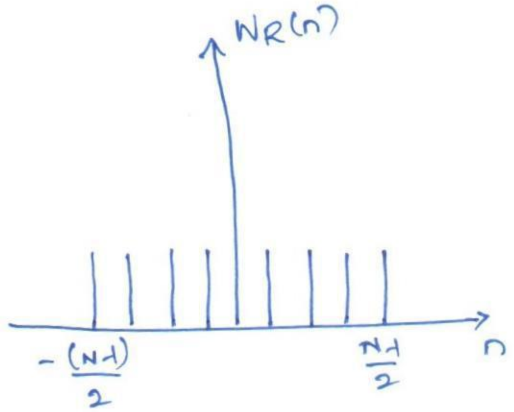
(i) Rectangular
(ii) Triangular (Barlett)
(iii) Hanning
(iv) Hamming
(v) Blackman
(vi) Kaiser

RECTANGULAR WINDOW:

The rectangular window sequence is given by

$$W_R(n) = 1 \quad \text{for} \quad -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{otherwise.}$$

For causal: $W_R(n) = 1$ for 0 to $N-1$.



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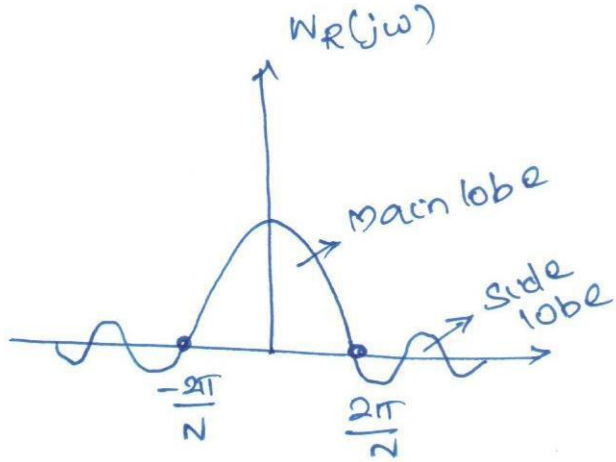
The spectrum of the rectangular window is given by

$$W_R(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} w_R(n)\, e^{-j\omega n}.$$

$$W_R(e^{j\omega}) = \frac{\sin \frac{\omega N}{2}}{\sin \omega/2}$$



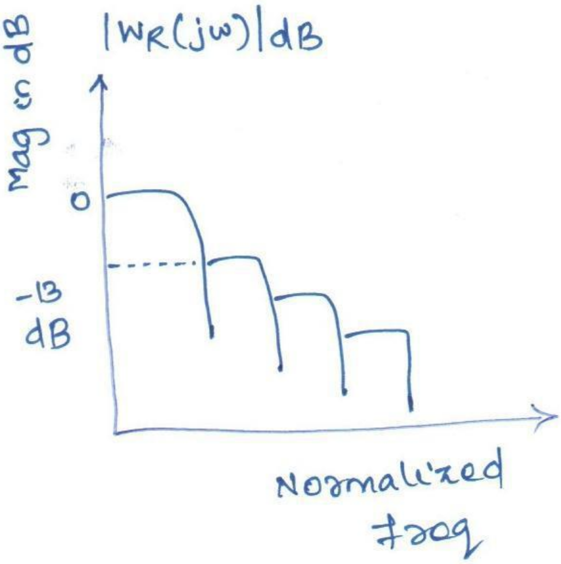
The spectrum has two important features

(i) Width of the mainlobe

(ii) Sidelobe amplitude

The mainlobe of the response is the portion that lies b/w the first two crossings. The width of the main lobe is $\frac{4\pi}{N}$

The side lobe amplitude is -13 dB.



Let the desired frequency response is $H_d(e^{j\omega})$.

$$h(n) = h_d(n)\, w_R(n)$$

In freq domain

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W_R(e^{j\omega})$$

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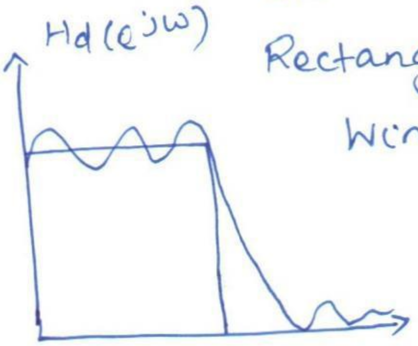
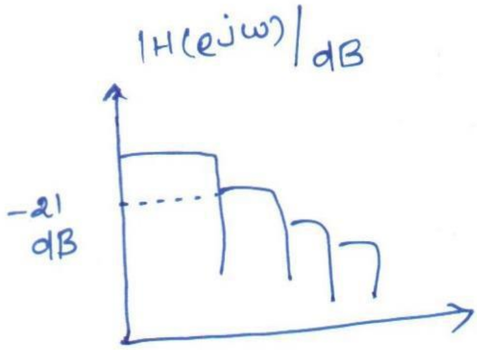
Magnitude response of LPF using Rectangular Window





The convolution of the desired response and window's response give rise to the ripples in both passband and stopband.

* The width of the transition region depends on the width of the mainlobe. As the filter length increases, the mainlobe becomes narrower and decrease the width of the transition region

* The amplitude of the ripples is depending on the amplitude of the sidelobes.

* The sidelobe amplitude is reduced by using a less abrupt truncation of filter coefficients. This can be achieved using a window function that tapers smoothly towards zero at both ends.

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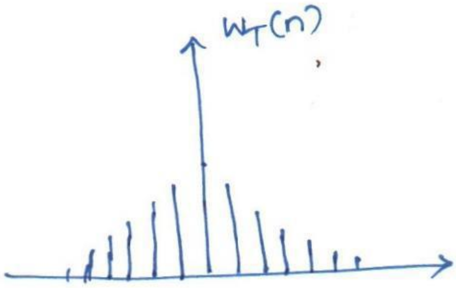
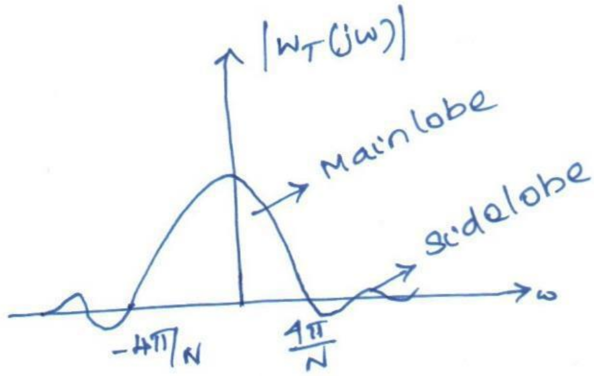
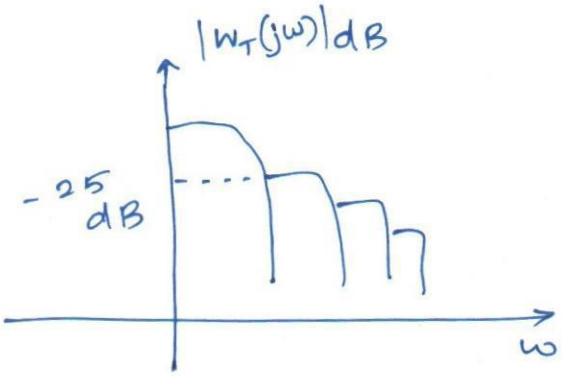
Triangular Window or Barlett Window:

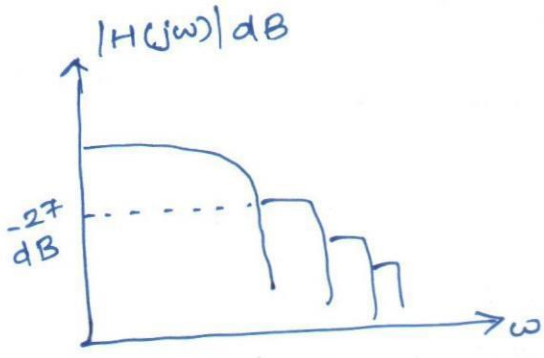
The triangular window has tapered sequences from the middle on either sides. the window function

$$w_T(n) = \begin{cases} 1 - \frac{2|n|}{N-1} & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$









* The sidelobe level is smaller than rectangular window
* The side lobe amplitude is -25 dB
* The mainlobe width is $8\pi/N$.
* This result indicates that there is a trade off between mainlobe width & sidelobe level.

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Raised cosine Window :-

Those windows are smoother at the ends. i.e smoothly truncate the fourier coefficients toward the ends of the filter.

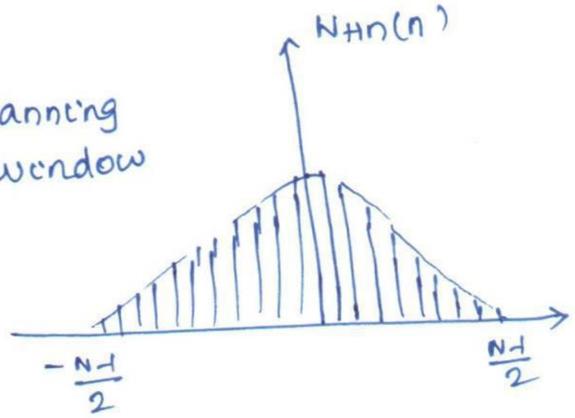
Hanning window :-

$$W_{Hn}(n) = 0.5 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

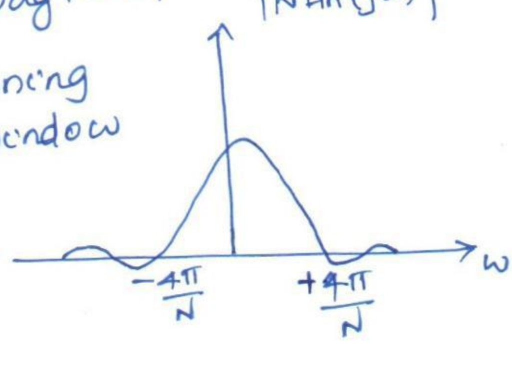
for causal $W_{Hn}(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right)$ for 0 to N-1

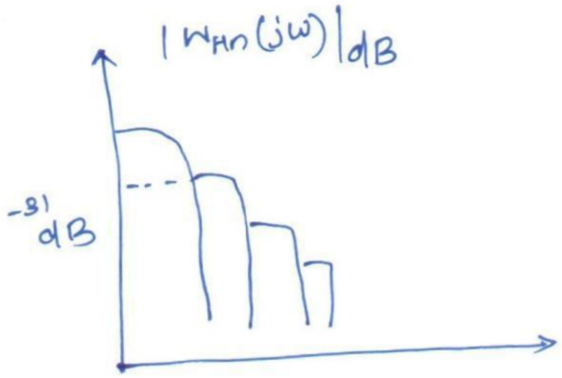
* The mainlobe width is $8\pi/N$
* The magnitude of the first sidelobe is -31 dB
* The largest peak is 44 dB

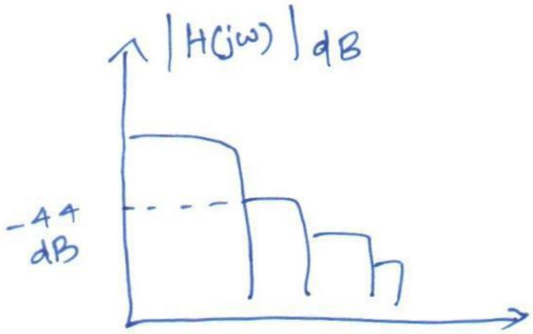
Hanning window



Mag. Response of Hanning window







Log Magnitude response of LPF

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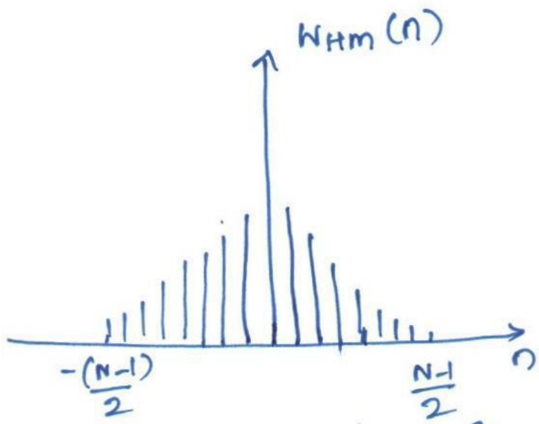
Hamming Window :-

$$W_{Hm}(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for} \quad -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

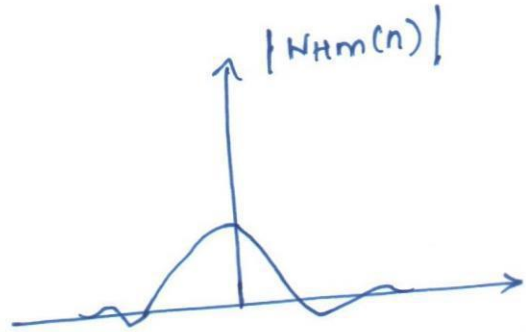
For casual

$$W_{Hm}(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for} \quad 0 \le n \le N-1$$

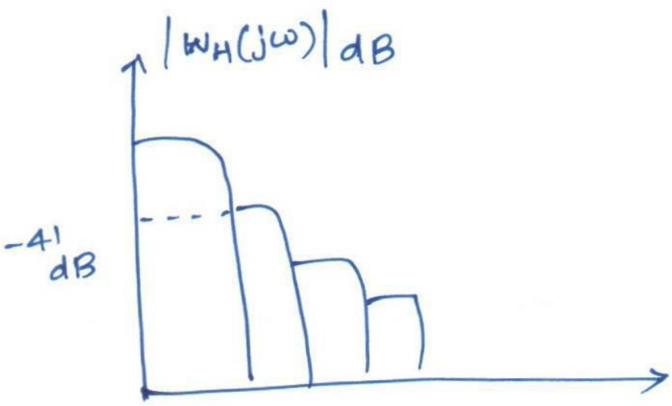
* The magnitude of the first sidelobe is $-41\,dB$.
* The stopband attenuation of LPF is $-51\,dB$
* The mainlobe width is $8\pi/N$.



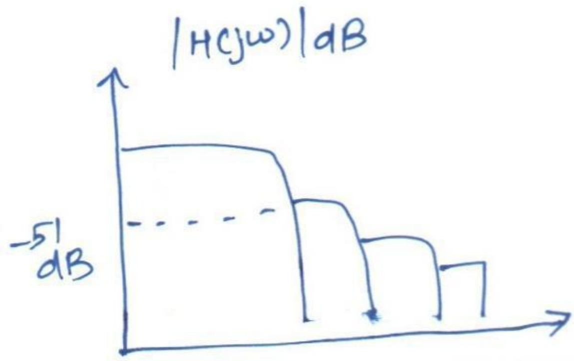
(a) Hamming Window sequence



(b) Magnitude response of Hamming window



(c) Log Magnitude response of hamming window



(d) Log Magnitude response of FIR LPF using hamming window

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Blackman Window:

$$w_B(n) = \begin{cases} 0.42 + 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \\ \end{cases}$$

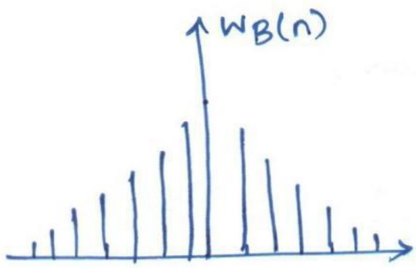
for $-\left(\frac{N-1}{2}\right) \leq n \leq \frac{N-1}{2}$

For causal.

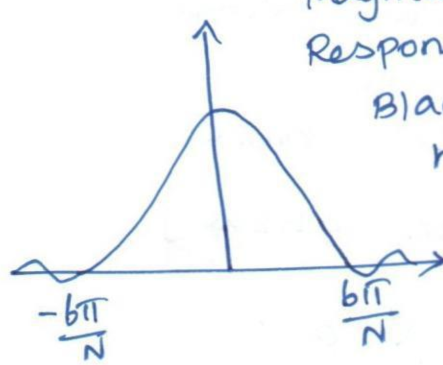
$$w_B(n) = 0.42 - 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1}$$

The additional cosine term reduces sidelobe but increases the mainlobe width to $12\pi/N$. The peak side lobe level is -58 dB. The side lobe attenuation of a lowpass filter using Blackman window is -74 dB.

Window Sequence

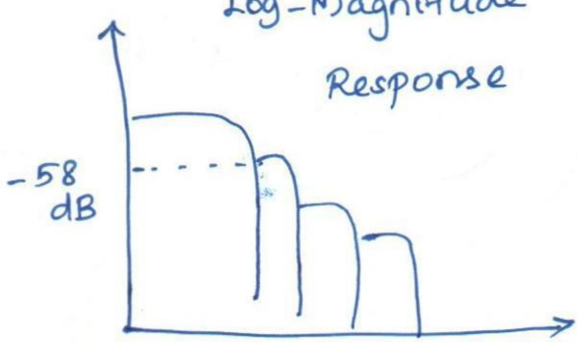


$|W_B(jw)|$



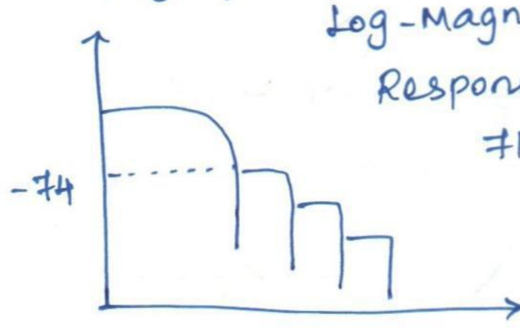
Magnitude Response of Blackman Window

$|W_B(jw)|$



Log-Magnitude Response

$|H(jw)|$



Log-Magnitude Response of FIR filter

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Summary of Window Parameters.

Window	Peak Amp of Side lobe (dB)	Mainlobe Width	Minimum Stop band Attenuation (dB)
Rectangular	−13	$4\pi/N$	−21
Triangular	−25	$8\pi/N$	−27
Hanning	−31	$8\pi/N$	−44
Hamming	−41	$8\pi/N$	−51
Blackman	−58	$12\pi/N$	−74

Filter Design

Ideal filter or zerophase [Non-Causal]	Linear phase [Causal]
$\alpha = 0$	$\alpha = \frac{N-1}{2}$

Let $N = 7$

α

0 1 2 3 4 5 6

−3 −2 −1 0 1 2 3

α

0 1 2 3 4 5 6

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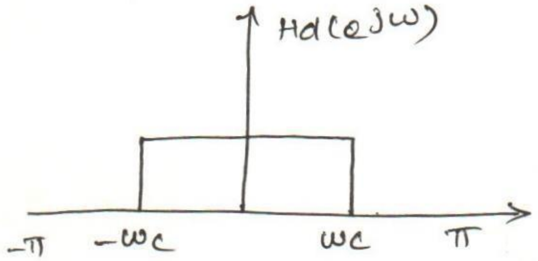
Steps to design FIR filter :- [Linear Phase].

Step1 :-

According to the given specifications, draw the freq response of the desired filter with limits [LPF, HPF BPF or BRF].

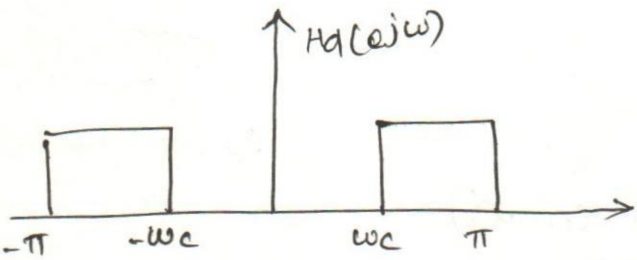
Linear phase filter $H_d(e^{j\omega}) = e^{-j\alpha\omega}$.

(i) LPF with ω_c :-



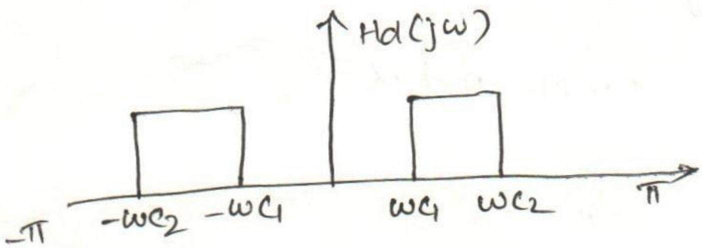
$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise.} \end{cases}$$

(ii) HPF with ω_c :-



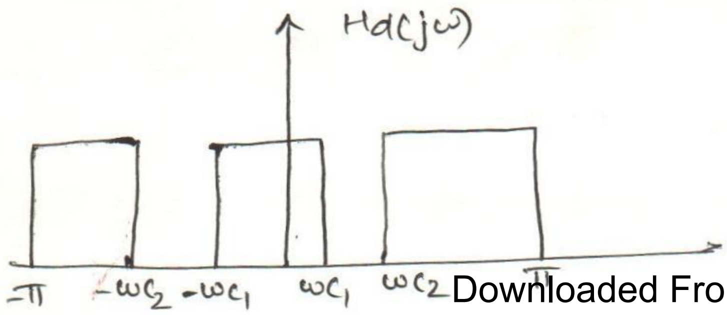
$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & \omega_c \le \omega \le \pi \\ & -\pi \le \omega \le -\omega_c \\ 0 & \text{otherwise} \end{cases}$$

(iii) BPF with ω_{c1} & ω_{c2} :-



$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & \omega_{c1} \le \omega \le \omega_{c2} \\ & -\omega_{c2} \le \omega \le -\omega_{c1} \\ 0 & \text{otherwise} \end{cases}$$

(iv) BRF with ω_{c1} & ω_{c2} :-



$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & -\omega_{c1} \le \omega \le \omega_{c1} \\ & \omega_{c2} \le \omega \le \pi \\ & -\pi \le \omega \le -\omega_{c2} \\ 0 & \text{otherwise.} \end{cases}$$

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Step 2 :-

$\alpha = \frac{N-1}{2}$ or $N = 2\alpha + 1$

Step 3 :-

Find the coefficients of $hd(n)$ using

$$hd(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Hd(e^{j\omega}) e^{j\omega n} d\omega .$$

Depending upon filter, the limits of integration changes.

(i) LPF :-

$$hd(n) = \left[\frac{\sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}\right] \quad \text{when } n \neq \alpha$$

$$hd(n) = \frac{\omega_c}{\pi} \quad \text{when } n = \alpha .$$

(ii) HPF :-

$$hd(n) = 1 - \frac{\omega_c}{\pi} \quad \text{when } n = \alpha$$

$$hd(n) = \frac{\sin[\pi(n-\alpha)] - \sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}$$

Since $\sin n\pi$ is always zero.

$$hd(n) = -\frac{\sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)} \quad \text{when } n \neq \alpha$$

(iii) BPF :-

$$hd(n) = \frac{\sin[\omega_{c2}(n-\alpha)] - \sin[\omega_{c1}(n-\alpha)]}{\pi(n-\alpha)} \quad \text{when } n \neq \alpha$$

$$hd(n) = \frac{\omega_{c2} - \omega_{c1}}{\pi} \quad \text{when } n = \alpha$$

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(iv) BRF :

$$h_d(n) = 1 - \left(\frac{\omega c_2 - \omega c_1}{\pi}\right) \quad \text{when } n = \alpha$$

$$h_d(n) = \frac{\sin[\omega c_1(n-\alpha)] - \sin[\omega c_2(n-\alpha)]}{\pi(n-\alpha)} \quad \text{when } n \neq \alpha$$

Step 4 :

Find the causal window coefficients $w(n)$ from 0 to N-1 by varying 'n' value.

(i) Rectangular Window :

$$w_R(n) = 1 \quad \text{for } 0 \text{ to } N-1$$

(ii) Hanning Window :

$$w_{Hn}(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } 0 \text{ to } N-1$$

(iii) Hamming Window :

$$w_{Hm}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } 0 \text{ to } N-1$$

(iv) Blackman Window :

$$w_B(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)$$

Step 5 : Find $h(n)$ – filter coefficients

$$h(n) = h_d(n) * w(n)$$

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Step 6 :-

Find the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Step 7 :

Find the frequency response

case(i) When N is odd & h(n) is symmetric

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\alpha\omega}$$

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \cos n\omega$$

case (ii) When N is odd & h(n) is antisymmetric

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\left[\pi/2 - \omega\left(\frac{N-1}{2}\right)\right]}$$

$$|H(e^{j\omega})| = 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \sin n\omega$$

case (iii) When N is even & h(n) is symmetric

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\alpha\omega}$$

$$|H(e^{j\omega})| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right) \cos\left(\omega\left(n - 1/2\right)\right)$$

case (iv) When N is even & h(n) is antisymmetric

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\left[\pi/2 - \omega\left(\frac{N-1}{2}\right)\right]}$$

$$|H(e^{j\omega})| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right) \sin\left(\omega\left(n - 1/2\right)\right).$$

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Problem No : 1 :

Design a filter with

$$H_d(e^{j\omega}) = e^{-j3\omega} \qquad -\pi/4 \le \omega \le \pi/4$$

$$= 0 \qquad \pi/4 < |\omega| \le \pi$$

using Hanning window with N=7.

Given data :-

* LPF
* N = 7
* $\omega_c = \pi/4$
* Hanning window.

Step 1 :

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Step 2 :- Find the value of α

$$\alpha = \frac{N-1}{2}$$

$$= \frac{7-1}{2} = 3.$$

Step 3 : Find $h_d(n)$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

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$$hd(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\alpha\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi j(n-\alpha)} \left[e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)} \right]$$

$$= \frac{\cancel{2}\cancel{j} \sin(\omega_c(n-\alpha))}{\cancel{2}\pi \cancel{j}(n-\alpha)}$$

$$\boxed{hd(n) = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)} \quad \text{for } n \neq \alpha}$$

when $n = \alpha$ $hd(n) = \frac{0}{0}$ Indeterminate. So Apply L hospital rule.

$$\boxed{hd(n) = \frac{\omega_c}{\pi}}$$

Calculate the values of hd(n) :-

For $n = 0$ to $N-1$ ie $n = 0$ to 6.

$$hd(n) = \frac{\left[\sin\left(\pi(n-3) \div 4\right)\right]}{\left[\pi(n-3)\right]}$$

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$hd(0) = 0.075 = hd(6)$

$hd(1) = 0.159 = hd(5)$

$hd(2) = 0.225 = hd(4)$

$hd(3) = 0.25$

Step 4 :-

Find the window coeffecient.

$$W_{Hn}(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } 0 \text{ to } N-1$$

$$W_{Hn}(n) = 0.5 - 0.5 \cos\left(\frac{2\pi n}{6}\right)$$

$W_{Hn}(0) = 0$

$W_{Hn}(1) = 0.25$

$W_{Hn}(2) = 0.75$

$W_{Hn}(3) = 1$

$W_{Hn}(4) = 0.75$

$W_{Hn}(5) = 0.25$

$W_{Hn}(6) = 0$

Step 5 :

The filter coeffecients are

$$h(n) = hd(n) * W_{Hn}(n)$$

$h(0) = 0 = h(6)$

$h(1) = 0.03975 = h(5)$

$h(2) = 0.16875 = h(4)$

$h(3) = 0.25$

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Step 6:

Find the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{6} h(n) z^{-n}$$

$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6}.$$

$$H(z) = 0.03975 z^{-1} + 0.165 z^{-2} + 0.25 z^{-3} + 0.165 z^{-4} + 0.03975 z^{-5}$$

$$H(z) = 0.03975 [z^{-1} + z^{-5}] + 0.168 [z^{-2} + z^{-4}] + 0.25 z^{-3}$$

Step 7:

Find the frequency response

$$H(j\omega) = |H(j\omega)| e^{-j\alpha\omega}$$

$$|H(j\omega)| = h\left(\frac{N-1}{2}\right) + 2 \sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \cos n\omega$$

when N is odd & h(n) is symmetric

$$|H(j\omega)| = h(3) + 2 \sum_{n=1}^{3} h(3-n) \cos n\omega$$

$$|H(j\omega)| = 0.25 + 2 h(2) \cos \omega + 2 h(1) \cos 2\omega + 2 h(0) \cos 3\omega$$

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$$|H(j\omega)| = 0.25 + 0.336 \cos\omega + 0.0795 \cos 2\omega .$$

Frequency response

$$|H(j\omega)| e^{-j\alpha\omega} = H(j\omega)$$

$$H(j\omega) = e^{-j3\omega} \left[0.25 + 0.336 \cos\omega + 0.0795 \cos 2\omega \right]$$

Problem No 2 :-

A bandpass FIR filter of length 7 is required. It is to have lower and upper cut-off frequencies of 3 khz, 6khz respectively and is intended to be used with a sampling frequency of 24 khz. Determine the filter coefficients using Hamming window. Consider the filter to be causal.

Step 1 :-

$f_{c1} = 3$ khz $\qquad f_{c2} = 6$ khz $\qquad F_s = 24$ khz

$$\omega_{c1} = \frac{2\pi f_{c1}}{F_s} = 2\pi \frac{3000}{24000}$$

$$\boxed{\omega_{c1} = \pi/4}$$

$$\omega_{c2} = \frac{2\pi f_{c2}}{F_s} = \frac{2\pi \times 6000}{24000}$$

$$\boxed{\omega_{c2} = \pi/2}$$

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Step 1 :

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & \omega_{c1} \le \omega \le \omega_{c2} \\ e^{-j\alpha\omega} & -\omega_{c2} \le \omega \le -\omega_{c1} \\ 0 & \text{otherwise} \end{cases}$$

Step 2 :-

$$\alpha = \frac{N-1}{2}$$

$$\alpha = \frac{7-1}{2} = 3 .$$

Step 3 :-

Find the fourier coeffecient

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega .$$

$$= \frac{1}{2\pi} \int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\alpha\omega} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{+\omega_{c1}}^{\omega_{c2}} e^{-j\alpha\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega(n-\alpha)} d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega(n-\alpha)} d\omega \right]$$

$$= \frac{1}{2\pi} \left\{ \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \Bigg|_{-\omega_{c2}}^{-\omega_{c1}} + \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} \Bigg|_{\omega_{c1}}^{\omega_{c2}} \right.$$

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$$= \frac{1}{2\pi j(n-\alpha)}\left[e^{-j\omega c_1(n-\alpha)} - e^{-j\omega c_2(n-\alpha)} + e^{j\omega c_2(n-\alpha)} - e^{+j\omega c_1(n-\alpha)} \right]$$

$$= \frac{1}{2\pi j(n-\alpha)}\left[2j\sin(\omega c_2(n-\alpha)) - 2j\sin(\omega c_1(n-\alpha)) \right]$$

$$\boxed{hd(n) = \frac{\sin(\omega c_2(n-\alpha)) - \sin(\omega c_1(n-\alpha))}{\pi(n-\alpha)} \quad \text{for } n \neq \alpha \qquad \text{when } n = \alpha \quad hd(n) = \frac{\omega c_2 - \omega c_1}{\pi}}$$

Now $\omega c_1 = \pi/4$ $\quad \omega c_2 = \pi/2$

$$hd(n) = \frac{\sin(\pi/2(n-\alpha)) - \sin(\pi/4(n-\alpha))}{\pi(n-\alpha)}$$

$hd(0) = -0.181 = hd(6)$

$hd(1) = -0.159 = hd(5)$

$hd(2) = 0.093 = hd(4)$

$hd(3) = 0.25$

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Step 4:

The Hamming window

$$w_{Hm}(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } 0 \text{ to } N-1$$

$$w_{Hm}(0) = 0.08 = w_{Hm}(6)$$

$$w_{Hm}(1) = 0.31 = w_{Hm}(5)$$

$$w_{Hm}(2) = 0.77 = w_{Hm}(4)$$

$$w_{Hm}(3) = 1$$

Step 5:

Find filter coeffecients $h(n) = hd(n) * w_{Hm}(n)$

$$h(0) = -0.0145 = h(6)$$

$$h(1) = -0.049 = h(5)$$

$$h(2) = 0.07161 = h(4)$$

$$h(3) = 0.25$$

Step 6:

The Transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(z) = -0.0145\left[1 + z^{-6}\right] - 0.049\left[z^{-1} + z^{-5}\right] + 0.07161\left[z^{-2} + z^{-4}\right] + 0.25 z^{-3}$$

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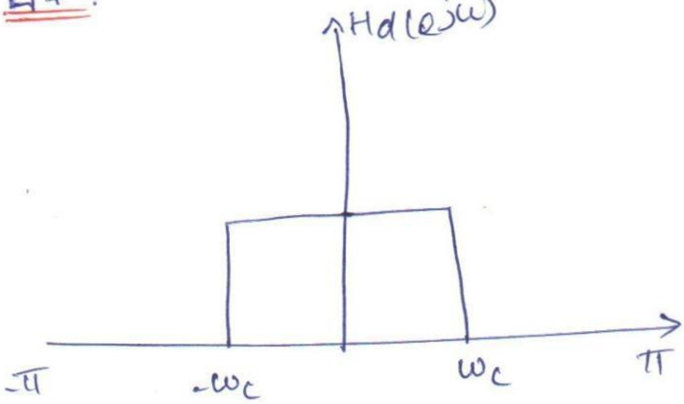
Design of Ideal FIR filter/zero phase using window technique [Non-causal].

Step 1:-

From the given specifications, draw the LPF, HPF, BPF or BRF and $\alpha = 0$

LPF:

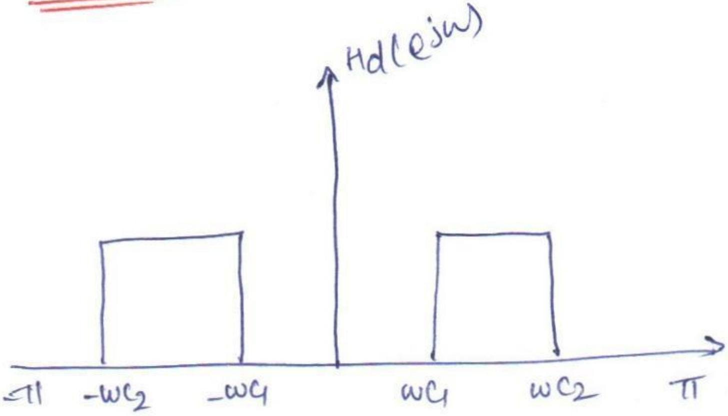


$$H_d(e^{j\omega}) = \begin{cases} 1 & -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

HPF:-

$$H_d(e^{j\omega}) = \begin{cases} 1 & \omega_c \le |\omega| \le \pi \\ 0 & \text{otherwise} \end{cases}$$

BPF:



$$H_d(e^{j\omega}) = \begin{cases} 1 & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0 & \text{otherwise} \end{cases}$$

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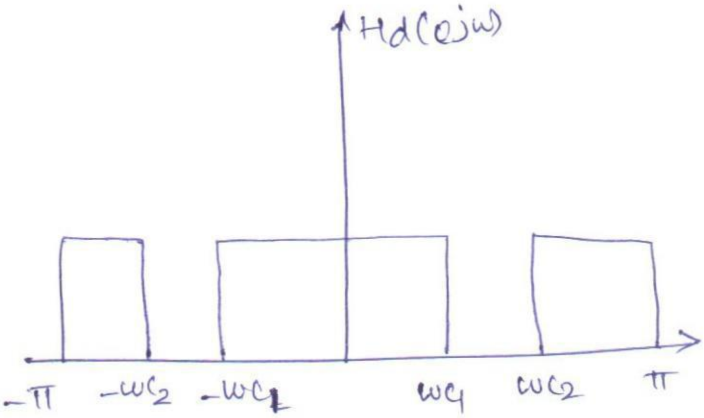
BPF or BSF:



$$H_d(e^{j\omega}) = \begin{cases} 1 & -\omega_{c1} \le \omega \le \omega_{c1} \\ 1 & \omega_{c2} \le |\omega| \le \pi \end{cases}$$

Step 2: Find the fourier coefficients

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega$$

Depending upon the filter, the limits of integration would change.

(i) **LPF:**

$$h_d(n) = \frac{\sin(\omega_c n)}{\pi n} \quad \text{when } n \neq 0 \quad (\alpha = 0)$$

$$h_d(n) = \frac{\omega_c}{\pi} \quad \text{when } n = 0$$

(ii) **HPF:**

$$h_d(n) = \frac{\sin \pi n - \sin \omega_c n}{\pi n}$$

Since $\sin n\pi = 0$

$$h_d(n) = \frac{-\sin \omega_c n}{\pi n} \quad \text{when } n \neq 0$$

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& $h_d(n) = 1 - \frac{\omega_c}{\pi}$ when $n = 0$

(iii) BPF:

$$h_d(n) = \frac{\sin \omega_{c2} n - \sin \omega_{c1} n}{\pi n} \quad \text{when } n \neq 0$$

$$h_d(n) = \frac{\omega_{c2} - \omega_{c1}}{\pi n} \quad \text{when } n = 0$$

(iv) BRF:

$$h_d(n) = \frac{\sin \omega_{c1} n - \sin \omega_{c2} n + \sin n\pi}{\pi n}$$

Since $\sin n\pi = 0$

$$h_d(n) = \frac{\sin \omega_{c1} n - \sin \omega_{c2} n}{\pi n} \quad \text{when } n \neq 0$$

& $h_d(n) = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$ when $n = 0$

* calculate $h_d(n)$ values from $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

For Eg if $N = 11$

calculate from -5 to 5

Step 3: Find the Non-causal window coefficients $w(n)$ from $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ by varying 'n' value

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(i) Rectangular window:

$$w_R(n) = 1 \quad \text{for} \quad -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

(ii) Hanning window:

$$W_{Hn}(n) = 0.5 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for} \quad -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

(iii) Hamming window:

$$W_{Hm}(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for} \quad -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

(iv) Blackman window:

$$W_B(n) = 0.42 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right)$$

$$\text{for} \quad -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

Step 4: Find the filter coefficients $h(n)$ (Non-causal)

$$h(n) = h_d(n) * w(n).$$

Step 5: To find the realizable filter & causal filter coefficients

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The Non-causal filter coefficients are practically impossible to realize the filter. So the causal filter coefficients can be obtained by shifting the sequence to right by 'α' samples.

Step 6: Find the transfer function using causal coefficients

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Step 7: Find the frequency response when N is odd & symmetrical

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\alpha\omega}$$

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \cos n\omega .$$

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Problem No 1 :

Design an ideal high pass filter with a frequency response

$$Hd(e^{jw}) = \begin{cases} 1 , & \frac{\pi}{4} \le |w| \le \pi \\ 0 , & |w| \le \pi/4 \end{cases}$$

Find h(n) and H(z) for N = 11 using Hamming window and plot the magnitude response.

Solution.

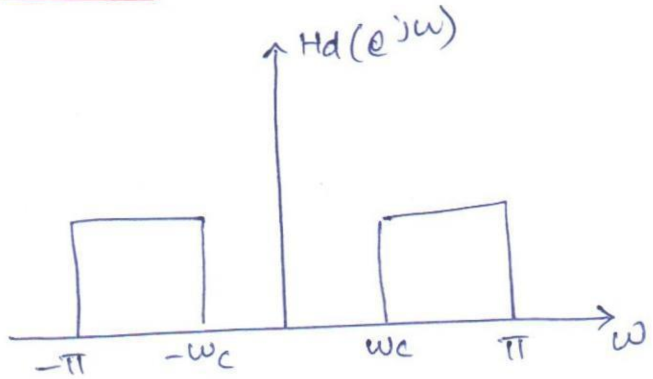
* Ideal high pass filter.

so $\alpha = 0$

* $w_c = \pi/4$.

* $N = 11$.

Step 1 :-



$$hd(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Hd(e^{jw}) e^{jwn} dw$$

$$hd(n) = \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/4} e^{jwn} dw + \int_{\pi/4}^{\pi} e^{jwn} dw \right]$$

$$hd(n) = - \frac{\sin w_c n}{\pi n} \quad \text{for} \quad n \neq 0 \quad (\text{since } \alpha = 0)$$

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$$hd(n) = 1 - \left(\frac{w_c}{\pi}\right) \quad \text{for } n = 0.$$

Step 2: Calculate Fourier coefficients.

* Calculate $hd(n)$ from $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

$N = 11$ so -5 to 5

$$hd(-5) = \frac{-\sin\left(\frac{\pi}{4}(-5)\right)}{\pi(-5)} = -0.045$$

$$hd(-4) = \frac{-\sin\left(\frac{\pi}{4}(-4)\right)}{\pi(-4)} = 0$$

$hd(-3) = -0.075$

$hd(-2) = -0.159$

$hd(-1) = -0.225$

$hd(0) = 0.75 \Rightarrow \left(1 - \frac{\pi}{4\pi}\right) \Rightarrow 1 - 0.25 \Rightarrow 0.75$

$hd(1) = -0.225$

$hd(2) = -0.159$

$hd(3) = -0.075$

$hd(+4) = 0$

$hd(+5) = -0.045$

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Step 3:

Calculate window coefficients

$$W_{Hm}(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \quad -\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2}$$

sub N = 11

$$W_{Hm}(n) = 0.54 + 0.46 \cos\left(\frac{\pi n}{5}\right)$$

$W_{Hm}(-5) = 0.08 = W_{Hm}(5)$

$W_{Hm}(-4) = 0.1678 = W_{Hm}(4)$

$W_{Hm}(-3) = 0.398 = W_{Hm}(3)$

$W_{Hm}(-2) = 0.682 = W_{Hm}(2)$

$W_{Hm}(-1) = 0.912 = W_{Hm}(1)$

$W_{Hm}(0) = 1$

Step 4:

Calculate filter coefficients

$h(n) = h_d(n) * W_{Hm}(n)$

$h(-5) = h_d(-5) * W_{Hm}(-5) = -0.045 * 0.08$

$h(-5) = -0.\cancel{0}0036 = h(5)$

$h(-4) = 0 = h(4)$

$h(-3) = -0.03 = h(3)$

$h(-2) = -0.1084 = h(2)$

$h(-1) = -0.2052 = h(1)$

$h(0) = 0.75$

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For converting to causal filter coefficients, shift by $\frac{N-1}{2} = \frac{11-1}{2} = 5$ samples right

$h(0) = h(10) = -0.0036$

$h(1) = h(9) = 0$

$h(2) = h(8) = -0.03$

$h(3) = h(7) = -0.1084$

$h(4) = h(6) = -0.2052$

$h(5) = 0.75$

Step 5:

Find the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}.$$

$$= -0.0036\left[z^0 + z^{-10}\right] - 0.03\left[z^{-2} + z^{-8}\right] - 0.2052\left[z^{-4} + z^{-6}\right]$$

$$-0.1084\left[z^{-3} + z^{-7}\right]$$

$$+ 0.75 z^{-5}$$

Step 6:

Find the magnitude response

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos n\omega$$

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sub N=11

$$|H(e^{j\omega})| = h(5) + \sum_{n=1}^{5} 2h(5-n)\cos n\omega$$

$$|H(e^{j\omega})| = h(5) + 2h(4)\cos\omega + 2h(3)\cos 2\omega + 2h(2)\cos 3\omega + 2h(1)\cos 4\omega + 2h(0)\cos 5\omega$$

$$|H(e^{j\omega})| = 0.75 - 0.4104\cos\omega - 0.2168\cos 2\omega - 0.06\cos 3\omega - 0.0072\cos 5\omega$$

ω	$\lvert H(e^{j\omega})\rvert$	$\lvert H(e^{j\omega})\rvert_{dB} = 20\log\lvert H(e^{j\omega})\rvert$
0	0.0556	-25.09
0.1π	0.149	-16.53
0.2π	0.376	-8.496
0.3π	0.6328	-3.9746
0.4π	0.8399	-1.5115
0.5π	0.9608	-0.347
0.6π	1.0108	0.093
0.7π	1.001	0.008
0.8π	0.989	-0.096
0.9π	1.0001	0.0008
1.0π	1.0108	0.09

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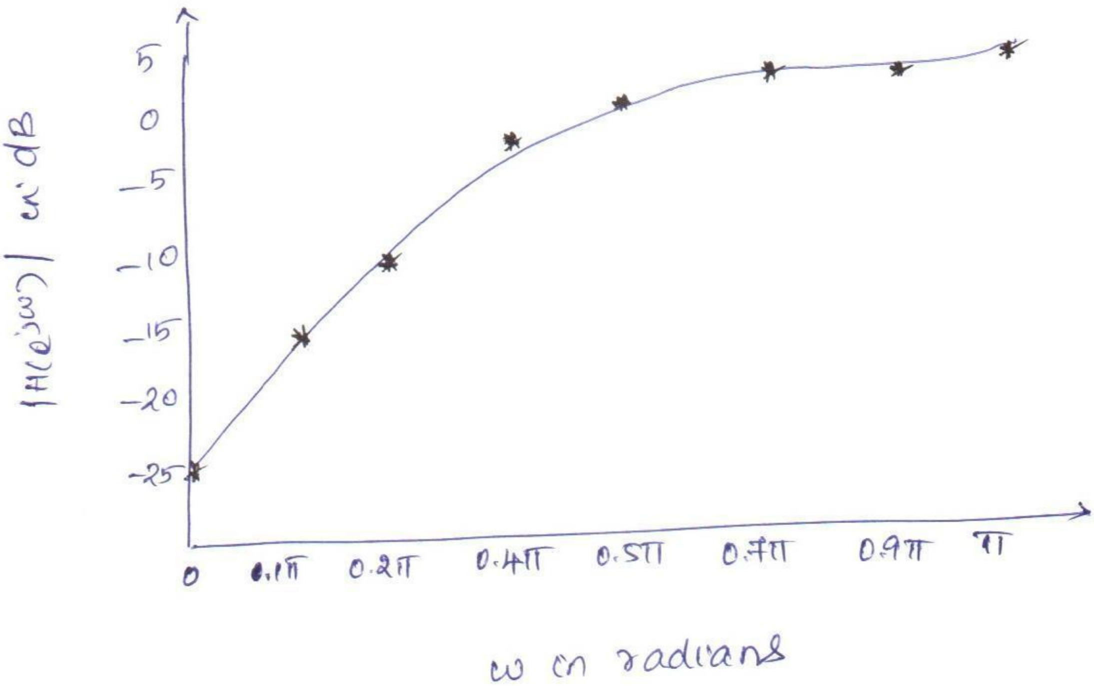

Magnitude Response of HPF.

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Design of FIR filters using Fourier series

The frequency response of an FIR filter can be represented by the Fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n}$$

where the fourier coefficients $h_d(n)$ are the desired impulse response sequence of the filter

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

Problem No 1 :

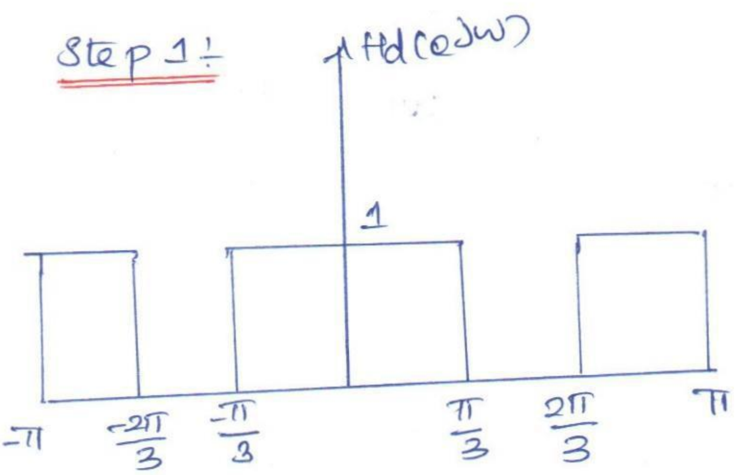
Design an ideal bandreject filter with a desired frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } |\omega| \le \pi/3 \text{ and } |\omega| \ge 2\pi/3$$
$$= 0 \quad \text{otherwise}$$

Find the value of $h(n)$ for $N=11$. Find $H(z)$ and plot the magnitude response.

Solution.

Step 1 :-



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Step 2:

$\alpha = 0$

Step 3:

$$hd(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Hd(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-2\pi/3} e^{j\omega n} d\omega + \int_{-\pi/3}^{\pi/3} e^{j\omega n} d\omega + \int_{2\pi/3}^{\pi} e^{j\omega n} d\omega\right]$$

After Integration

$$hd(n) = \frac{\sin n\pi + \sin \pi/3\, n - \sin 2\pi/3\, n}{\pi n}$$

$\sin n\pi = 0$

$$hd(n) = \frac{\sin \pi/3\, n - \sin 2\pi/3\, n}{\pi n} \quad \text{when } n \neq 0$$

Generally for BRF

$$hd(n) = \frac{\sin \omega_{c1} n - \sin \omega_{c2} n}{\pi n} \quad \text{when } n \neq 0$$

$$hd(n) = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) \quad \text{when } n = 0.$$

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Calculate hd(n) values from $-\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

i.e -5 to 5

$$hd(-5) = \frac{\left[\sin\left[(\pi \div 3)(-5)\right] - \sin\left[(2\pi \div 3)(-5)\right]\right]}{\pi(-5)}$$

$hd(-5) = 0$

$hd(-4) = -0.1378$

$hd(-3) = 0$

$hd(-2) = 0.2757$

$hd(-1) = 0$

$$hd(0) = 1 - \left(\frac{\omega c_2 - \omega c_1}{\pi}\right) = 1 - \left(\frac{2\pi/3 - \pi/3}{\pi}\right)$$

$$= 1 - \frac{1}{3} = 0.667$$

$hd(1) = 0$

$hd(2) = 0.2757$

$hd(3) = 0$

$hd(4) = -0.1378$

$hd(5) = 0$

Filter coeff = Fourier coeff

$\therefore \boxed{hd(n) = h(n)}$

The filter coefficients of the causal filter can be obtained by shifting $\frac{N-1}{2} = 5$ samples right.

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$h(0) = 0$

$h(1) = -0.1378$

$h(2) = 0$

$h(3) = 0.2757$

$h(4) = 0$

$h(5) = 0.667$

$h(6) = 0$

$h(7) = 0.2757$

$h(8) = 0$

$h(9) = -0.1378$

$h(10) = 0$

Step 4: Find the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(1) z^{-1} + h(3) z^{-3} + h(5) z^{-5} + h(7) z^{-7}$$

$$h(9) z^{-9}$$

$$= -0.1378 \left[z^{-1} + z^{-9} \right] + 0.667 z^{-5}$$

$$+ 0.2757 \left[z^{-3} + z^{-7} \right].$$

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Step 5: Find the Magnitude response

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2}-n\right)\cos n\omega$$

$$= h(5) + 2\sum_{n=1}^{5} h(5-n)\cos n\omega$$

$$= h(5) + 2\left[h(4)\cos\omega + h(3)\cos 2\omega + h(2)\cos 3\omega + h(1)\cos 4\omega + h(0)\cos 5\omega\right]$$

$$= 0.667 + 2\left[0.2757\cos 2\omega - 0.1378\cos 4\omega\right]$$

$$= 0.667 + 0.5514\cos 2\omega - 0.2756\cos 4\omega.$$

ω	$\lvert H(e^{j\omega})\rvert$	$\lvert H(e^{j\omega})\rvert$ dB
0	0.9428	−0.511
0.1π	1.0279	0.229
0.2π	1.0603	0.506
0.3π	0.719	−2.86
0.4π	0.135	−17.39
0.5π	−0.16	−15.39
0.6π	0.1357	−17.39
0.7π	0.7195	−2.86
0.8π	1.060	0.506
0.9π	1.0279	0.239
1.0π	0.9428	−0.511

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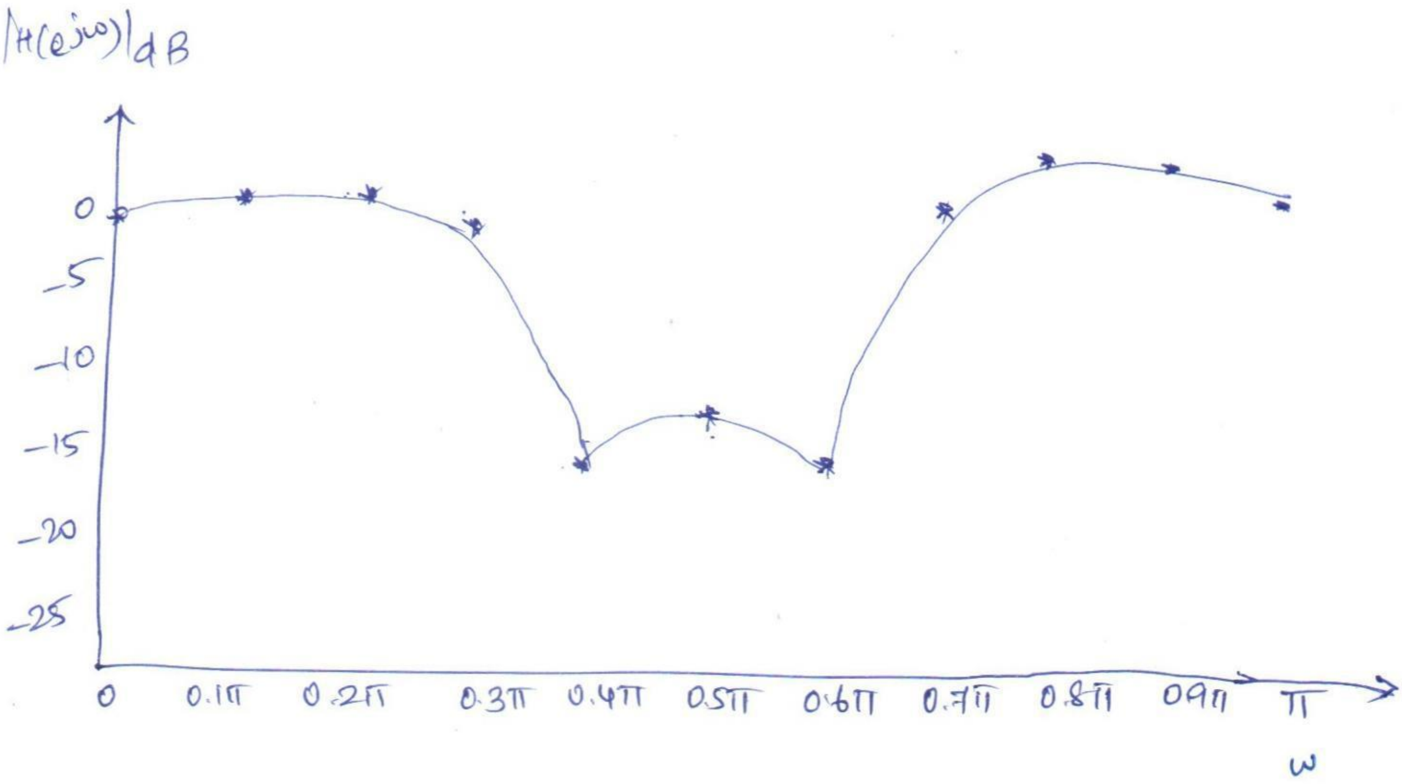

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Designing of FIR filters using frequency Sampling Method :-

In this method, a set of samples can be determined by sampling a desired frequency points response $H_d(e^{j\omega})$ at N points taking ω_k, & uniformly spaced around the circle.

The samples of desired frequency response are identified as DFT coefficients. Hence the Inverse DFT of this set of samples gives the filter coefficients.

There are 2 designs.

(i) Type-I (ii) Type II

Type-I Design :-

In type I design, the set of freq samples includes the sample at frequency $\omega=0$

* sample the desired frequency response at 'N' points taking

$$\omega_k = \frac{2\pi k}{N} \qquad k=0,1,\ldots N-1.$$

$$H(k) = H_d(e^{j\omega}) \Big|_{\omega=\omega_k = \frac{2\pi k}{N}}$$

$$H(k) = H_d\left(e^{j\frac{2\pi k}{N}}\right)$$

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The samples can be expressed in the form

$$H(K) = |H(K)| e^{-j\alpha\omega_K}$$

$$H(K) = |H(K)| e^{-j\alpha \times \frac{2\pi K}{N}}.$$

The filter coefficients can be obtained by.

$$h(n) = \frac{1}{N} \sum_{K=0}^{N-1} H(K) e^{j\frac{2\pi nK}{N}} \qquad n = 0, 1, \ldots N-1.$$

If h(n), the impulse response is a real valued signal, the frequency samples H(K) must satisfy the requirement of symmetry.

$$H(N-K) = H^*(K).$$

Using this
The filter coefficients can be written as.

$$\boxed{h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} \operatorname{Re}\left(H(K) e^{j\frac{2\pi nK}{N}}\right)\right]}$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} \operatorname{Re}\left(|H(K)| e^{-j\alpha \times \frac{2\pi K}{N}} e^{j\frac{2\pi nK}{N}}\right)\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} \operatorname{Re}\left(|H(K)| e^{j\frac{2\pi K(n-\alpha)}{N}}\right)\right]$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} |H(K)| \cos\left(\frac{2\pi K(n-\alpha)}{N}\right)\right].$$

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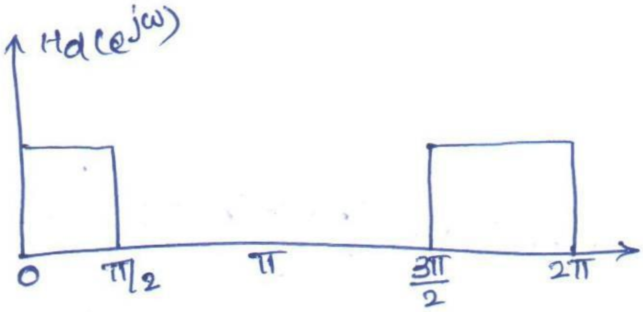
Problem No: 1

Determine the filter coefficient $h(n)$ by frequency sampling method for $N=7$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\left(\frac{N-1}{2}\right)\omega} & 0 \le |\omega| \le \pi/2 \\ 0 & \pi/2 \le |\omega| \le \pi \end{cases}$$

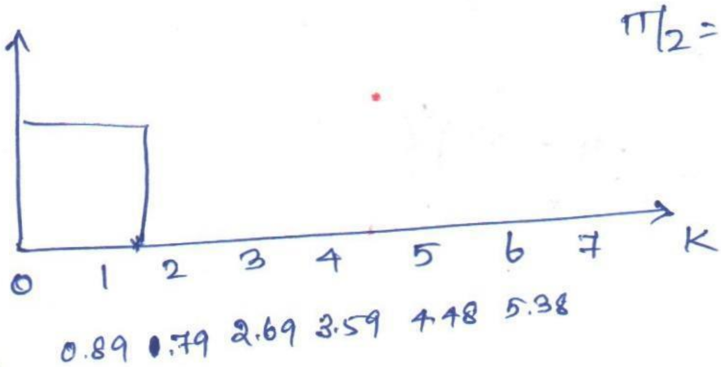
Step 1 :



$$H(K) = |H(K)| e^{-j\alpha\omega_k}$$

where $\omega_k = \frac{2\pi K}{N} = \frac{2\pi K}{7}$.

$\pi/2 = 1.57$



$$|H(K)| = \begin{cases} 1 & \text{for } k=0,1 \\ 0 & \text{for } k=2,3 \end{cases}$$

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Formula for calculating filter coeff

$$h(n) = \frac{1}{N}\left\{ H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right\}$$

After simplification

$$h(n) = \frac{1}{N}\left\{ |H(0)| + 2\sum_{k=1}^{\frac{N-1}{2}} |H(k)|\cos\left(\frac{2\pi k(n-\alpha)}{N}\right)\right\}$$

sub $N=7$ & $\alpha=3$

$$h(n) = \frac{1}{7}\left\{ 1 + 2\sum_{k=1}^{3} |H(k)|\cos\left(\frac{2\pi k(n-3)}{7}\right)\right\}.$$

$$h(n) = \frac{1}{7}\left\{ 1 + 2|H(1)|\cos\left(\frac{2\pi(n-3)}{7}\right) + \cancel{2|H(2)|\cos}^{\,0}\left(\frac{4\pi(n-3)}{7}\right) + \cancel{2|H(3)|\cos}^{\,0}\left(\frac{6\pi(n-3)}{7}\right)\right\}$$

$$\boxed{h(n) = \frac{1}{7}\left\{ 1 + 2\cos\left(\frac{2\pi(n-3)}{7}\right)\right\}}$$

$h(0) = -0.1146$

$h(1) = 0.0793$

$h(2) = 0.321$

$h(3) = 0.4286$

$h(4) = 0.321$

$h(5) = 0.0793$

$h(6) = -0.1146$

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To find Transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(z) = -0.1146\left[z^{0} + z^{-6}\right] + 0.0793\left[z^{-1} + z^{-5}\right]$$
$$+ 0.321\left[z^{-2} + z^{-4}\right] + 0.4286\, z^{-3}.$$

Problem NO : 2

Using sampling method, design a BPF with the following lower cut off frequency 1000 hz and upper cut-off frequency 3000 hz. The sampling freq is 8000 Hz. Determine the filter coefficients for N = 7.

Step (i)

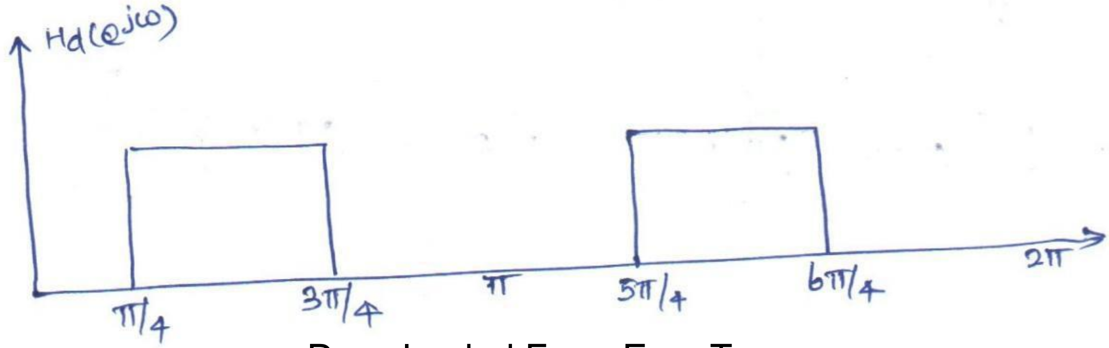
$N = 7$, $\alpha = \boxed{3}$

Convert hz to rad.

$$\omega_{c1} = \frac{2\pi f_{c1}}{F_s} = \frac{2\pi \times 1000}{8000} = \pi/4 = \boxed{0.785}\ \text{r/sec.}$$

$$\omega_{c2} = \frac{2\pi f_{c2}}{F_s} = \frac{2\pi \times 3000}{8000} = 3\pi/4 = \boxed{2.356}\ \text{rad/sec.}$$

BPF :



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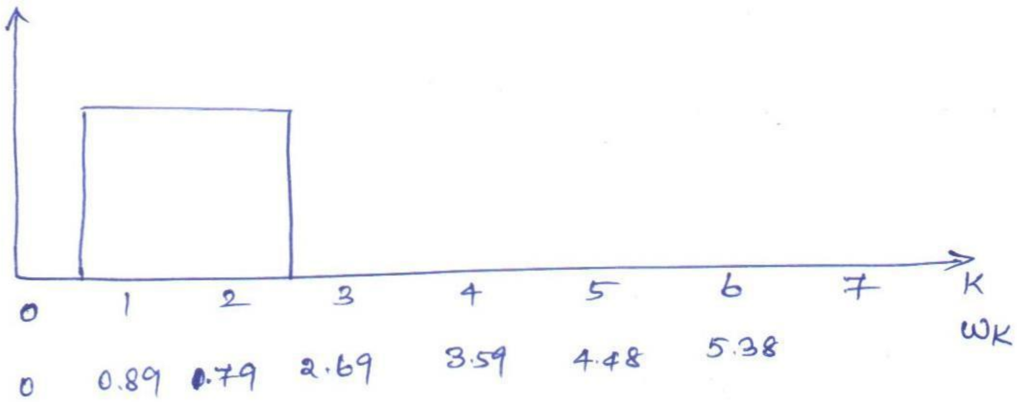
$$H(k) = |H(k)| e^{-j\alpha\omega_k}$$

$$\omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{7}.$$



$$\boxed{|H(k)| = \begin{cases} 1 & \text{for} \quad k = 1, 2 \\ 0 & \text{for} \quad k = 0, 3 \end{cases}}$$

Formula for calculating filter coeff.

$$h(n) = \frac{1}{N}\left\{ H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k) e^{\frac{j2\pi nk}{N}}\right]\right\}$$

After simplification

$$h(n) = \frac{1}{N}\left\{ |H(0)| + 2\sum_{k=1}^{\frac{N-1}{2}} |H(k)| \cos\left(\frac{2\pi k(n-\alpha)}{N}\right)\right\}$$

sub $N = 7, \ \alpha = 3$

$$h(n) = \frac{1}{7}\left\{ |H(0)| + 2\sum_{k=1}^{3} |H(k)| \cos\left(\frac{2\pi k(n-3)}{7}\right)\right\}$$

$$h(n) = \frac{1}{7}\left\{ 2\cos\left(\frac{2\pi(n-3)}{7}\right) + 2\cos\left(\frac{4\pi(n-3)}{7}\right)\right\}$$

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$h(0) = -0.0793$

$h(1) = -0.321$

$h(2) = 0.1146$

$h(3) = 0.5714$

$h(4) = 0.1146$

$h(5) = -0.3210$

$h(6) = -0.0793$

Find the Transfer function.

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$H(z) = -0.0793 \left[1 + z^{-6}\right] - 0.321 \left[z^{-1} + z^{-5}\right] + 0.1146 \left[z^{-2} + z^{-4}\right] + 0.5714 z^{-3}$$

Problem. NO : 3

Determine the coefficients of a linear phase FIR filter of length M = 15 has a symmetric unit sample response and a frequency response that satisfies the conditions'

$$H\left(\frac{2\pi k}{15}\right) = \begin{cases} 1 & \text{for } k = 0,1,2,3 \\ 0.4 & \text{for } k = 4 \\ 0 & \text{for } k = 5,6,7 \end{cases}$$

Solution.

$N = 15$

$$\alpha = \frac{N-1}{2} = 7.$$

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Formula.

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left(H(k)e^{j\frac{2\pi nk}{N}}\right)\right\},$$

After simplification

$$h(n) = \frac{1}{N}\left\{|H(0)| + 2\sum_{k=1}^{\frac{N-1}{2}} |H(k)|\cos\left(\frac{2\pi k(n-\alpha)}{N}\right)\right\}$$

Sub $N=15$, $\alpha=7$

$$h(n) = \frac{1}{15}\left\{1 + 2\sum_{k=1}^{7} |H(k)|\cos\left(\frac{2\pi k(n-7)}{15}\right)\right\}.$$

$$h(n) = \frac{1}{15}\left\{1 + 2\cos\left(\frac{2\pi(n-7)}{15}\right) + 2\cos\left(\frac{4\pi(n-7)}{15}\right) + 2\cos\left(\frac{6\pi(n-7)}{15}\right) + \cos\left(\frac{8\pi(n-7)}{15}\right)\right\}.$$

Sub 'n' value

$h(0) = -0.0141 = h(14)$

$h(1) = -0.0019 = h(13)$

$h(2) = 0.04 = h(12)$

$h(3) = 0.0122 = h(11)$

$h(4) = -0.0913 = h(10)$

$h(5) = -0.0181 = h(9)$

$h(6) = 0.3133 = h(8)$

$h(7) = 0.52$

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$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}.$$

$$H(z) = -0.0141\left[z^{0} + z^{-14}\right] - 0.0019\left[z^{-1} + z^{-13}\right] + 0.04\left[z^{-2} + z^{-12}\right] + 0.0122\left[z^{-3} + z^{-11}\right] + -0.0913\left[z^{-4} + z^{-10}\right] - 0.0181\left(z^{-5} + z^{-9}\right] + 0.313\left[z^{-6} + z^{-8}\right] + 0.52 z^{-1}$$

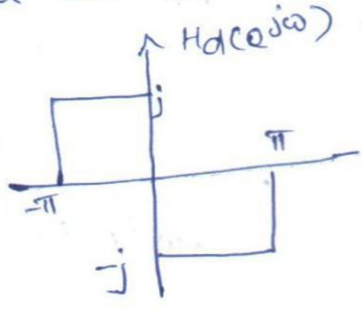
Problem NO.4 !

Design a Hilbert transformer having frequency response

$$H_d(e^{j\omega}) = j \quad \text{for } -\pi \le \omega \le 0$$
$$= -j \quad \text{for } 0 \le \omega \le \pi$$

using rectangular window and $N = 11$. Plot the frequency response.

Solution. For linear phase, introduce a delay of α samples.



$$H_d(e^{j\omega}) = \begin{cases} je^{-j\alpha\omega} & \text{for } -\pi \le \omega \le 0 \\ -je^{-j\alpha\omega} & \text{for } 0 \le \omega \le \pi \end{cases}$$

step 2:

$$\alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$$

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$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$h_d(n) = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} je^{-j\alpha\omega} e^{j\omega n} d\omega + \int_{0}^{\pi} -je^{-j\alpha\omega} e^{j\omega n} d\omega.\right.$$

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{0} je^{j\omega(n-\alpha)} d\omega - \int_{0}^{\pi} je^{+j\omega(n-\alpha)} d\omega.\right.$$

$$= \frac{j}{2\pi}\left[\left.\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right|_{-\pi}^{0} - \left.\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right|_{0}^{\pi}\right.$$

$$= \frac{1}{2\pi(n-\alpha)}\left[e^{0} - e^{-j\pi(n-\alpha)} - e^{j\pi(n-\alpha)} + e^{0}\right]$$

$$= \frac{1}{2\pi(n-\alpha)}\left[2 - 2\cos\pi(n-\alpha)\right]$$

$$\boxed{h_d(n) = \frac{1-\cos\pi(n-\alpha)}{\pi(n-\alpha)} \quad \text{when } n \neq \alpha}$$

Apply L'Hopital rule

when $n=\alpha$

$$\boxed{h_d(n) = 0.}$$

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Step 4:

Find hd(n) for n=0 to N-1

$$hd(n) = \frac{1 - \cos\pi(n-5)}{\pi(n-5)} \quad \text{when } n \neq \infty$$

$hd(0) = -0.1273$

$hd(1) = 0$

$hd(2) = -0.2122$

$hd(3) = 0$

$hd(4) = -0.6366$

$hd(5) = 0$

$hd(6) = 0.6366$

$hd(7) = 0$

$hd(8) = 0.2122$

$hd(9) = 0$

$hd(10) = 0.1273$

It is anti-symmetric

$h(n) = -h(N-1-n)$.

Step 5:

Rectangular window

$w_R(n) = 1$ for $n=0$ to 10.

Step 6:

$h(n) = hd(n) * w_R(n)$

$h(0) = -0.1273 = -h(10)$

$h(1) = 0 = -h(9)$

$h(2) = -0.2122 = -h(8)$

$h(3) = 0 = -h(7)$

$h(4) = -0.6366 = -h(6)$

$h(5) = 0$

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Step 7: Transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}.$$

$$H(z) = -0.1273\left[1 - z^{-10}\right]$$
$$- 0.2122\left[z^{-2} - z^{-8}\right]$$
$$- 0.6366\left[z^{-4} - z^{-6}\right]$$

Step 8: Frequency response.

when N is odd and h(n) is antisymmetric

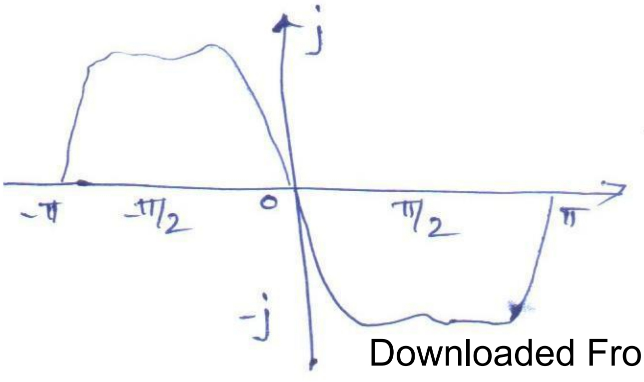
$$H(e^{j\omega}) = e^{-j(\alpha\omega)} e^{j\pi/2} |H(e^{j\omega})|$$

$$|H(e^{j\omega})| = h\frac{(N-1)}{2} + 2\sum_{n=1}^{\frac{N-1}{2}} h\left(\frac{N-1}{2} - n\right) \sin\omega.$$

$$= 0 + 2\sum_{n=1}^{5} h(5-n) \sin\omega$$

$$= 2h(4)\sin\omega + 2h(3)\cancel{\sin 2\omega}^{0} + 2h(2)\sin 3\omega$$
$$+ 2h(1)\cancel{\sin 4\omega}^{0} + 2h(0)\sin 5\omega.$$

$$|H(e^{j\omega})| = -1.2732 \sin\omega - 0.4244 \sin 3\omega$$
$$- 0.2546 \cos 5\omega.$$



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Problem. No. 4

· Design a differentiator using Hamming window with $N=8$

$$H_d(e^{j\omega}) = \begin{cases} j\omega & -\pi \le \omega \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

To get the linear phase differentiator introduce a delay of α samples

$$H_d(e^{j\omega}) = \begin{cases} j\omega e^{-j\alpha\omega} & -\pi \le \omega \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Step: 1

$$\alpha = \frac{N-1}{2} = \frac{8-1}{2} = 7/2$$

Step: 2 find the value of $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega\, e^{-j\alpha\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega\, e^{j\omega(n-\alpha)}\, d\omega$$

$u = \omega$ $\qquad dv = e^{j\omega(n-\alpha)}$

$u' = 1$ $\qquad v' = \dfrac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}$

$$v'' = \frac{e^{j\omega(n-\alpha)}}{-(n-\alpha)^2}$$

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$$= \frac{j}{2\pi}\left[\omega \frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)} + \frac{e^{j\omega(n-\alpha)}}{(n-\alpha)^2} \right]_{-\pi}^{\pi}$$

$$= \frac{j}{2\pi}\left[\pi \frac{e^{j\pi(n-\alpha)}}{j(n-\alpha)} + \pi \frac{e^{-j\pi(n-\alpha)}}{j(n-\alpha)} + \frac{e^{j\pi(n-\alpha)}}{(n-\alpha)^2} - \frac{e^{-j\pi(n-\alpha)}}{(n-\alpha)^2} \right]$$

$$= \frac{j}{2\pi}\left[\pi \frac{2\cos\pi(n-\alpha)}{j(n-\alpha)} + \frac{2j\sin\pi(n-\alpha)}{(n-\alpha)^2} \right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\pi\cos\pi(n-\alpha) - \frac{\sin\pi(n-\alpha)}{(n-\alpha)} \right]$$

$$= \frac{1}{(n-\alpha)}\left[\cos\pi(n-\alpha) - \frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)} \right]$$

$$hd(n) = \frac{\cos\pi(n-\alpha)}{(n-\alpha)} - \frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)^2}$$

when N is odd the sin term becomes zero

when N is even the cos term becomes zero

Here N is even so, $hd(n) = -\frac{\sin(\pi(n-\alpha))}{\pi(n-\alpha)^2}$

Step: 3

$hd(0) = -0.025$

$hd(1) = 0.050$

$hd(2) = -0.141$

$hd(3) = 1.273$

$hd(4) = -1.273$

$hd(5) = 0.141$

$hd(6) = -0.050$

$hd(7) = 0.025$

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Step: 4 To find window coefficient

$$w_{Hm}(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) \quad \text{for } 0 \le n \le N-1$$

$$= 0.54 - 0.46 \cos\left(\frac{2\pi n}{7}\right) \quad \text{for } 0 \le n \le 7.$$

$w_{Hm}(0) = 0.08$ $\quad$ $w_{Hm}(4) = 0.954$

$w_{Hm}(1) = 0.253$ $\quad$ $w_{Hm}(5) = 0.642$

$w_{Hm}(2) = 0.642$ $\quad$ $w_{Hm}(6) = 0.253$

$w_{Hm}(3) = 0.954$ $\quad$ $w_{Hm}(7) = 0.08.$

Step: 5

$$h(n) = h_d(n) * w_{Hm}(n)$$

$h(0) = -0.002$ $\quad$ $h(4) = -1.214$

$h(1) = 0.012$ $\quad$ $h(5) = 0.090$

$h(2) = -0.090$ $\quad$ $h(6) = -0.012$

$h(3) = 1.214$ $\quad$ $h(7) = 0.002.$

Step: 6

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= 0.002\left[z^{-7} - 1\right] + 0.012\left[z^{-1} - z^{-6}\right] + 0.090\left[z^{-5} - z^{-2}\right] + 1.214\left[z^{-3} - z^{-4}\right].$$

Step: 7 Freq response.

$$H(j\omega) = |H(j\omega)|\, e^{j(\pi/2 - \alpha\omega)}$$

$$|H(j\omega)| = \sum_{n=1}^{N/2} 2h\left(N/2 - n\right) \sin\left(\omega\left(n - 1/2\right)\right)$$

$$= \sum_{n=1}^{4} 2h(4-n) \sin\left(\omega\left(n - 1/2\right)\right)$$

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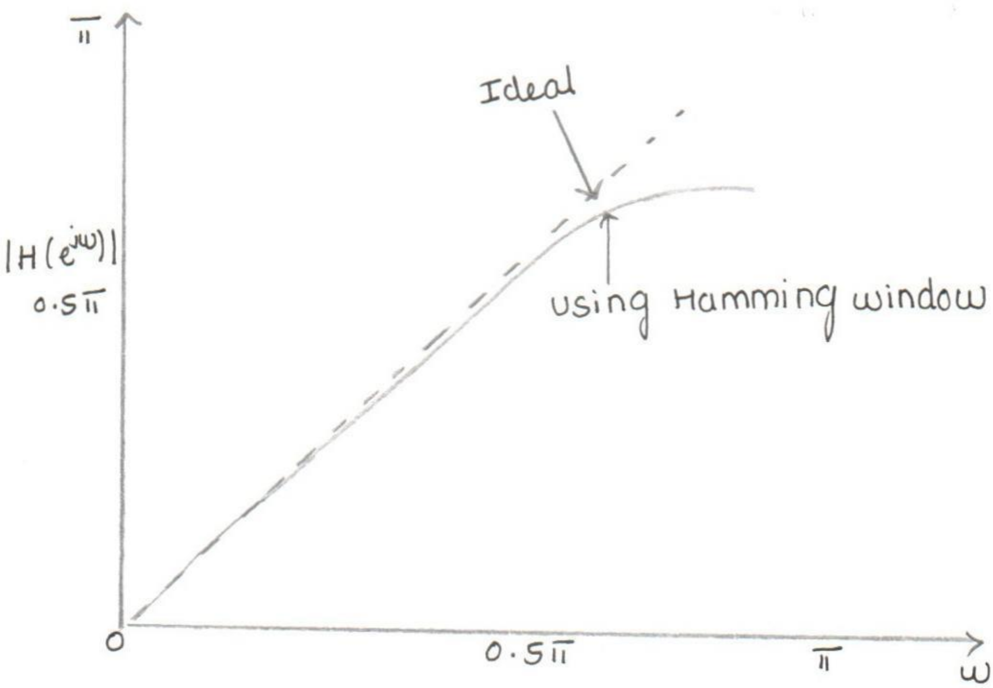
$$= 2h(3)\sin\omega(0.5) + 2h(2)\sin\omega(1.5) + 2h(1)\sin\omega(2.5) + 2h(0)\sin\omega(3.5)$$

$$= 2.428\sin(0.5\omega) - 0.18\sin(1.5\omega) + 0.024\sin(2.5\omega) - 0.004\sin(3.5\omega)$$

ω (in radians)	0	$\pi/9$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$H(e^{j\omega})$	0	0.347	0.521	0.784	1.048	1.57	2.078	2.30	2.481	2.642



frequency response of differentiator using Hamming window.

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Realization of FIR Systems :-

The FIR systems can be realized in different ways. They are

(i) Transverse or Direct form Realization

(ii) Cascade Realization

(iii) Linear phase Realization

(iv) Lattice Structure

(v) Polyphase Realization.

It needs adders, multipliers and delay elements for realization.

Direct Form Realization of FIR systems :-

Problem No 1 :

Obtain the direct form realization for the following transfer function.

$$H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$$

Expanding the transfer function

$$H(z) = 1 - \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} + \frac{5}{64}z^{-3} - \frac{3}{16}z^{-4}$$

No of delay element : 4

$$N-1 = 4$$

$$N = 5$$

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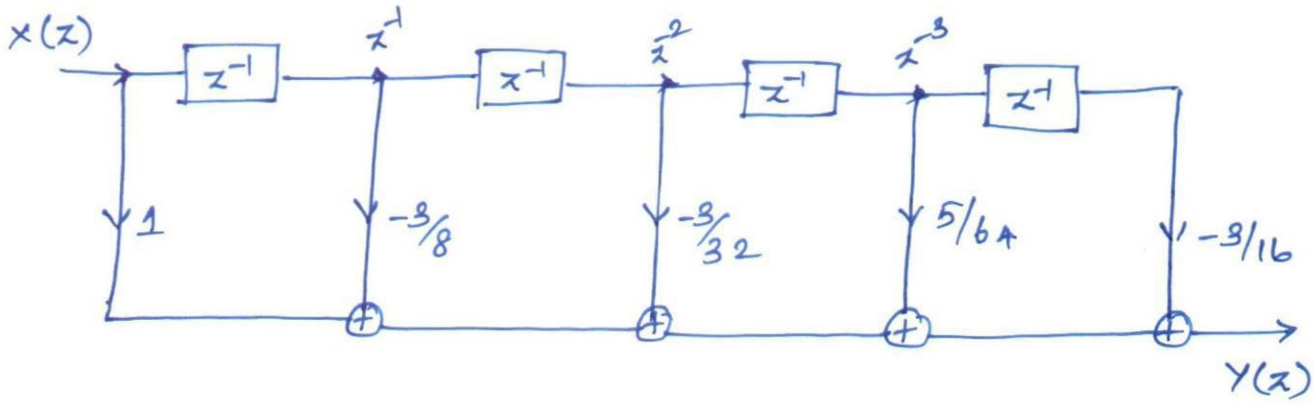


N-1 delay elements

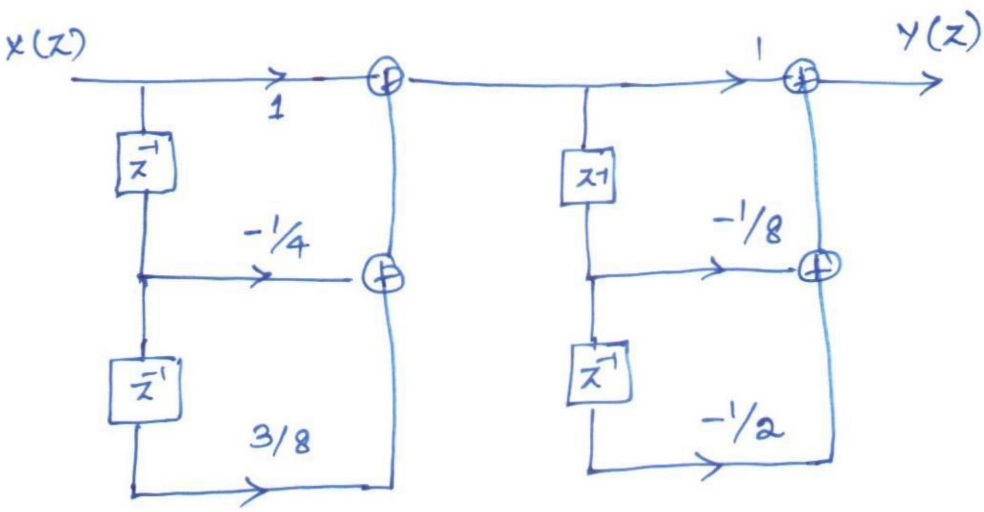
N multipliers

N-1 adders.

Problem No 2:

Obtain the cascade form realisation.

$$H(z) = \left(1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2}\right)\left(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2}\right)$$



Linear phase Realization (Minimum Multiplier)

If the impulse response is symmetric,

$h(n) = h(N-1-n)$, then the symmetry property is used to reduce the multipliers.

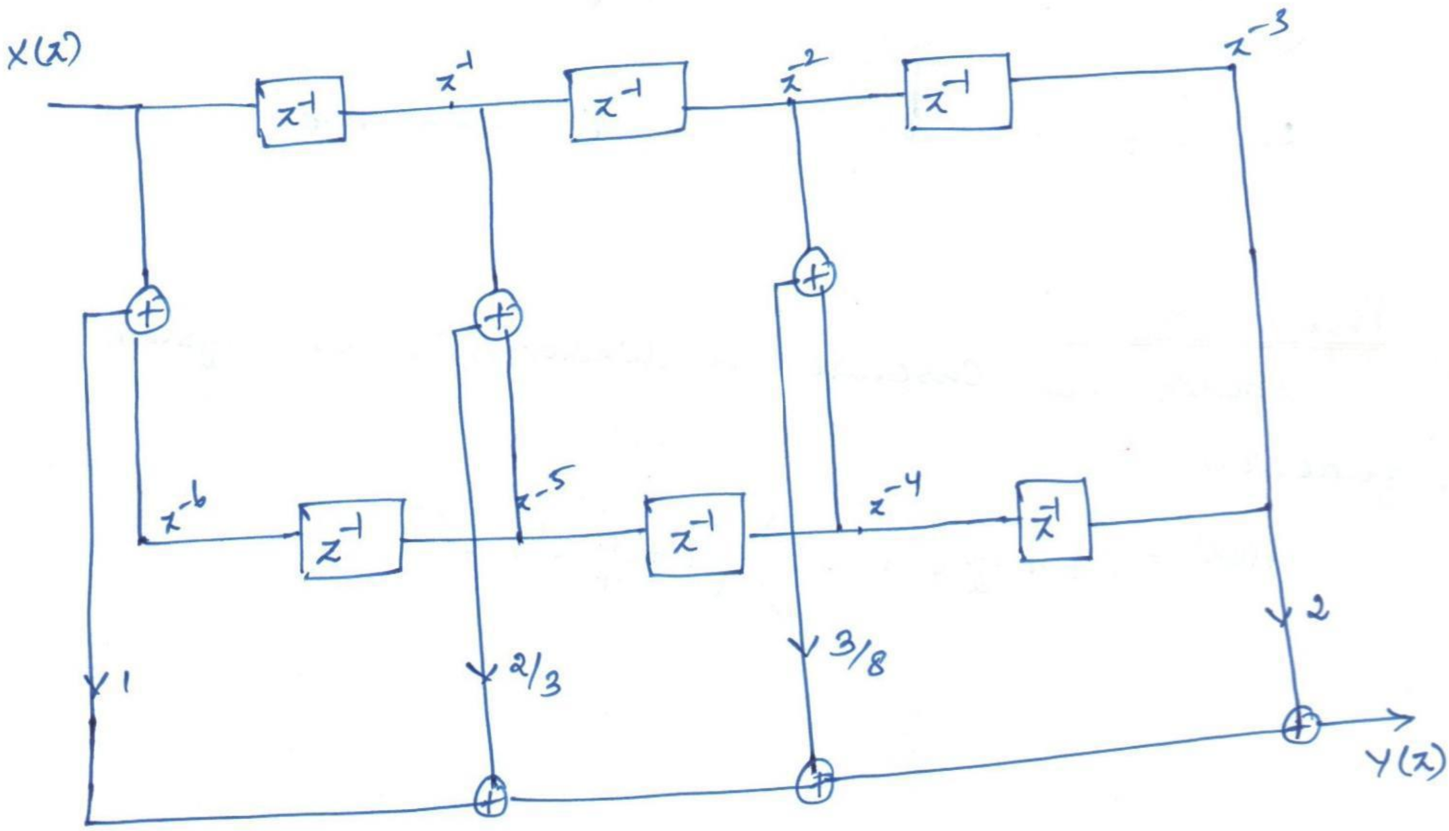
Problem No 1:

Obtain the linear phase realization

$$H(z) = 1 + \frac{2}{3}z^{-1} + \frac{3}{8}z^{-2} + 2z^{-3} + \frac{3}{8}z^{-4} + \frac{2}{3}z^{-5} + z^{-6}$$

N.o of delay element = 6

$N-1 = 6$, $N = 7$



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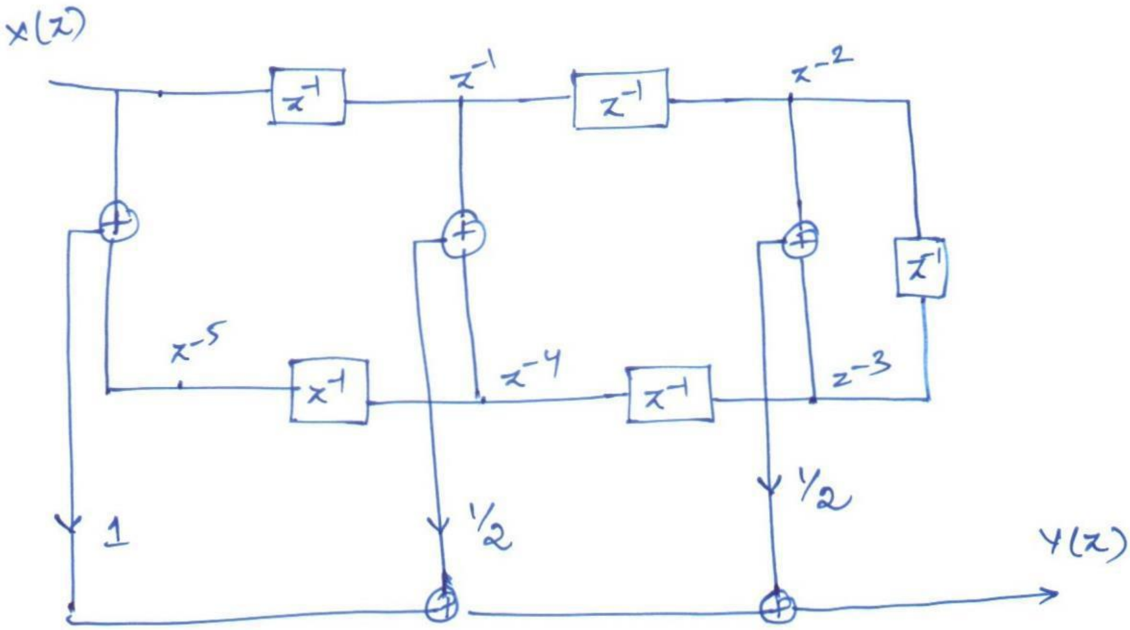
Problem No 4: Obtain the linear phase realization of the following H(z)

$$H(z) = 1 + \tfrac{1}{2}z^{-1} + \tfrac{2}{3}z^{-2} + \tfrac{2}{3}z^{-3} + \tfrac{1}{2}z^{-4} + z^{-5}$$

No of delay element = 5

$N-1 = 5 \quad N = 6$



Problem No 5:

Obtain the cascade realization of the system function

$$H(z) = \left(1 + \tfrac{1}{2}z^{-1} + z^{-2}\right)\left(1 + \tfrac{1}{4}z^{-1} + z^{-2}\right)$$

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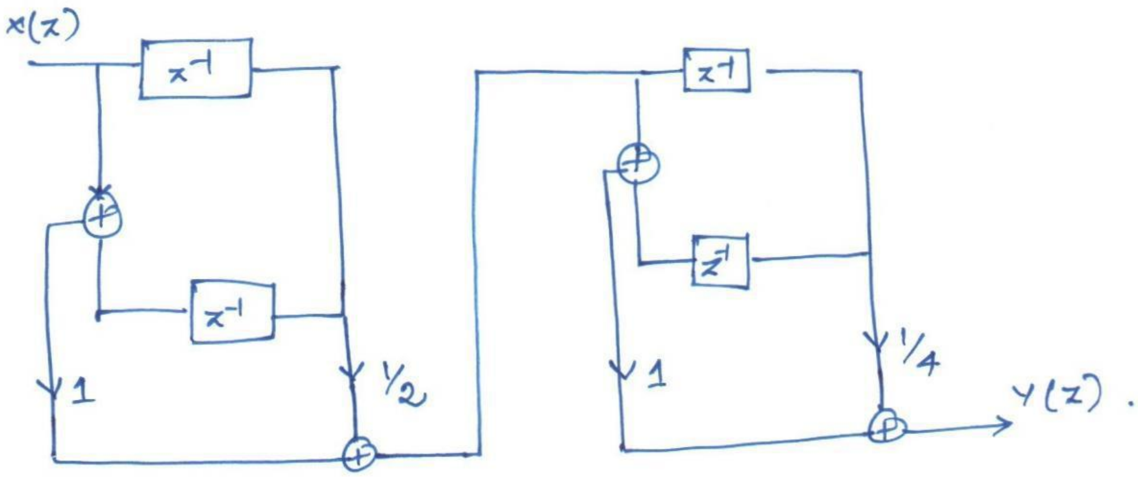

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UNIT - IV

Finite word length effects

→ Fixed Point & Floating Point Number representations

→ comparison

→ Truncation and Rounding errors

→ Quantization Noise

→ Derivation for quantization noise power

→ coefficient quantization error

→ Product Quantization error

→ overflow error

→ Round off Noise power

→ Limit cycle Oscillations due to product roundoff and overflow error

→ signal scaling.

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Types of Number Representation:

There are three forms used to represent the numbers in a digital computer

(i) Fixed point representation

(ii) Floating point representation

(iii) Block floating point representation.

Fixed point representation

In this the position of the binary point is fixed. The bit to the right represent the fractional part of the number and those to the left represent the integer part.

$$(1.75)_{10} \Rightarrow (01.1100)_2$$

The -ve numbers are represented in three forms

(i) Sign magnitude form

(ii) one's complement form

(iii) Two's complement form.

Sign-Magnitude form!

The MSB is set to 1 to represent the sign.

For eg. $(-0.5)_{10}$.

$0.5 \Rightarrow (0.100)_2$

$-0.5 \Rightarrow (1.100)_2$

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One's complement form:

* The +ve number is same as signmagnitude but the negative number is obtained by taking one's complement of the +ve number

For eg -0.875

$+0.875 \Rightarrow (0.111)_2$

→ complementing all the bits

$-0.875 \Rightarrow (1.000)_2$

Two's complement form:

* The +ve numbers are represented as in sign magnitude and one's complement form But the negative number is obtained by taking two's complement of the +ve number

For eg. -0.875

$+0.875 \Rightarrow (0.111)_2$

→ complement the bits and add '1' to LSB

⇓

$(-0.875) \Rightarrow (1.001)_2$

Problem No 1:

Express $-5/8$ in signmagnitude, one's complement and two's complement.

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$(+5/8) \Rightarrow (0.101)_2$

$(-5/8)$ in sign magnitude $\Rightarrow (1.101)_2$

$(-5/8)$ in 1's complement $\Rightarrow (1.010)_2$

$(-5/8)$ in 2's complement $\Rightarrow$ 1.010 (complementing)

$$\begin{array}{r} 1.010 \\ +1 \\ \hline 1.011 \\ \hline \end{array}$$

$\Rightarrow (1.011)_2$.

Floating Point Representation

→ it is employed to represent larger numbers range in a given binary word size.

It is represented as

$$F = 2^c . M$$

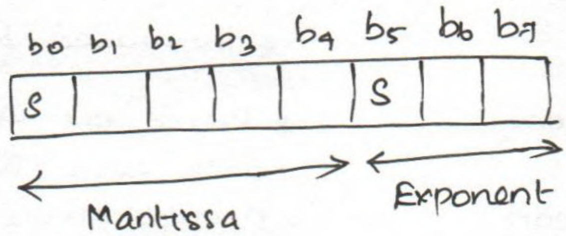
$M \rightarrow$ mantissa and the range is $1/2 \leq M \leq 1$

$c \rightarrow$ Exponent (it is either +ve or -ve).

Let us consider 8 bit representation.

five bit for mantissa and three bit exponent.

Left most bit is used for sign.



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The range of numbers

$\pm 7.8125 \times 10^{-3}$ to ± 15.5

Problem No 2 :

convert $(+0.125)_{10}$ to binary in floating point representation

step 1 : convert to binary

step 2 : Binary point is moved to a position such that MSB of mantissa is one and the exponent is adjusted accordingly.

$(+0.125)_{10} \Rightarrow (0.001)_2 \Rightarrow 0.0010 \times 2^0$

Move the binary point to right so that MSB is one.

$$0.1 \times 2^{-2}$$

$$\Downarrow$$

$$0.1000 \times 2^{-10}$$

0	1	0	0	0	1	1	0

Comparison of Fixed point & Floating point Arithmetic

Fixed point	Floating point
* Fast	* slow
* Economical	* costlier
* Small dynamic range	* Increased Dynamic range
* Round off errors occur only for addition	* Round off errors occur with both addition and multiplication.
* overflow occur in addition	* overflow does not arise
* used in small computers	* used in larger computers.

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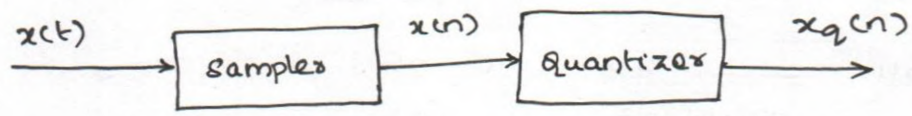
Quantization :-

The process of converting a discrete-time continuous amplitude signal into a digital signal by expressing each sample value as a finite number of digits is called quantization.

The error introduced in representing the continuous valued signal by a finite set of discrete level is called quantization error or quantization noise.

The quantization error is a sequence which is defined as the difference b/w the quantized value and the actual sample value.

$$e(n) = x_q(n) - x(n).$$



Let us assume a sinusoidal signal varying b/w +1 and −1 having a dynamic range 2.

This signal is sampled and quantized to

$b+1$ bits

where $b \rightarrow$ n.o of data bits

So the n.o of levels available for quantizing $x(n)$ is 2^{b+1}

The interval b/w successive level is

$$q = \frac{\text{range}}{\text{n.o of levels}} = \frac{2}{2^{b+1}} = \frac{2}{2^b \cdot 2^1} = 2^{-b}$$

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where q is known as quantization step size.

If $b = 3$.

$$q = \frac{2}{2^4} = 2^{-3} = 0.125.$$

The methods of quantization is

(i) Truncation

(ii) Rounding.

Truncation:

Truncation is the process of reducing the size of binary numbers by discarding all bits less significant than the least significant bit that is retained.

for eg.

$$\underset{(8)}{0.00110011} \xrightarrow[\text{to 4 bits}]{\text{truncate}} \underset{(4)}{0.0011}$$

$$\underset{(7)}{1.0011011} \xrightarrow[\text{to 3 bits}]{\text{truncate}} \underset{(3)}{1.001}$$

Problem No 1:

Perform the quantization of 0.875_{10} to 2 bit by truncation.

$$0.875 \xrightarrow[\text{to binary}]{\text{convert}} (0.1110)_2 \xrightarrow[\text{to 2 bits}]{\text{truncate}} (0.11)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (0.75)_{10}$$

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Error range due to truncation in fixed point Number system :-

Let x be the unquantized fixed point binary number.

Let x_T be the quantized fixed point binary number.

The error due to quantization is

$$e = x_T - x.$$

This error will be negative or zero.

The error made by truncating a number to b bits satisfy the inequality

$$0 \geq x_T - x > -2^{-b}.$$

In fixed point number system, the effect of truncation on positive numbers are same in all the three representations.

The error due to truncation of negative number depends on the type of representation.

Range of error in two's complement Representation :-

The effect of truncation on a negative number is to increase the magnitude of the negative number and so the truncation error is always negative and it satisfies the inequality

$$0 \geq x_T - x > -2^{-b}.$$

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One's complement Representation :-

The magnitude of the given number decreases with truncation and hence the error is positive and satisfy the inequality

$$0 \leq x_T - x < 2^{-b}$$

In sign magnitude representation also the error is positive and satisfy the inequality

$$0 \leq x_T - x < 2^{-b}.$$

Error due to truncation in floating point number system :

In this representation, the mantissa of the number alone is truncated.

If $x = 2^c . M$

then after truncation

$$x_T = 2^c . M_T$$

$$e = x_T - x = 2^c M_T - 2^c M$$

$$e = 2^c (M_T - M)$$

Two's complement Representation.

The range of Mantissa is

$$0 \geq M_T - N > -2^{-b} \quad \text{---} (1)$$

$$M_T - M = \frac{e}{2^c}$$

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sub this in (1)

$$0 \geq \frac{e}{2^c} > -2^{-b}$$

$$0 \geq e > -2^c \cdot 2^{-b} \quad \text{———} \quad (2)$$

Relative error $\varepsilon = \dfrac{x_T - x}{x} = \dfrac{e}{x}$

$$e = \varepsilon x$$

$$0 \geq \varepsilon x > -2^c \cdot 2^{-b}$$

$$x = 2^c \cdot M$$

$$0 \geq \varepsilon\, 2^c \cdot M > -2^c \cdot 2^{-b}$$

$$0 \geq \varepsilon M > -2^{-b}.$$

Max error occur at $M = 1/2$

$$0 \geq \varepsilon > -2 \cdot 2^{-b}$$

For -ve number

sub $M = -1/2$

$$0 \geq -\varepsilon/2 > -2^{-b}$$

$$0 \leq \varepsilon < 2 \cdot 2^{-b}$$

One's complement Representation :

For +ve number

$$0 \geq \varepsilon > -2 \cdot 2^{-b}$$

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For negative mantissa value the error is

$$0 \leq N_T - M < 2^{-b}$$

$$0 \leq e < 2^{c} \cdot 2^{-b}$$

$$0 \leq \varepsilon x < 2^{c} \cdot 2^{-b}$$

$$0 \leq \varepsilon 2^{c} M < 2^{c} \cdot 2^{-b}$$

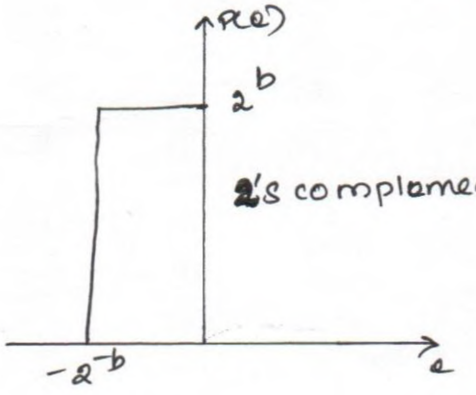
$$0 \leq \varepsilon M < 2^{-b}$$

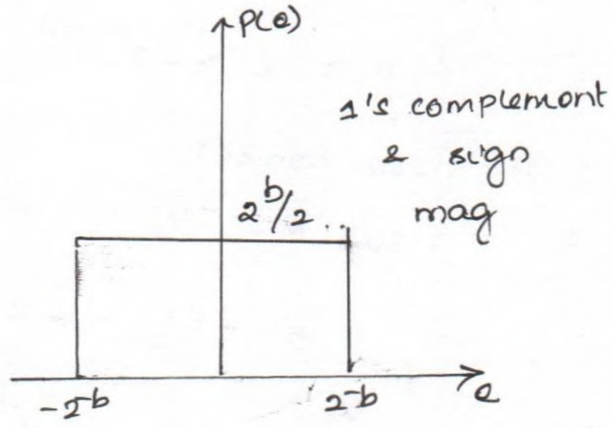
$M = -1/2$

$$0 \leq -\varepsilon/2 < 2^{-b}$$

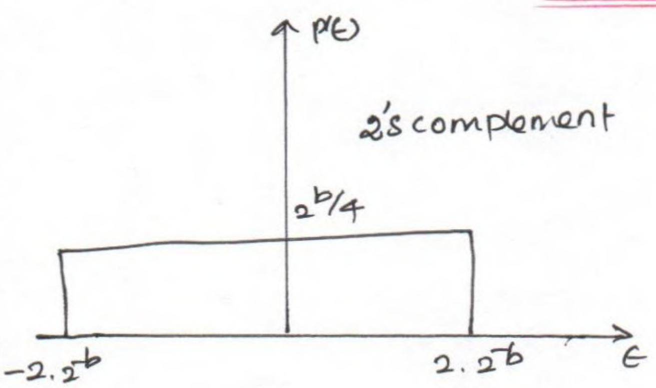
$0 \geq \varepsilon > -2 \cdot 2^{-b}$ → it is same as +ve number.

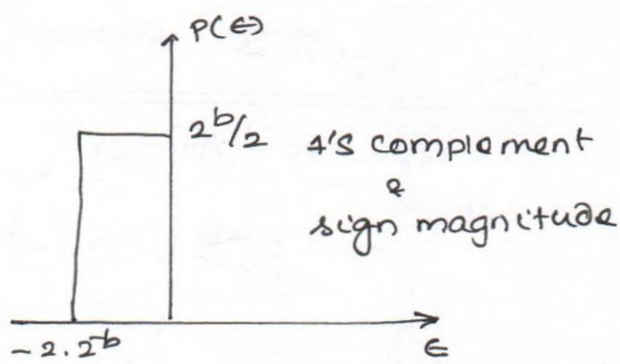
Fixed point





Floating point





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Error range due to rounding in fixed point :-

The error due to rounding to b bits produce an error $e = x_T - x$ which satisfies the inequality

$$\frac{-2^{-b}}{2} \leq x_T - x \leq \frac{2^{-b}}{2}$$

This is because with rounding, if the value lies halfway b/w two levels, it can approximated to either nearest higher level or by the nearest lower level.

Floating Point :-

In this only the mantissa is affected

so $$-\frac{2^{-b}}{2} \leq M_T - M \leq \frac{2^{-b}}{2} \qquad e = x_T - x$$

$$e = (M_T - M)\, 2^c$$

$$M_T - M = \frac{e}{2^c}$$

$$-2^{-b}/2 \leq \frac{e}{2^c} \leq 2^{-b}/2$$

$$-2^c\, 2^{-b}/2 \leq e \leq 2^{-b}/2 \cdot 2^c$$

$$\varepsilon = \frac{x_T - x}{x} = \frac{e}{x}$$

$$e = \varepsilon x$$

where $x = M\, 2^c$.

Sub in the above expression

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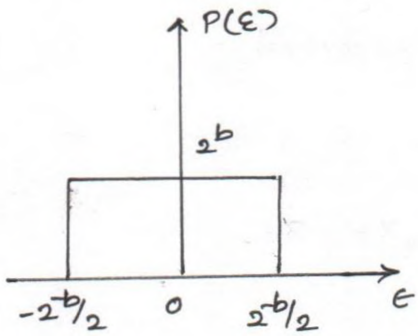
$$-2^{c}\, 2^{-b}/2 \le \varepsilon\, 2^{c} M \le 2^{c}\, 2^{-b}/2$$

$$-2^{-b}/2 \le \varepsilon M \le 2^{-b}/2$$

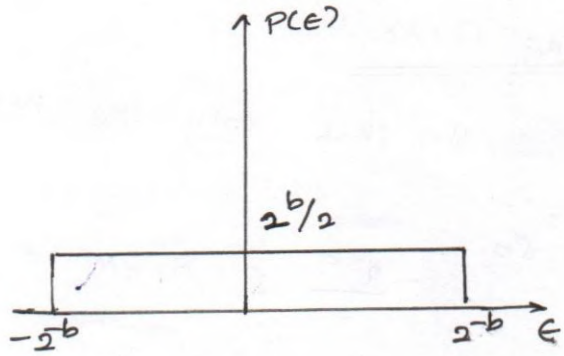
if $M = 1/2$,

$$-2^{-b} \le \varepsilon \le 2^{-b}$$

The propability density function for rounding is

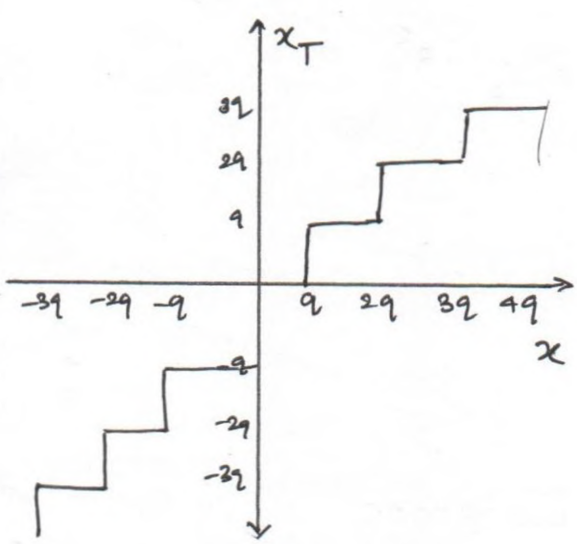


(i) fixed point

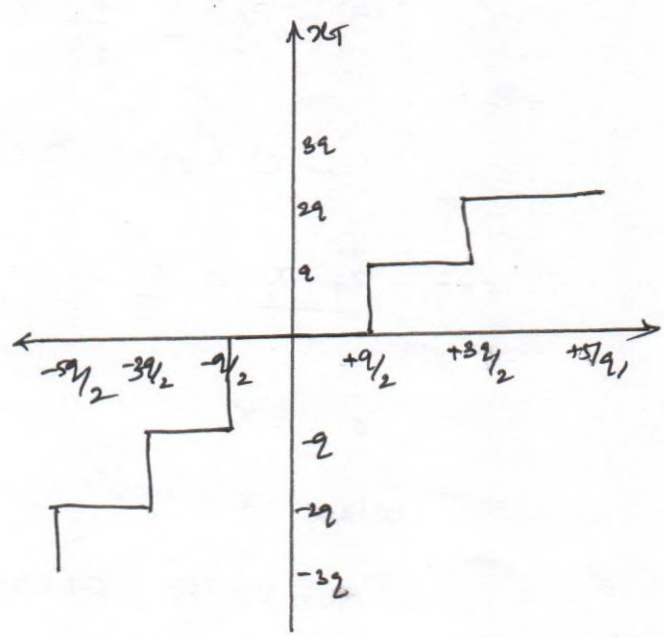


(ii) floating point

I/p & o/p characteristics



(i) Truncation



(ii) Rounding.

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Coefficient Quantization Errors :

* The filter coefficients are evaluated with infinite precision. They are limited by the word length of the register
* The filter coefficients are quantized to the word size of the register.
* The location of poles and zeros of the digital filter directly depends on the value of filter coefficients.
* So the quantization will modify the values of poles and zeros. This will create deviation in the frequency response of the system.
* The sensitivity of the filter frequency response characteristics to quantization of the filter coefficients is minimized by realizing the filter in cascade form since it has large n.o of poles and zeros as an interconnection of II order section.

Problem No 1 :

For the second order IIR filter

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

. Study the effect of shift in pole location with 3 bit coefficient representation in direct form and cascade form.

cascade form :

$$H(z) = \underbrace{\frac{1}{1-0.5z^{-1}}}_{H_1(z)} \cdot \underbrace{\frac{1}{1-0.45z^{-1}}}_{H_2(z)}$$

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The original poles are

$\boxed{P_1 = 0.5}$ $\boxed{P_2 = 0.45}$

The Quantization is done by truncating into 3 bits

$$0.5 \xrightarrow[\text{to binary}]{\text{convert}} (0.1000)_2 \xrightarrow[\text{to 3 bits}]{\text{truncate}} (0.100)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (0.5)_{10}$$

$$0.45 \xrightarrow[\text{to binary}]{\text{convert}} (0.0111)_2 \xrightarrow[\text{truncation}]{\text{3 bit}} (0.011)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (0.375)_{10}$$

The new poles in cascade form after truncation

$\boxed{P_{c1} = 0.5}$ $\boxed{P_{c2} = 0.375}$

<u>Direct form</u>:

$$H(z) = \frac{1}{1 - 0.95z^{-1} + 0.225z^{-2}}$$

Quantization of coefficient by truncation

$$0.95 \xrightarrow[\text{to binary}]{\text{convert}} (0.1111)_2 \xrightarrow[\text{to 3 bit}]{\text{truncate}} (0.111)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (0.875)_{10}$$

$$0.225 \xrightarrow[\text{to binary}]{\text{convert}} (0.0011)_2 \xrightarrow[\text{to 3 bit}]{\text{truncate}} (0.001)_2 \xrightarrow[\text{to decimal}]{\text{convert}} (0.125)_{10}$$

$$H(z) = \frac{1}{1 - 0.875z^{-1} + 0.125z^{-2}}$$

The new poles in direct form structure

$\boxed{P_{d1} = 0.695}$ $\boxed{P_{d2} = 0.179}$

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Conclusion :

* The poles deviate very much in direct form compared to cascade form. So the cascade realization is better.

Problem No 2 :

Find the effect of coefficient quantization on pole locations of the given second order system when it is realized in direct form I and in cascade form. Assume a word length of 4 bits through truncation.

$$H(z) = \frac{1}{1-0.9z^{-1}+0.2z^{-2}}$$

word length = 4

so $b+1 \Rightarrow 3+1$

$\downarrow$ data bits so 3 bit truncation.

Directform :

The original poles are

$\boxed{P_1 = 0.5}$ $\boxed{P_2 = 0.4}$

$$H(z) = \frac{1}{1-0.9z^{-1}+0.2z^{-2}}$$

$$0.9 \xrightarrow[\text{to Binary}]{} (0.1110)_2 \xrightarrow[\text{3 bit truncation}]{} (0.111)_2 \xrightarrow[\text{to Dec}]{} 0.875$$

$$0.2 \xrightarrow[\text{to Binary}]{} (0.0011)_2 \xrightarrow[\text{3 bit truncation}]{} (0.001)_2 \xrightarrow[\text{to Decimal}]{} 0.125$$

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The new transfer function

$$H(z) = \frac{1}{1-0.875z^{-1}+0.125z^{-2}}$$

The new poles

$\boxed{P_{d1} = 0.6958}$ $\boxed{P_{d2} = 0.1798}$

<u>cascade form:</u>

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.4z^{-1})}$$

$$0.5 \xrightarrow[\text{Binary}]{} (0.1000)_2 \xrightarrow[\text{3 bit truncation}]{} (0.100)_2 \xrightarrow[\text{Decimal}]{} (0.5)_{10}$$

$$0.4 \xrightarrow[\text{Binary}]{} (0.0111)_2 \xrightarrow[\text{3 bit truncation}]{} (0.011)_2 \xrightarrow[\text{Decimal}]{} (0.375)_{10}$$

The new poles are

$\boxed{P_{c1} = 0.5}$ $\boxed{P_{c2} = 0.375}$

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.375z^{-1})}.$$

<u>Overflow limit cycle Oscillations:</u>

In fixed point addition of two binary numbers the overflow occurs when the sum exceeds the finite word length of the register used to store them. The overflow results in the filter o/p to oscillate b/w min and max amplitudes.

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such limit cycles is referred to as overflow limit cycle.

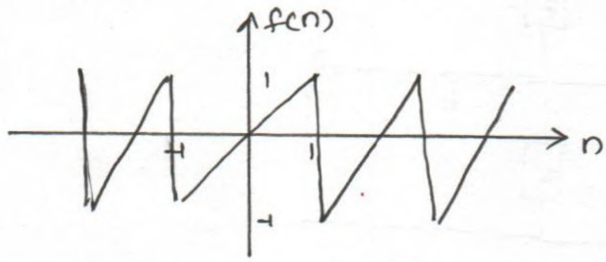
Let us consider two positive numbers n_1 and n_2

$$n_1 = 0.111 \rightarrow 7/8$$
$$n_2 = 0.110 \rightarrow 6/8$$
$$n_1 + n_2 = 1.101 \rightarrow -5/8 \text{ in sign magnitude}$$

In this, when two +ve numbers are added the sum is wrongly interpreted as -ve number

$2^{-1} = 0.5$
$2^{-3} = 0.125$
$0.625 \rightarrow 5/8$

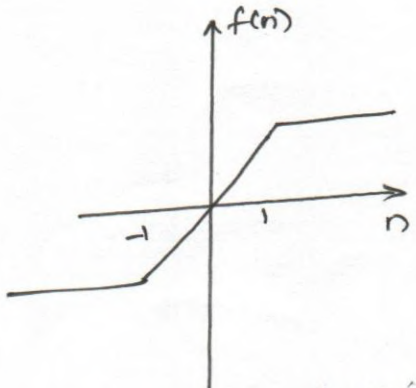
The transfer characteristics of an adder is



where n is the o/p of the adder.
f(n) is the corresponding o/p

The overflow occurs if the total i/p is out of range (-1,1). The overflow oscillations can be eliminated if saturation arithmetic is performed.

Hence when an overflow is detected, the sum of adder is set equal to maximum value and when an underflow is detected, the sum is set equal to minimum value.



(i) Saturation Arithmetic adder

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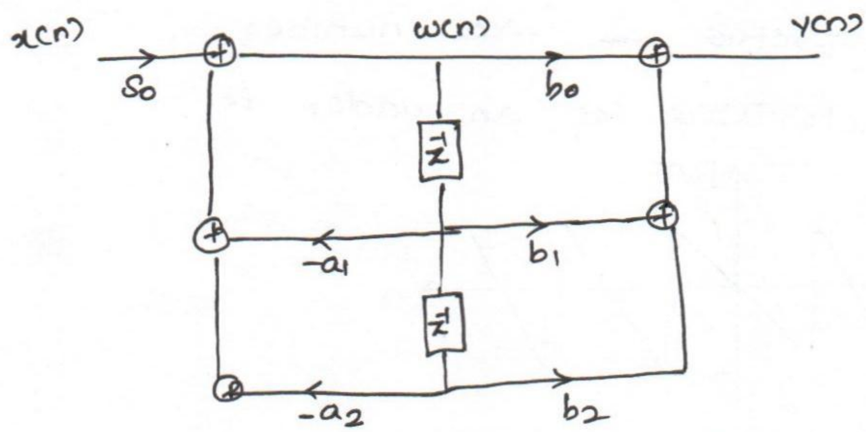
Signal scaling to prevent overflow:

Saturation arithmetic eliminates limit cycles due to overflow, but it causes undesirable signal distortion due to the non linearity of clippers.

In order to limit the amount of non-linear distortion, the i/p signal is scaled.

Let us consider a second order IIR filter



A scale factor S_0 is introduced b/w the i/p $x(n)$ and the adder 1.

The transfer function is

$$H(z) = S_0 \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$\frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = S_0 \frac{N(z)}{D(z)}.$$

Let $\frac{W(z)}{X(z)} = S_0 \frac{1}{D(z)}$

$$W(z) = S_0 X(z) \cdot \frac{1}{D(z)}$$

Let $S(z) = \frac{1}{D(z)}$

So $W(z) = S_0 X(z) \cdot S(z)$

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using inverse formula

$$w(n) = S_0 \cdot \frac{1}{2\pi} \int S(e^{j\theta})\, X(e^{j\theta})\, e^{jn\theta}\, d\theta.$$

Squaring the above term

$$w^2(n) = S_0^2 \frac{1}{4\pi^2} \int |S(e^{j\theta})|^2 |X(e^{j\theta})|^2 d^2\theta$$

Using schwartz inequality

$$w^2(n) \le S_0^2 \left[\frac{1}{2\pi} \int |S(e^{j\theta})|^2 d\theta\right] \left[\frac{1}{2\pi} \int |X(e^{j\theta})|^2 d\theta.\right.$$

Applying parseval's theorem we get

$$w^2(n) \le S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi} \int |S(e^{j\theta})|^2 d\theta.$$

Let $z = e^{j\theta}$; $dz = j e^{j\theta} d\theta$; $dz = j z d\theta$

$$d\theta = \frac{1}{jz} dz$$

$$d\theta = \frac{1}{j} z^{-1} dz$$

So Now

$$w^2(n) \le S_0^2 \sum_{n=0}^{\infty} x^2(n) \frac{1}{2\pi j} \int S(z)\, S(z^{-1})\, z^{-1} dz.$$

To avoid overflow the condition is

$$w^2(n) \le \sum_{n=0}^{\infty} x^2(n)$$

For this to be satisfied

$$S_0^2 \frac{1}{2\pi j} \int S(z)\, S(z^{-1})\, z^{-1} dz = 1$$

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$$S_0^2 = \frac{1}{\frac{1}{2\pi j}\int S(z)S(z^{-1})z^{-1}dz}$$

$$S_0^2 = \frac{1}{I}$$

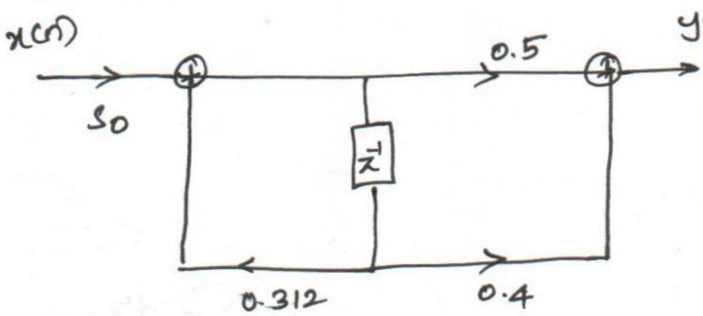
where $I = \frac{1}{2\pi j}\int S(z)\, S(z^{-1})\, z^{-1}dz$

$$S_0 = \frac{1}{\sqrt{I}}.$$

Problem No 1 :

The transfer function of the filter is

$H(z) = \dfrac{0.5 + 0.4z^{-1}}{1 - 0.312z^{-1}}$. find scaling factor S_0.



$$S_0^2 = \frac{1}{I}$$

$$I = \frac{1}{2\pi j}\int S(z)\, S(z^{-1})\, z^{-1}dz \qquad S(z) = \frac{1}{D(z)}$$

$$S(z) = \frac{1}{1-0.312z^{-1}} = \frac{z}{z-0.312}$$

$$S(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.312}$$

$$S(z)\, S(z^{-1})\, z^{-1} = \frac{z^{-1}}{(z-0.312)(z^{-1}-0.312)}$$

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The poles $z = 0.312$, $z = 1/0.312$

Residue at $z = 0.312$

$$\Rightarrow \cancel{(z - 0.312)} \frac{z^{-1}}{\cancel{(z - 0.312)}(z^{-1} - 0.312)} \Bigg|_{z = 0.312}$$

$$\Rightarrow \frac{0.312^{-1}}{0.312^{-1} - 0.312}$$

$$\Rightarrow 1.1078$$

$$S_0 = \frac{1}{\sqrt{I}} = \frac{1}{\sqrt{1.1078}} = \boxed{0.9501}$$

Zero i/p limit cycle oscillations:

In recursive systems, when the i/p is zero or some non-zero constant value, the non-linearities due to finite precision arithmetic operation may cause periodic oscillations. During periodic oscillations, the o/p y(n) of a system will oscillate b/w a finite +ve and -ve value for increasing n or the o/p will become constant for increasing n. Such oscillations are called limit cycles.

If the system o/p enters a limit cycle, it will continue to remain in limit cycle even when the i/p is made zero. Hence, these limit cycles are also called zero i/p limit cycles.

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Dead band :

The limit cycle occur as a result of the quantization effects in multiplications.

The amplitude of the o/p during a limit cycle are confined to a range of values that is called the dead band of the filter.

Let us consider a single pole IIR system

$$y(n) = \alpha y(n-1) + x(n)$$

After rounding

$$y_q(n) = Q[\alpha y(n-1)] + x(n)$$

During limit cycle oscillations

$$Q[\alpha y(n-1)] = \begin{cases} y(n-1) & \text{for } \alpha > 0 \\ -y(n-1) & \text{for } \alpha < 0 \end{cases}$$

By the definition of rounding

$$|Q[\alpha y(n-1)] - \alpha y(n-1)| \leq \frac{2^{-b}}{2}$$

$$|y(n-1) - \alpha y(n-1)| \leq \frac{2^{-b}}{2}$$

$$y(n-1)[1-|\alpha|] \leq \frac{2^{-b}}{2}$$

$$y(n-1) \leq \frac{2^{-b}/2}{1-|\alpha|}.$$ → This is the dead band of the filter

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Problem No 2 :

A digital system is characterised by the difference equation

$y(n) = 0.8\, y(n-1) + x(n)$ where $x(n) = 0$ and $y(-1) = 10$

Determine the deadband of the system.

n	$x(n)$	$y(n-1)$	$y(n) = \alpha y(n-1) + x(n)$	$Q[y(n)]$
0	0	10	8	8
1	0	8	6.4	6
2	0	6	4.8	5
3	0	5	4.0	4
4	0	4	3.2	3
5	0	3	2.4	2
6	0	2	1.6	2
7	0	2	1.6	2

Dead band

$$l \le \frac{0.5}{1-|\alpha|}$$

$$l \le \frac{0.5}{1-0.8}$$

$$l \le \frac{0.5}{0.2}$$

$$l \le \boxed{2.5}$$

The dead band lies in the range $[-2.5, 2.5]$

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Problem No 1:

Explain the characteristics of a limit cycle oscillation with respect to the system

$$y(n) = 0.5\, y(n-1) + x(n)$$. Determine the dead band of the filter.

Assume 3 bit rounding & $x(n) = 0.875$ for $n=0$
$= 0$ otherwise

$\alpha = 0.5$

n	x(n)	y(n-1)	$\alpha y(n-1)$ Dec	$\alpha y(n-1)$ Bin	$Q[\alpha y(n-1)]$ Bin	$Q[\alpha y(n-1)]$ Dec	$y(n) = x(n) + Q[\alpha y(n-1)]$
0	0.875	0	0	0.0000	0.000	0	0.875
1	0	0.875	0.4375	0.0111	0.100	0.5	0.5
2	0	0.5	0.25	0.0100	0.010	0.25	0.25
3	0	0.25	0.125	0.0010	0.001	0.125	0.125
4	0	0.125	0.625	0.0001	0.001	0.125	0.125
5	0	0.125	0.625	0.0001	0.001	0.125	0.125

$$\text{Dead Band} = \frac{2^{-b}/2}{1-|\alpha|} = \frac{2^{-3}/2}{1-0.5}$$

$$= 0.125$$

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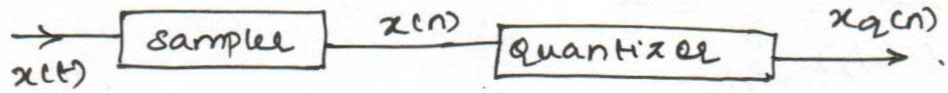
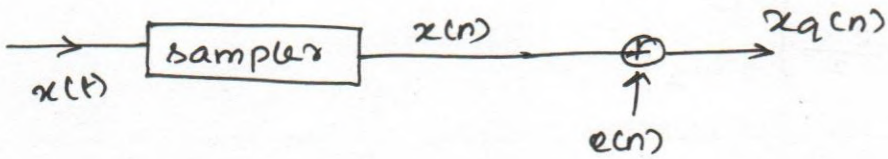
Steady, state Input Noise Power:



In digital processing of analog signals, the quantization error is commonly called as additive noise



$$x_q(n) = x(n) + e(n)$$

$$e(n) = x_q(n) - x(n)$$

In case of rounding, the e(n) lies b/w $-q/2$ and $q/2$ with equal probability and the mean value is zero.

The e(n) has the following properties

(i) e(n) is a sample sequence of a stationary random process

(ii) e(n) is uncorrelated with x(n)

(iii) e(n) is a whitenoise process with uniform amplitude probability distribution.

$$E(x) = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} x \, dx$$

The variance of e(n) is

$$\sigma_e^2 = E[e^2(n)] - E^2[e(n)].$$

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To calculate $E[e(n)]$:

$$E[e(n)] = \frac{1}{q/2-(-q/2)} \int_{-q/2}^{q/2} e\,de$$

$$= \frac{1}{q} \int_{-q/2}^{q/2} e\,de$$

$$= 1/q \left[\frac{e^2}{2}\right]_{-q/2}^{q/2}$$

$$\boxed{E[e(n)] = 0}$$

To calculate $E[e^2(n)]$

$$E[e^2(n)] = \frac{1}{q/2-(-q/2)} \int_{-q/2}^{q/2} e^2\,de$$

$$= \frac{1}{q} \left[\frac{e^3}{3}\right]_{-q/2}^{q/2}$$

$$= \frac{1}{3q}\left[\frac{q^3}{8} + \frac{q^3}{8}\right]$$

$$= \frac{2q^3}{24q} = \boxed{\frac{q^2}{12}}$$

$$\sigma_e^2 = E[e^2(n)] - E^2[e(n)]$$

$$\sigma_e^2 = \frac{q^2}{12}$$

if $q = 2^{-b}$ then

$$\sigma_e^2 = \frac{2^{-2b}}{12}.$$

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Let the i/p signal be $x(n)$ and it's variance is σ_x^2

Then the ratio of signal power to noise power

$$SNR = \frac{\sigma_x^2}{\sigma_e^2} = \frac{\sigma_x^2}{q^2/12}$$

$$= \frac{12\sigma_x^2}{q^2}$$

$$SNR = 12\sigma_x^2 \, 2^{+2b}$$

when expressed in dB

$$SNR = 10 \log_{10} \frac{\sigma_x^2}{\sigma_e^2}$$

$$= 10 \log_{10} (12\sigma_x^2 \, 2^{+2b})$$

$$= 6.02\, b + 10.79 + 10 \log_{10} \sigma_x^2$$

if the i/p signal is $Ax(n)$

$$SNR = 10 \log_{10} \frac{A^2\sigma_x^2}{\sigma_e^2}$$

$$= 6.02\, b + 10.79 + 10\log_{10}\sigma_x^2 + 20 \log_{10} A$$

if $A = \frac{1}{4\sigma_x}$

$$\boxed{SNR = (6.02\, b - 1.24)\, dB}.$$

Steady state o/p Noise Power:

Let $\epsilon(n)$ be the o/p noise due to quantization of the i/p

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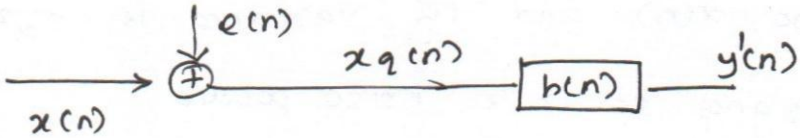

$$y'(n) = x_q(n) * h(n)$$

$$= [x(n) + e(n)] * h(n)$$

using property

$$y'(n) = x(n) * h(n) + e(n) * h(n)$$

$$y'(n) = y(n) + \varepsilon(n)$$

where $\varepsilon(n) = e(n) * h(n) = \sum_{k=0}^{\infty} h(k)\, e(n-k)$

The autocorrelation sequence for the o/p error signal is

$$\gamma_{\varepsilon_0}(m) = E\left[\varepsilon_0^*(n)\ \varepsilon_0(n+m)\right]$$

$$\gamma_{\varepsilon_0}(m) = E\left[\sum_{k=0}^{\infty} h(k) e^*(n-k) \sum_{k=0}^{\infty} h(k)\, e(n+m-k)\right]$$

$$= \sum_{k=0}^{\infty} h^2(k)\ E\left[e^*(n-k)\ e(n+m-k)\right]$$

$$\gamma_{\varepsilon_0}(m) = \sum_{k=0}^{\infty} h^2(k)\ \gamma_{ee}(m)$$

Now.

$$\gamma_{\varepsilon_0}(m) = \sigma_{\varepsilon_0}^2$$

$$\gamma_{ee}(m) = \sigma_e^2$$

The o/p Noise power is

$$\boxed{\sigma_{\varepsilon_0}^2 = \sigma_e^2 \sum_{k=0}^{\infty} h^2(k)}$$

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Using parseval's relation

$$\sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \int H(z) H(z^{-1}) z^{-1} dz.$$

$$\sigma_{\epsilon o}^2 = \sigma_e^2 \frac{1}{2\pi j} \int H(z) H(z^{-1}) z^{-1} dz.$$

This integration is evaluated using the method of residues, taking only the poles that lie inside the unit circle.

Problem No 1:

Find the steady state variance of the noise in the o/p due to quantization of i/p for the first order filter,

Solution:

$$y(n) = a y(n-1) + x(n)$$

Taking z transform

$$Y(z) = a z^{-1} Y(z) + X(z)$$

$$Y(z)[1 - a z^{-1}] = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}}$$

$$H(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}.$$

$$\sigma_{\epsilon o}^2 = \sigma_e^2 \frac{1}{2\pi j} \int H(z) H(z^{-1}) z^{-1} dz$$

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$$H(z)H(z^{-1})z^{-1} = \frac{\cancel{z}}{z-a}\,\frac{\cancel{z^{-1}}}{z^{-1}-a}\cdot z^{-1}$$

$$= \frac{z^{-1}}{(z-a)(z^{-1}-a)}$$

The poles are $z=a$, $z^{-1}=a$
$z = 1/a$

if $a<1$ then $z=1/a$ lies outside the circle.

Residue at $z=a$:-

$$\Rightarrow (z-a)\; H(z)\; H(z^{-1})z^{-1} \Big|_{z=a}$$

$$\Rightarrow \cancel{(z-a)}\; \frac{z^{-1}}{\cancel{(z-a)}(z^{-1}-a)} \Big|_{z=a}$$

$$\Rightarrow \frac{a^{-1}}{a^{-1}-a}$$

$$\Rightarrow \frac{1}{1-a^2}$$

The o/p Noise power is

$$\sigma_{\epsilon o}^2 = \boxed{\sigma_e^2\; \frac{1}{1-a^2}}.$$

Problem No 2:

The o/p of an A/D convertor is applied to a digital filter whose system function

$$H(z) = \frac{z(0.5)}{z-0.5}$$

. find the o/p Noise power, when the i/p signal is quantized to have eight bits.

given $b+1 = 8$ (including sign bit).

if the range is not given

Assume range = 2.

$$q = \frac{\text{range}}{\text{N.o of levels}}$$

$$= \frac{2}{2^8} = 2^{-7}.$$

$$\sigma_e^2 = \frac{q^2}{12} = \frac{(2^{-7})^2}{12} = \frac{2^{-4}}{12} = \boxed{5.086 \times 10^{-6}}$$

The o/p Noise power is given by

$$\sigma_{Eo}^2 = \sigma_e^2 \frac{1}{2\pi j} \int H(z) H(z^{-1}) z^{-1} dz$$

$$H(z) H(z) z^{-1} = \frac{0.5z}{z-0.5} \cdot \frac{0.5z^{-1}}{z^{-1}-0.5} \cdot z^{-1}$$

$$= \frac{0.25 z^{-1}}{(z-0.5)(z^{-1}-0.5)}$$

Residue at $z=0.5$

$$\cancel{(z-0.5)} \frac{0.25 z^{-1}}{\cancel{(z-0.5)}(z^{-1}-0.5)} \Bigg|_{z=0.5}$$

$$\Rightarrow 0.25 \frac{(0.5)^{-1}}{0.5^{-1}-0.5}$$

$$\Rightarrow \frac{0.25}{0.75} \Rightarrow 1/3$$

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The o/p Noise power is

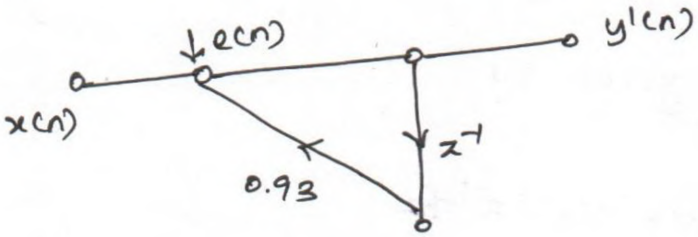
$$\sigma_{\varepsilon 0}^2 = \sigma_e^2 \left[1/3 \right]$$

$$= 5.086 \times 10^{-6} / 3$$

$$\sigma_{\varepsilon 0}^2 = 1.6954 \times 10^{-6}.$$

Problem No 3:

For the recursive filter, the i/p $x(n)$ has a peak value of 10 V represented by 6 bits. Compute the variance of o/p due to A/D conversion



Given.

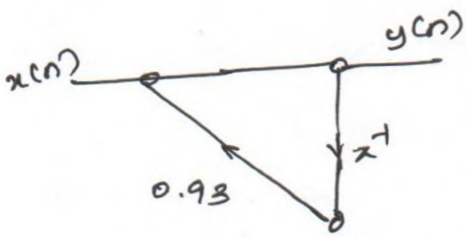
$R = 10\,V$

$b+1 = 6$ bits

$$q = \frac{R}{2^{b+1}} = \frac{10}{2^6} = 0.15625$$

$$\sigma_e^2 = \frac{q^2}{12} = \frac{(0.15625)^2}{12} = 2.0345 \times 10^{-3}.$$

The difference equation is foundout by



$$y(n) = x(n) + 0.93\, y(n-1)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - 0.93 z^{-1}}$$

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The Noise power

$$H(z) = \frac{z}{z-0.93} \qquad H(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.93}$$

$$H(z)\, H(z^{-1})\, z^{-1} = \frac{z^{-1}}{(z-0.93)(z^{-1}-0.93)}$$

Residue at $z=0.93$

$$\Rightarrow (z-0.93)\frac{z^{-1}}{(z-0.93)(z^{-1}-0.93)}\Bigg|_{z=0.93}$$

$$\Rightarrow \frac{1}{1-(0.93)^2}$$

$$\Rightarrow 7.4019$$

$$\sigma_{\varepsilon_0}^2 = \sigma_e^2 \; 7.4019$$

$$= 7.4019 \times 2.0345 \times 10^{-3}$$

$$\sigma_{\varepsilon_0}^2 = 0.0151.$$

Product Quantization Error :-

In fixed point arithmetic, the product of two b bit numbers result in 2b bits. In dsp applications it is necessary to round this product to a b-bit number, which produce an error known as product quantization error or product round off noise.

In realization structures of digital system multipliers are used to multiply the signal by constants.

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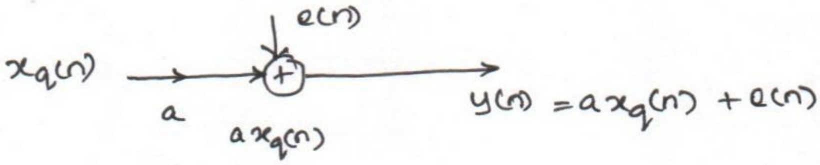
The multiplication is modelled as an infinite precision multipliers followed by an adder where round off noise is added to the product so that over all result equals some quantization level

The round off noise sample is a zero mean random variable with a variance $\frac{2^{-2b}}{12}$ where b is the n.o of bits



Following assumptions are made regarding the statistical independence of the various noise sources

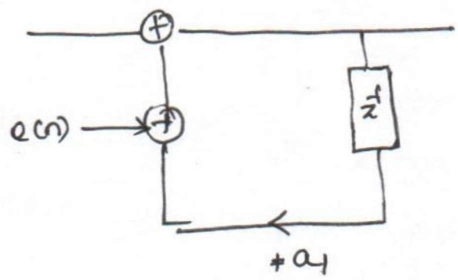
* The error sequence e(n) is uncorrelated with the signal sequence x(n).

* e(n) is a white noise.

* Each noise source are uncorrelated.

Product Quantization Noise model for first order system:

$y(n) = a_1 y(n-1) + x(n)$



$$\sigma_{E_0}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h_k^2(n).$$

Quantization Noise model for II order system:

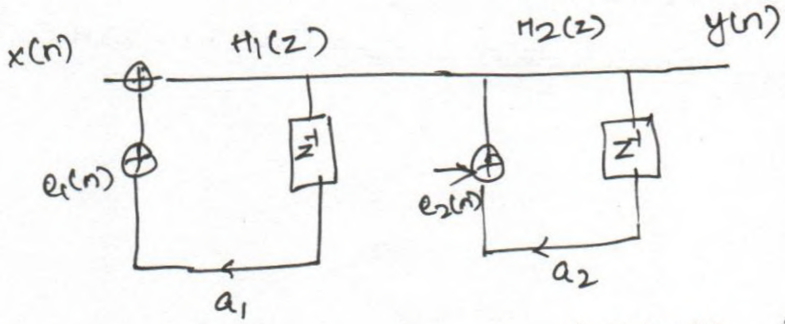
$y(n) = a_1 y(n-1) + a_2 y(n-2) + x(n)$

In cascade form

$$H(z) = \frac{1}{(1-a_1 z^{-1})(1-a_2 z^{-1})}$$

$$H_1(z) = \frac{1}{1-a_1 z^{-1}}$$

$$H_2(z) = \frac{1}{1-a_2 z^{-1}}$$



There are two noise sources in this realization. The noise sources are added at different points and they do not see the same noise transfer function.

ie The noise transfer function seen by $e_1(n) = H_1(z) H_2(z)$

& The noise transfer function seen by $e_2(n) = H_2(z)$

$$\sigma_{\epsilon o T}^2 = \sigma_{\epsilon o1}^2 + \sigma_{\epsilon o2}^2$$

If there is k noise source

$$\sigma_{\epsilon o}^2 = \sum_k \sigma_{\epsilon o}^2$$

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Problem No 1 :

In the IIR system given below the products are rounded to 4 bits (including sign bit). The system function is

$$H(z) = \frac{1}{(1-0.35z^{-1})(1-0.62z^{-1})}$$

Find the o/p round off noise power in (a) Direct form Realization

(b) Cascade form Realization - I $H(z) = H_1(z)H_2(z)$

(c) Cascade form Realization - II $H(z) = H_2(z)H_1(z)$

(a) Direct form :-

$$H(z) = \frac{1}{(1-0.35z^{-1})(1-0.62z^{-1})}$$

$$H(z) = \frac{1}{(1-0.97z^{-1}+0.217z^{-2})}$$

I/p variance Noise :

$$\sigma_e^2 = \frac{q^2}{12}.$$

if range is not given. Assume $R = 2V$

$$q = \frac{\text{range}}{\text{N.o of levels}} = \frac{2}{2^{b+1}} = \frac{2}{2^4} = 1/8$$

$$\sigma_e^2 = \frac{(1/8)^2}{12}$$

$$\boxed{\sigma_e^2 = 1.3021\times10^{-3}}$$

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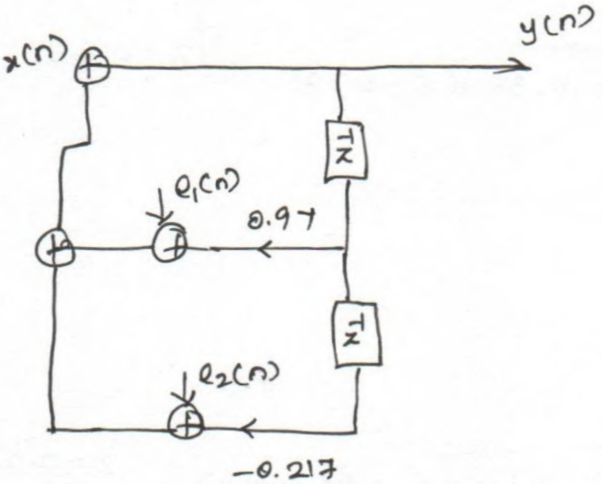
Direct form realization



The total noise power is

$$\sigma_{\varepsilon oT}^2 = \sigma_{\varepsilon o1}^2 + \sigma_{\varepsilon o2}^2$$

The noise transfer function seen by $e_1(n)$ is $H(z)$

The NTF seen by $e_2(n)$ is $H(z)$

Both are same

so $\sigma_{\varepsilon o1}^2 = \sigma_{\varepsilon o2}^2$

To find $\sigma_{\varepsilon o1}^2$:

$$\sigma_{\varepsilon o1}^2 = \sigma_e^2 \frac{1}{2\pi j} \int H(z) H(z^{-1}) z^{-1} dz$$

$$H(z) = \frac{1}{(1-0.35z^{-1})(1-0.62z^{-1})} = \frac{z}{(z-0.35)(z-0.62)}$$

$$H(z^{-1}) = \frac{z^{-1}}{(z^{-1}-0.35)(z^{-1}-0.62)}$$

The poles are $z = 0.35$ ✓ $z = 0.62$ ✓ $z = 1/0.35$ × $z = 1/0.62$ ×

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Residue at $z = 0.35$

$$= \cancel{(z-0.35)} \frac{z^{-1}}{\cancel{(z-0.35)}(z-0.62)(z^{-1}-0.35)(z^{-1}-0.62)} \Bigg|_{z=0.35}$$

$$= -1.8867$$

Residue at $z = 0.62$

$$= \cancel{(z-0.62)} \frac{z^{-1}}{(z-0.35)\cancel{(z-0.62)}(z^{-1}-0.35)(z^{-1}-0.62)} \Bigg|_{z=0.62}$$

$$= 4.7640$$

Sum of residues. $= 4.764 - 1.8867$

$= 2.8773$ //.

$$\sigma_{\varepsilon 01}^2 = \sigma_e^2 (\text{sum of residues})$$

$$= 1.3021 \times 10^{-3} \times 2.8733$$

$$\boxed{\sigma_{\varepsilon 01}^2 = 3.7465 \times 10^{-3}}$$

Since $\sigma_{\varepsilon 02}^2 = 3.7465 \times 10^{-3} = \sigma_{\varepsilon 01}^2$

$$\sigma_{\varepsilon 0T}^2 = \sigma_{\varepsilon 01}^2 + \sigma_{\varepsilon 02}^2$$

$$\boxed{\sigma_{\varepsilon 0T}^2 = 7.493 \times 10^{-3}}$$ (Direct form).

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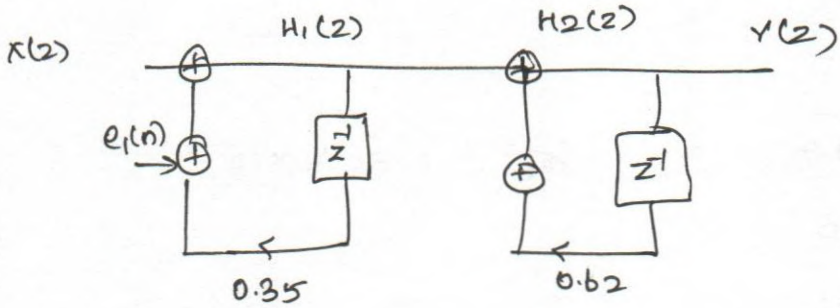
(b) Cascade Realization I:

Order of cascading $H_1(z)\, H_2(z)$

$$H(z) = \frac{1}{(1-0.35z^{-1})(1-0.62z^{-1})}$$

$$H_1(z) = \frac{1}{(1-0.35z^{-1})} \qquad H_2(z) = \frac{1}{(1-0.62z^{-1})}$$



The NTF seen by $e_1(n)$ is $H(z) = H_1(z) H_2(z)$

The NTF seen by $e_2(n)$ is $H_2(z)$.

$$\sigma_{\varepsilon oT}^2 = \sigma_{\varepsilon o1}^2 + \sigma_{\varepsilon o2}^2$$

To find $\sigma_{\varepsilon o1}^2$:

$$\sigma_{\varepsilon o1}^2 = 3.7465 \times 10^{-3} \quad \text{(Refer direct form)}$$

To find $\sigma_{\varepsilon o2}^2$:

$$H_2(z) = \frac{z}{z-0.62}$$

$$H_2(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.62}$$

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Residue at $z = 0.62$

$$\Rightarrow (\cancel{z-0.62}) \frac{z^{-1}}{(z^{-1}-0.62)(\cancel{z-0.62})} \Bigg|_{z=0.62}$$

$$\Rightarrow \frac{1}{1-0.62^2}$$

$$= 1.6244$$

$$\sigma_{\varepsilon o2}^2 = \sigma_e^2 \times 1.6244 \qquad \therefore \sigma_e^2 = 1.3021 \times 10^{-3}$$

$$\sigma_{\varepsilon o2}^2 = 2.1151 \times 10^{-3}$$

Total Noise power

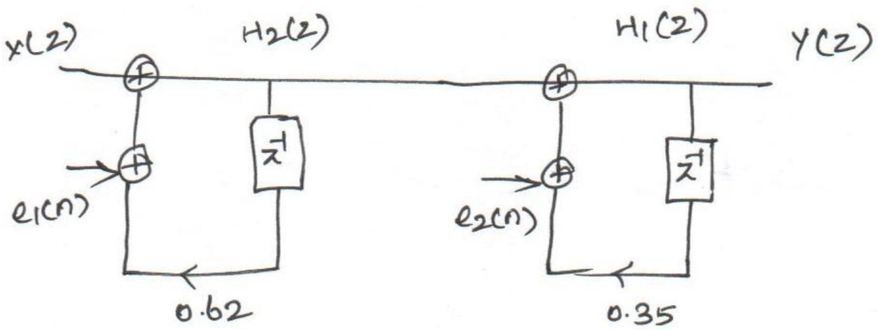
$$\sigma_{\varepsilon oT}^2 = \sigma_{\varepsilon o1}^2 + \sigma_{\varepsilon o2}^2$$

$$= 3.7465 \times 10^{-3} + 2.1151 \times 10^{-3}$$

$$\boxed{\sigma_{\varepsilon oT}^2 = 5.8616 \times 10^{-3}}$$

(c) Cascade Realization II

$$H(z) = H_2(z) H_1(z)$$



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The NTF seen by $e_1(n)$ = $H_1(z)H_2(z)$

The NTF seen by $e_2(n)$ = $H_1(z) = \frac{1}{1-0.35z^{-1}}$

To find $\sigma_{\varepsilon 01}^2$:

$$\boxed{\sigma_{\varepsilon 01}^2 = 3.7465 \times 10^{-3}}$$

To find $\sigma_{\varepsilon 01}^2$

$$H_1(z) = \frac{z}{z-0.35}$$

$$H_1(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.35}$$

Residue at $z = 0.35$

$$\Rightarrow \cancel{(z-0.35)} \frac{z^{-1}}{(z^{-1}-0.35)\cancel{(z-0.35)}} \Bigg|_{z=0.35}$$

$$\Rightarrow 1.1396$$

$$\sigma_{\varepsilon 02}^2 = \sigma_e^2 \; 1.1396$$

$$= 1.3021 \times 10^{-3} \times 1.1396$$

$$\boxed{\sigma_{\varepsilon 02}^2 = 1.4839 \times 10^{-3}}$$

Total Noise power

$$\sigma_{\varepsilon oT}^2 = \sigma_{\varepsilon 01}^2 + \sigma_{\varepsilon 02}^2$$

$$= 3.7465 \times 10^{-3} + 1.4839 \times 10^{-3}$$

$$\boxed{\sigma_{\varepsilon oT}^2 = 5.2304 \times 10^{-3}}$$

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The product round off noise is less in case (II) when compared to case (I) and also direct form realization.

Problem 2 :

A causal IIR filter is defined by the difference equation

$y(n) = 0.9\, y(n-1) + x(n)$. The computed values are rounded to one decimal place. show that the filter exhibits dead band effect.

solution

$$y(n) = 0.9\, y(n-1) + x(n).$$

for causal system $y(n) = 0$ for $n < 0$

so the i/p $x(n)$ should be assumed

$$l \le \frac{0.5}{1-|\alpha|}$$

$$l \le \frac{0.5}{1-0.9} \quad ; \quad l \le 5$$

So choose i/p $x(n)$ greater than 5

Let $x(n) = \begin{cases} 8 & \text{at } n=0 \\ 0 & \text{otherwise} \end{cases}$

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n	$x(n)$	$y(n-1)$	$y(n) = 0.9y(n-1) + x(n)$	$Q[y(n)]$
0	8	0	8	8
1	0	8	7.2	7
2	0	7	6.3	6
3	0	6	5.4	5
4	0	5	4.5	5
5	0	5	4.5	5 → dead band!

Problem No 3:

Determine the dead band of the filter if 8 bits are used for representation

$$y(n) = 0.2y(n-1) + 0.5\, y(n-2) + x(n)$$

$$b+1 = 8$$

$$b = 7.$$

$a_1 = 0.2 \qquad a_2 = 0.5$

For a II order system

$$\text{Dead band} = \frac{\pm\, 2^{-b/2}}{1-|a_2|}$$

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$$= \frac{\pm 2^{-b/2}}{1-0.5}$$

$$= \pm \frac{2^{-8}}{0.5}$$

$$= \boxed{\pm 0.0078125}$$

The dead band interval $[-0.0078125 \quad +0.0078125]$.

Floating Point Addition & Multiplication

$F_1 = 2^{c_1} * M_1$

$F_2 = 2^{c_2} * M_2$

$$F_1 * F_2 = 2^{c_1+c_2}(M_1 * M_2)$$

$F_1 + F_2 \rightarrow$ addition

Addition is possible only when both exponents are equal. The exponent should be made equal before addition.

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Problem No. 4:

Determine the deadband of the filter

$$y(n) = -0.5\, y(n-1) + x(n).$$

Assume $x(n) = 0.875$ for $n=0$
$= 0$ otherwise $\quad \alpha = -0.5$

Assume $b=3$.

$$db \le \frac{2^{-b/2}}{1-|\alpha|} \le \frac{2^{-3/2}}{1-0.5} \le 0.125.$$

n	x(n)	y(n-1)	$-0.5\,y(n-1)$ $\alpha y(n-1)$ Dec	Bin	$Q[\alpha y(n-1)]$ Bin	Dec	$y(n)=x(n)+Q[\alpha y(n-1)]$
0	0.875	0	0	0	0	0	0.875
1	0	0.875	−0.4375	1.0111	1.100	−0.5	−0.5
2	0	−0.5	+0.25	0.0100	0.010	0.25	+0.25
3	0	0.25	−0.125	1.0010	1.001	−0.125	−0.125
4	0	−0.125	+0.625	0.0001	0.001	0.125	+0.125
5	0	0.125	−0.625	1.0001	1.001	−0.125	−0.125

It oscillates b/w −0.125 and +0.125

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UNIT - V.

Multirate signal Processing:-

The process of converting a signal from a given rate to a different rate is called sampling rate conversion.

The system that employ sampling rate in the processing of digital signals are called multi-rate digital signal processing systems.

The areas where multirate signal processing is used are.

(i) In high quality data acquisition & storage systems.

(ii) In audio signal processing. For eg CA is sampled at 44.1 Khz but DAT (digital audio tape) is sampled at 48 Khz. Conversion b/w DAT & CA needs multirate rate systems.

(iii) In video, PAL & NTSC run at different rates.

(iv) In transmultiplexers.

(v) Narrow band filtering for ECG and EEG.

(vi) Radar systems.

(vii) Signal compression.

The various advantages of multirate systems are

(i) computational requirements are less
(ii) Storage for filter coefficients are less
(iii) finite arithmetic effects are less
(iv) Filter order required are less
(v) sensitivity to filter coefficient lengths are less.

Methods used for sampling rate conversion :-

Two methods are used for sampling rate conversion.

First method :-

In the first method, digital signal is converted into analog signal by using digital to analog converter (DAC). Then analog signal is converted into digital signal by using ADC.

adv :

New sampling rate can be arbitrarily selected.

Disadv :

Distortion & quantization effect in ADC.

Second Method :

In this method, sampling rate conversion is performed in digital domain.

Decimation :-

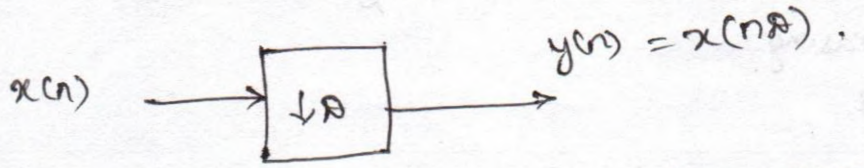
The process of reducing the sampling rate by a factor D is known as decimation or downsampling.

Let F_x be the sampling frequency of the i/p signal. $x(n)$.

$x(n)$ is downsampled by factor 'D'

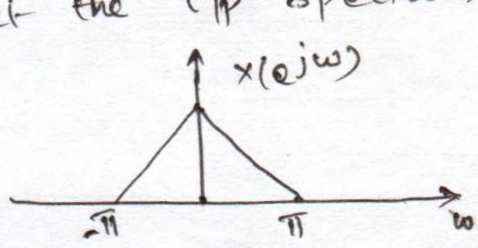
Let F_y be the o/p sampling frequency.

$$F_y = \frac{F_x}{D}$$



The $y(n)$ can be obtained by simply keeping every D^{th} sample and removing $(D-1)$ in b/w samples.

Let the i/p spectrum of $x(n)$ is $X(e^{j\omega})$.



$0 \le |\omega| \le \pi$

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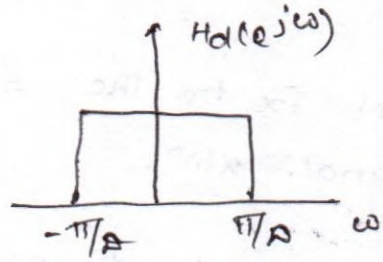
$$0 \le 2\pi \frac{f}{F_x} \le \pi$$

$0 \le f \le \frac{F_x}{2}$. But there will be aliasing error after downsampling with folding frequency $\frac{F_x}{2D}$ (π/D).

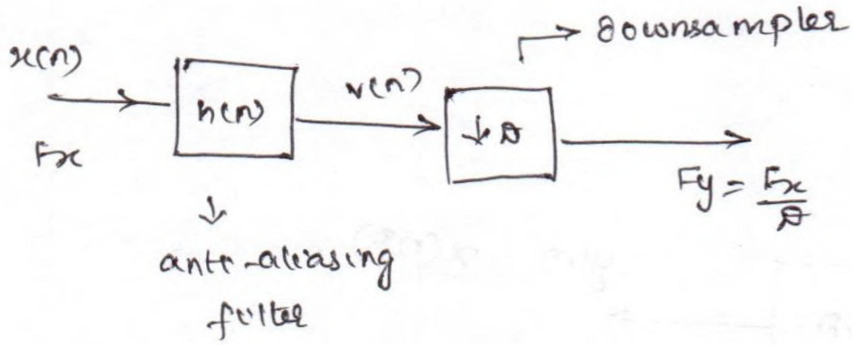
Inorder to avoid aliasing error the signal should be bandlimited to $\pm \pi/D$. So the i/p signal $x(n)$ is passed through LPF with impulse response $h(n)$. This filter is known as anti-aliasing filter.

The ideal magnitude Response of filter

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \pi/D \\ 0 & \text{otherwise} \end{cases}$$



Block diagram:



The o/p of the filter is $v(n)$

$$v(n) = x(n) * h(n)$$

$$v(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

The $v(n)$ is downsampled by factor D. Then

$$y(m) = v(mD)$$

$$y(m) = \sum_{k=0}^{\infty} h(k)\, x(mD-k)$$

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The operation on $x(n)$ will be linear and time variant.

For eg.

$$x(n) = \{ x(0),\ x(1)\ x(2),\ x(3) \ldots \}$$

after $\downarrow 2$

$$x'(n) = \{ x(0),\ x(2),\ x(4),\ x(6) \ldots \}.$$

So we can define

$$\tilde{v}(n) = \begin{cases} v(n) & n = 0, \pm D, \pm 2D \\ 0 & \text{otherwise.} \end{cases}$$

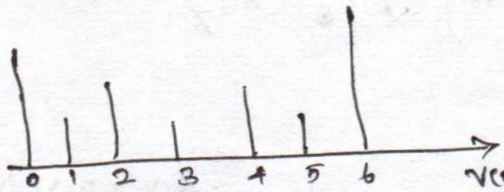
$\tilde{v}(n) = v(n)\, p(n)$ where $p(n)$ is the periodic train of impulses. The fourier series representation of impulse train is

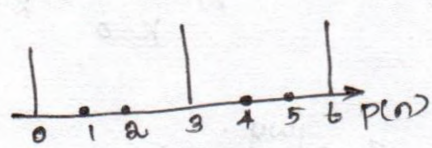
$$p(n) = \frac{1}{D} \sum_{k=0}^{D-1} e^{j\frac{2\pi k n}{D}}$$

Let the o/p $y(m)$

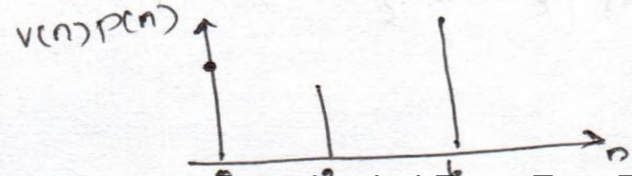
$$y(m) = \tilde{v}(m)$$

$$= v(mD)\, p(mD)$$





$\boxed{D = 3.}$



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The z-transform of the o/p sequence.

$$Y(z) = \sum_{m=-\infty}^{\infty} y(m) z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \tilde{v}(mD) z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \tilde{v}(m) z^{-m/D}$$

$$= \sum_{m=-\infty}^{\infty} v(m) p(m) z^{-m/D}$$

sub p(m) value

$$Y(z) = \frac{1}{D} \sum_{m=-\infty}^{\infty} v(m) \sum_{k=0}^{D-1} e^{\frac{j2\pi mk}{D}} z^{-m/D}$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} \sum_{m=-\infty}^{\infty} v(m) \left(e^{-j\frac{2\pi k}{D}} z^{1/D}\right)^{-m}$$

$$Y(z) = 1/D \sum_{k=0}^{D-1} V\left(e^{-j\frac{2\pi k}{D}} z^{1/D}\right)$$

$$\because \quad V(z) = X(z) H(z) = x(n) * h(n)$$

Then

$$Y(z) = 1/D \sum_{k=0}^{D-1} H_D\left(e^{-j\frac{2\pi k}{D}} z^{1/D}\right) \times \left(e^{-j\frac{2\pi k}{D}} z^{1/D}\right)$$

sub $z = e^{j\omega y}$.

$$Y(z) = 1/D \sum_{k=0}^{D-1} H_D\left(e^{-j\frac{2\pi k}{D}} e^{\frac{j\omega y}{D}}\right) \times \left(e^{-j\frac{2\pi k}{D}} e^{\frac{j\omega y}{D}}\right)$$

$$Y(e^{j\omega_y}) = \frac{1}{D} \sum_{k=0}^{D-1} H_d\left(e^{j\left(\frac{\omega_y - 2\pi k}{D}\right)}\right) X\left(e^{j\frac{\omega_y - 2\pi k}{D}}\right)$$

$$\boxed{Y(\omega_y) = \frac{1}{D} \sum_{k=0}^{D-1} H_d\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(e \frac{\omega_y - 2\pi k}{D}\right)}$$

If $H(\omega)$ is designed properly, then aliasing effect will be eliminated

$$Y(\omega_y) = \frac{1}{D} H_D\left(\frac{\omega_y}{D}\right) X\left(\frac{\omega_y}{D}\right)$$

$$Y(\omega_y) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right) \quad \text{for } 0 \le |\omega_y| \le \pi .$$

Note:-

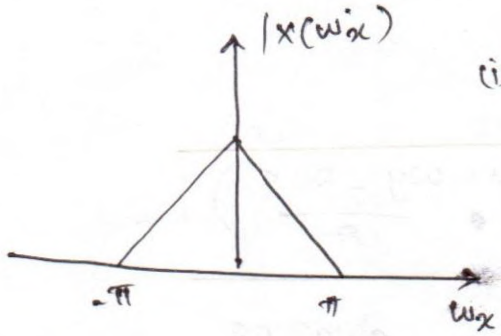
$$\omega_y = \frac{2\pi F}{F_y} \qquad \omega_x = \frac{2\pi F}{F_x}$$

$$\omega_y = \frac{2\pi F D}{F_x}$$

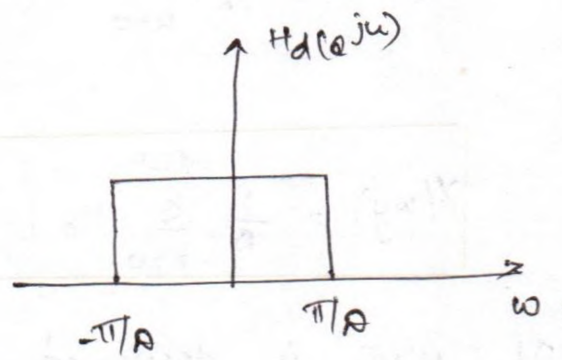
$$\boxed{\omega_y = D\,\omega_x .}$$

The fourier of o/p spetrum $Y(e^{j\omega})$ is the stretched version of sum of uniformly shifted and scaled by a factor of $1/D$. $X(e^{j\omega})$ and

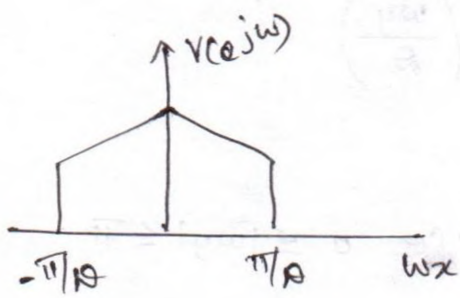
Spectrum of signals when $x(n)$ is decimated by D.



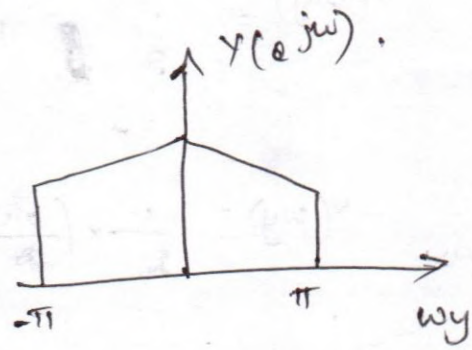
(i) i/p spectrum



(iii) The filter o/p spectrum



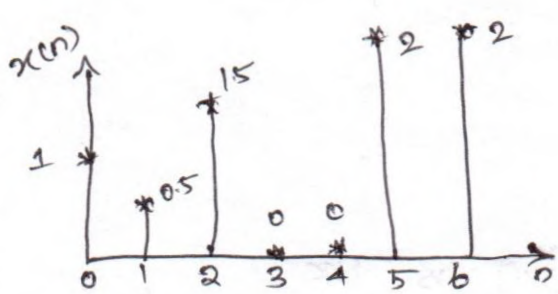
(iv) Final o/p spectrum



Problem No 1:

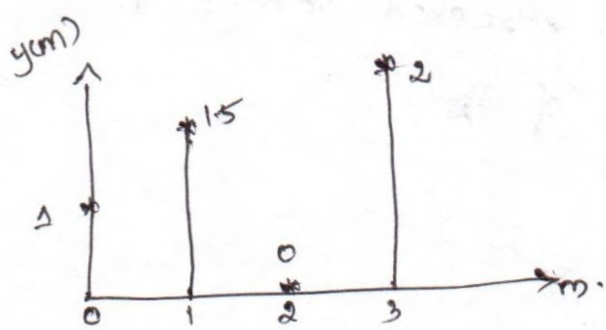
Draw the decimated signal of $x(n)$ where $D = 2$.

$x(n) = \{1, 0.5, 1.5, 0, 0, 2, 2\}$.



$D = 2$ — $D-1$ sample is eliminated.

only every 2^{nd} sample is taken.



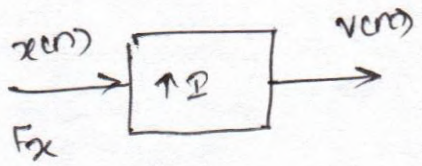
$y(m) = \{1, 1.5, 2\}$.

Interpolation:

The process of increasing the sampling rate by a factor of I is called interpolation.

Let Fx be the i/p signal sampling frequency.

$$\boxed{Fy = I\,Fx}$$



$Fy = Fx\,I.$

$$v(m) = \begin{cases} x\left(\frac{m}{I}\right) & \text{for } m = 0, \pm I, \pm 2I \ldots \\ 0 & \text{otherwise.} \end{cases}$$

The z-transform of the o/p sequence

$$V(z) = \sum_{m=-\infty}^{\infty} v(m)\, z^{-m}.$$

$$= \sum_{m=-\infty}^{\infty} x\left(\frac{m}{I}\right) z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, z^{-mI}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \left(z^{I}\right)^{-m}$$

$$\boxed{V(z) = X\left(z^{I}\right)}.$$

The spectrum of x(m) is obtained by evaluating on the unit circle. Sub $z = e^{j\omega_y}$

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So $V(e^{j\omega_y}) = X(e^{j\omega_y I})$

$$V(\omega_y) = X(\omega_y I).$$

<u>Relationship b/w ω_y & ω_x :-</u>

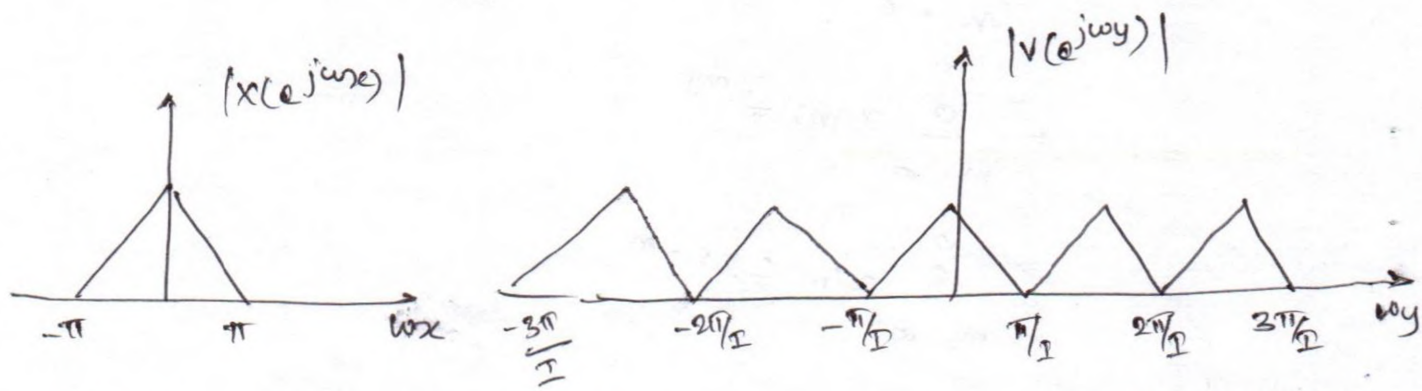
$$\omega_y = \frac{2\pi F}{F_y}$$

$$\omega_y = \frac{2\pi F}{F_x I}$$

$$\boxed{\omega_y = \frac{\omega_x}{I}}$$

Adding $I-1$ zero samples b/w successive samples of $x(n)$ results in a signal whose spectrum is an I fold periodic repetition of i/p spectrum.

The i/p spectrum be $X(e^{j\omega_x})$

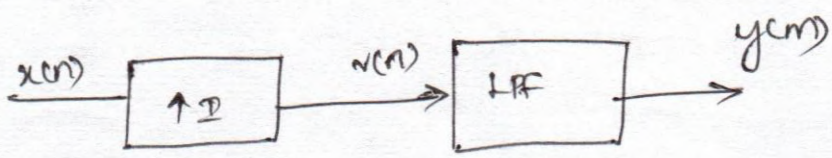


Since only the frequency of $x(n)$ in the range of $0 \le |\omega_y| \le \pi/I$ is unique and the other images of the i/p spectrum should be eliminated. So the upsampled signal is passed to LPF. This filter is known as anti-imaging filter.

The frequency response of the filter:

$$H_I(\omega_y) = \begin{cases} c & 0 \le |\omega_y| \le \pi/I \\ 0 & \text{otherwise} \end{cases}$$

Block diagram of interpolator is



$$Y(\omega_y) = \begin{cases} V(\omega_y)\, H_I(\omega_y) \end{cases}$$

$$Y(\omega_y) = V(\omega_y)\, c$$

$$= X(\omega_y I)\, c.$$

Selection of c value :-

$$y(m) = \begin{cases} x\left(\frac{m}{I}\right) & \text{for } m = 0, \pm I, \pm 2I \ldots \\ 0 & \text{otherwise} \end{cases}$$

For easy analysis:

$y(0) = x(0)$.

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$$y(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega y})\, e^{j\omega y m}\, d\omega y. \qquad \because m=0, \; e^0=1$$

$$y(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega y})\, d\omega y$$

$$= \frac{1}{2\pi}\int_{-\pi/I}^{\pi/I} X(\omega y I)\, c\, d\omega y.$$

$$\omega y = \frac{\omega x}{I}$$

$$d\omega y = \frac{1}{I}\, d\omega x.$$

$$= \frac{c}{I\, 2\pi}\int_{-\pi/I}^{\pi/I} X(\omega x)\, d\omega x.$$

$$y(0) = \frac{c}{I}\, \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega x)\, d\omega x$$

$$= \frac{c}{I}\, x(0)$$

when $\boxed{c = I}$ $\quad y(0) = x(0).$

So the c value should be Interpolation value.

The o/p sequence y(m) can be expressed as a convolution of the sequence x(n) with h(n)

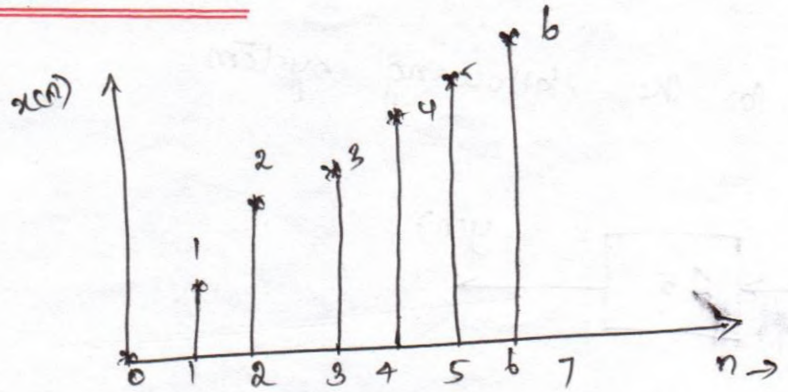
$$y(m) = \sum_{k=-\infty}^{\infty} h(m-k)\,v(k)$$

Since $v(k) = 0$ except at multiples of I.

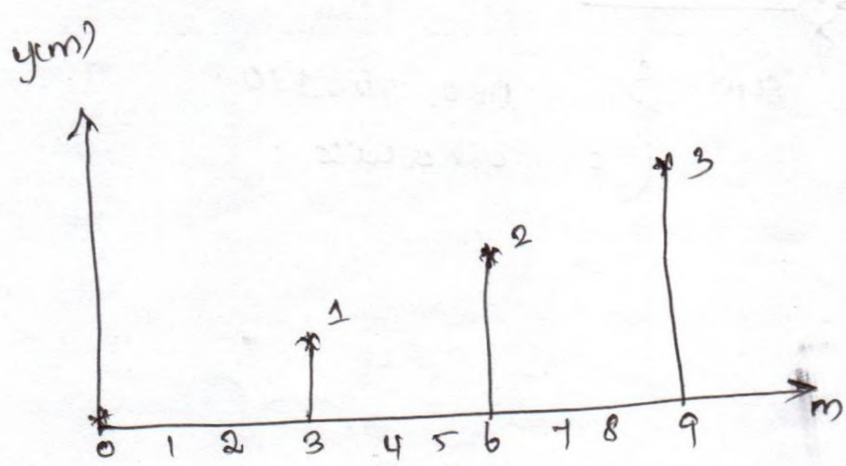
$$v(kI) = x(k)$$

$$\boxed{y(m) = \sum_{k=-\infty}^{\infty} h(m-kI)\,x(k)}$$

Problem No 2:-



Draw the interpolated signal by a factor 3, ie $I = 3$



ie $I-1$ zeros are inserted b/w 2 samples.

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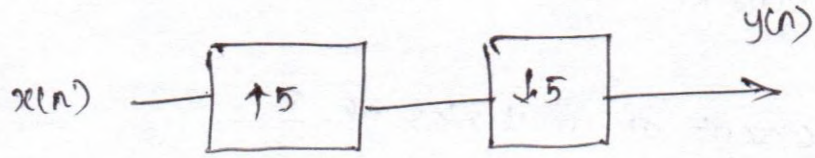
Problem No : 3

Obtain the expression for the o/p in terms of $x(n)$ for the multirate system



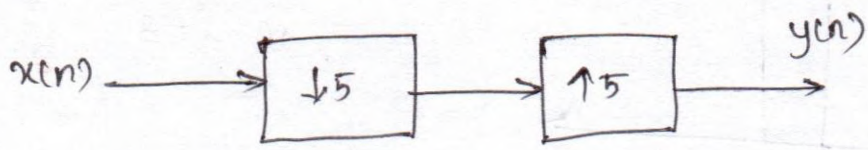
upsampling of system 1 is cancelled by downsampling of system 2.

$$y(n) = x(n)$$

$$x(n) \longrightarrow y(n)$$

Problem No : 4

obtain the expression for the following system



$$s(n) = \begin{cases} 1 & n = 0, \pm 5, \pm 10, \\ 0 & \text{otherwise.} \end{cases}$$

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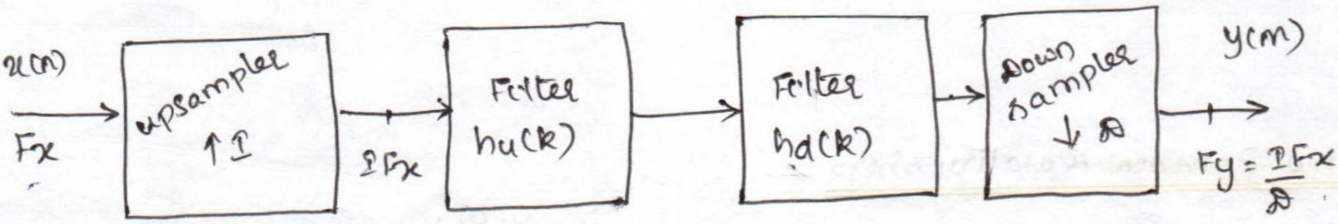
Sampling Rate conversion by a Rational Factor I/D!

The process of converting a signal from the given rate to a different rate is known as sampling rate conversion.

The sampling rate conversion by a rational factor I/D can be done by cascading interpolator with a decimator.

To preserve the desired spectral characteristics of x(n), the interpolation process is done first and then the decimation.



i/p x(n) is given to upsampler block and the o/p of upsampler block is given to hu(k) and hd(k). The o/p of the filter is given to downsampler and the final o/p is y(m).

The two filters are operated at the same rate, namely IFx, and hence can be combined into a single low pass filter with impulse response h(k). The frequency response $H(w_v)$ of the combined filter must incorporate the filtering operations for both

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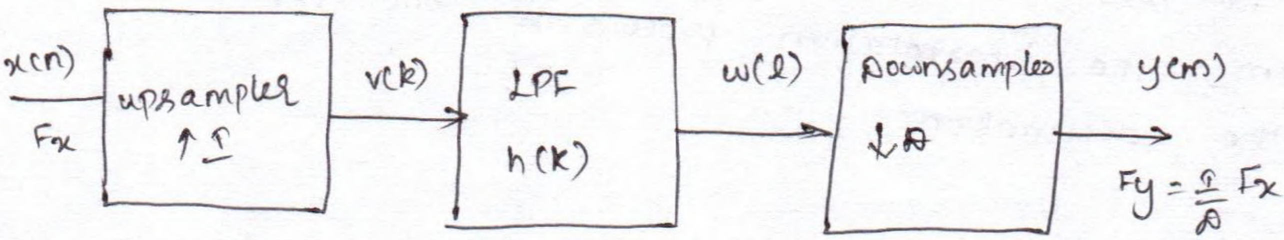
interpolation and decimation.

$$H(\omega_v) = \begin{cases} I & 0 \le |\omega_v| \le \min(\pi/D, \pi/I) \\ 0 & \text{otherwise} \end{cases}$$

where $\omega_v = \dfrac{2\pi F}{F_v} = \dfrac{2\pi F}{I F_x} = \dfrac{\omega_x}{I}$.



Time domain Relationship:

The o/p of the up-sampler is $v(k)$.

$$v(k) = \begin{cases} x\left(\frac{n}{I}\right) & \text{at } n = 0, \pm I, \pm 2I \ldots \\ 0 & \text{otherwise} \end{cases}$$

& the o/p of the filter is

$$w(l) = \sum_{k=-\infty}^{\infty} h(l-k)\, v(k),$$

Since $v(k) = 0$ except at multiples of I.

$$w(l) = \sum_{k=-\infty}^{\infty} h(l-kI)\, v(kI)$$

$\because v(kI) = x(k)$

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So $w(l)$ can be written as

$$w(l) = \sum_{k=-\infty}^{\infty} h(l - kI)\, x(k).$$

The o/p of the downsampler is

$$y(m) = w(mD)$$

$$\boxed{y(m) = \sum_{k=-\infty}^{\infty} h(mD - kI)\, x(k)}.$$

Frequency Domain Relationship:-

Let $X(e^{j\omega x})$ be the i/p spectrum

then the o/p of the filter

$$V(e^{j\omega y}) = X(e^{j\omega y I})$$

$$W(e^{j\omega y}) = X(e^{j\omega y I})\, H_d(e^{j\omega y I})$$

$$W(e^{j\omega y}) = I\, X(e^{j\omega y I})$$

The o/p spectrum,

$$Y(e^{j\omega y}) = \frac{1}{D} \sum_{k=0}^{D-1} W\left(e^{j\left(\frac{\omega y - 2\pi k}{D}\right)}\right)$$

So

$$Y(e^{j\omega y}) = \begin{cases} \frac{I}{D}\, X\left(\frac{\omega y}{D}\right) & 0 \le |\omega y| \le \min\left(\pi, \frac{\pi D}{I}\right) \\ 0 & \text{otherwise} \end{cases}$$

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Multistage implementation of sampling rate conversion :-

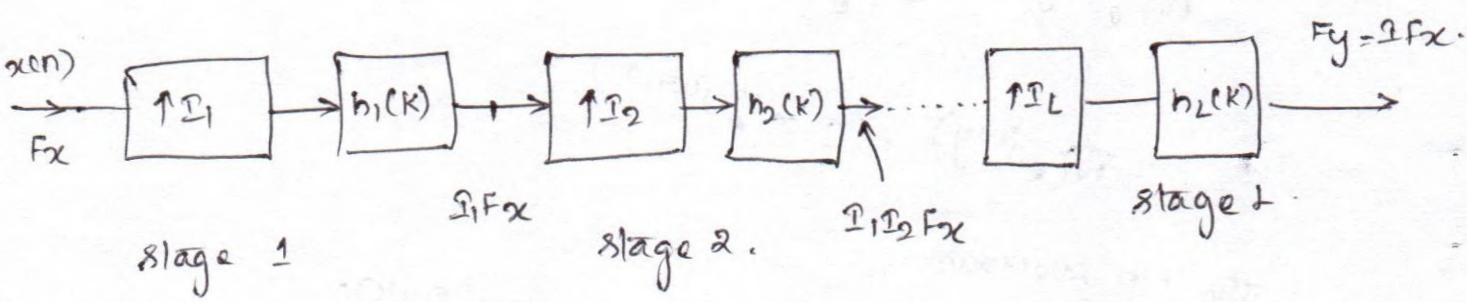
Altering the sampling rate by a large factor can be achieved exactly, but the implementation would require a more n.o of polyphase filters and computationally inefficient.

Let us consider, $D >> 1$ and $I >> 1$, then the sampling rate conversion is done in multiple stages.

At first consider $I >> 1$, the value I can be factored into a product of positive integers as

$$I = \prod_{i=1}^{L} I_i$$

It can be implemented by cascading L stages of interpolation and filters.



Consider $D >> 1$, the value D can be factored into a product of +ve integers as

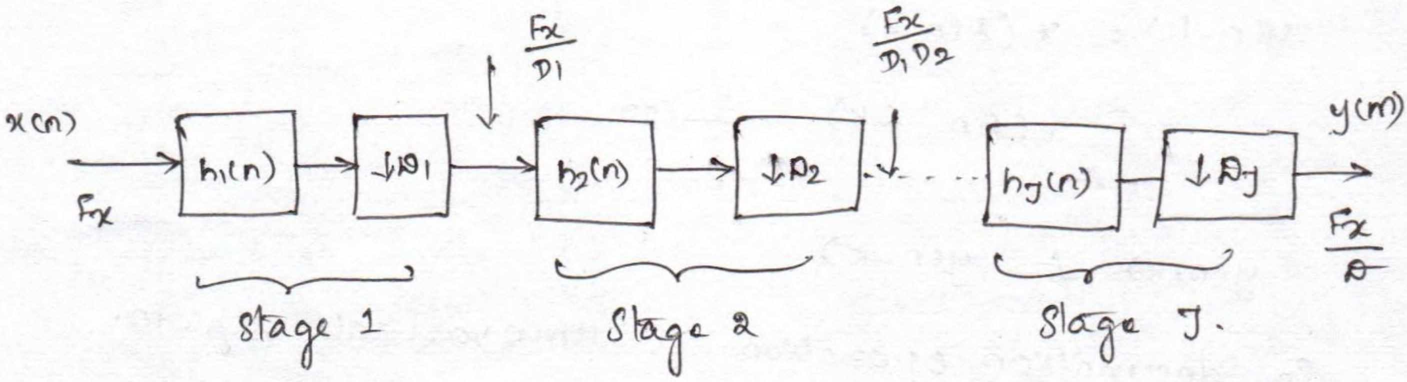
$$D = \prod_{i=1}^{J} D_i$$

∴ The sampling rate at the o/p of the i^{th} stage is

$$F_i = \frac{F_{i-1}}{D_i} \qquad i = 1, 2 \ldots J$$

where $F_0 = F_x$.

It can be implemented as cascade of J stages of filtering and decimation



Let us define the desired passband & transition band in the overall decimator:

Passband : $0 \le F \le F_{pc}$

Transition band : $F_{pc} \le F \le F_{sc}$.

Aliasing can be avoided in the band $0 \le F \le F_{sc}$ is avoided by selecting the frequency bands of each filter stage as.

Passband : $0 \le F \le F_{pc}$.

Transition band : $F_{pc} \le F \le F_i - F_{sc}$

Stopband : $F_i - F_s \le F \le \frac{F_{i-1}}{2}$

where $\boxed{F_i = \frac{F_{i-1}}{D_i}}$.

Problem No : 5 :

check whether the decimator and interpolator is time variant (or) time invariant

(i) Decimator :

$y(n) = x(Dn)$

$y(n,k) = x(Dn-k)$ —— (1)

$y(n-k) = x(D(n-k)$

$= x(Dn-Dk)$ —— (2)

$y(n,k) \neq y(n-k)$

So decimation operation is time variant system.

(ii) Interpolator :

$y(n) = x(n/I)$

$y(n,k) = x(n/I - k)$ —— (1)

$y(n-k) = x\left(\frac{n-k}{I}\right)$ —— (2)

$y(n,k) \neq y(n-k)$

So interpolation operation is time variant system.

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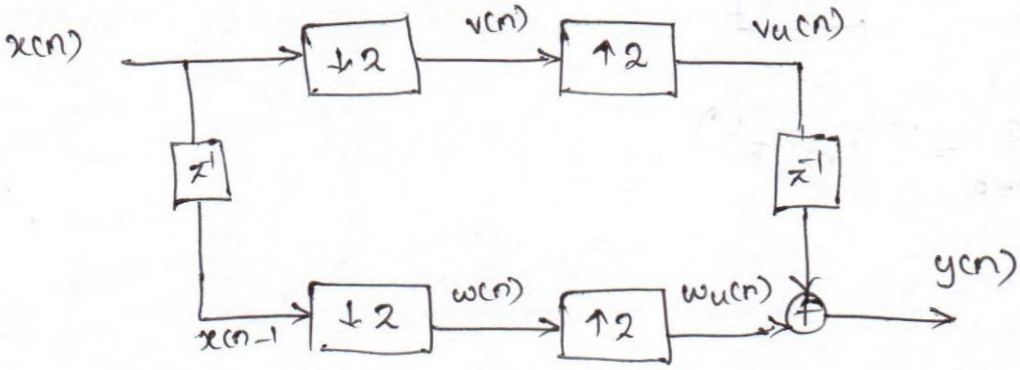
Problem No 6 :-

Find the o/p of the multirate system.



The o/p $y(n) = wu(n) + vu(n-1)$

Let $x(n) = \{ \ldots \; x(-1), x(0), x(1), x(2) \ldots \}$

$v(n) = x(2n)$ [Take only every 2^{nd} sample].

$v(n) = \{ \ldots x(-2), x(0) \; x(2), x(4), x(6) \ldots \}$

$vu(n) = v(n/2)$ · [add (2-1) zeros b/w samples]

$vu(n) = \{ \ldots x(-2), 0, x(0), 0, x(2), 0, x(4), 0, x(6) \ldots \}$.

$vu(n-1) = \{ x(-2), 0, x(0), 0, x(2), 0, x(4), 0, x(6) \ldots \}$

Now

$x(n-1) = \{ \ldots \; x(-1), x(0), x(1), x(2) \ldots \}$

$w(n) = x(2(n-1))$

$w(n) = \{ \ldots \; x(-1), \; x(1), \; x(3), \; x(5) \ldots \}$

$w_u(n) = \{ x(-1), 0, x(1), 0, x(3), 0, x(5) \ldots \}$

$\uparrow$

$y(n) = v_u(n-1) + w_u(n)$

$y(n) = \{ \underset{\uparrow}{x(-1)}, x(0), x(1), x(2), x(3) \ldots \}$

$$\boxed{y(n) = x(n-1)}$$

Problem No 7 :

Find whether the downsampler is linear or not ?

The Decimation is

$$y(n) = x(nD)$$

$$y_1(n) = x_1(nD)$$

$$y_2(n) = x_2(nD)$$

The weighted sum of o/p is

$$y_3(m) = a_1 y_1(n) + a_2 y_2(n) \quad \text{——(I)}$$

$$= a_1 x_1(nD) + a_2 x_2(nD)$$

The o/p due to weighted sum of i/p is

$$y_4(n) = a_1 x_1(nD) + a_2 x_2(nD) \quad \text{——(II)}$$

The system is linear

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Problem No: 8.

Find whether the interpolator is linear or not

The interpolation is

$$y(n) = x\left(\frac{n}{I}\right)$$

Let $y_1(n) = x_1\left(\frac{n}{I}\right)$

$$y_2(n) = x_2\left(\frac{n}{I}\right)$$

The weighted sum of o/p is

$$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

$$= a_1 x_1\left(\frac{n}{I}\right) + a_2 x_2\left(\frac{n}{I}\right)$$

The o/p due to weighted sum of i/p is

$$y_4(n) = a_1 x_1\left(\frac{n}{I}\right) + a_2 x_2\left(\frac{n}{I}\right)$$

Thus the system is linear

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Adaptive channel Equalization:-

Adaptive equalizer is used to compensate for the distortion caused by the transmission medium (channel).

Symbols are transmitted through the channel and corrupted by additive complex valued white noise. The received signal is processed by the equaliser to generate estimate $(\hat{d}(n))$.

The equaliser possesses the following two modes of operation.

(i) A Training mode, during which a known i/p signal is used as a reference signal.

(ii) A decision directed mode during which the output of decision device replaces the reference sequence.

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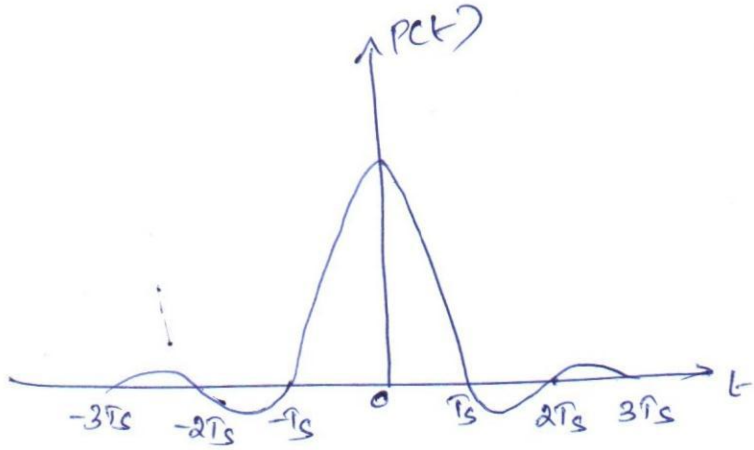
In the training mode, a known data sequence d(n) is transmitted. Then the equalizer o/p $\hat{d}(n)$ is compared with d(n) and an error is generated. This error signal is used to adjust the coefficients of the equalizer.

The digital sequence of information symbols a(n) is fed to the transmitting filter whose output is

$$s(t) = \sum_{k=0}^{\infty} a(k)\, p(t - kT_s)$$, where p(t) is the impulse response of the filter and T_s is the time interval b/w information symbols.

symbol rate = $1/T_s$.



(i) Pulse shape for the symbol at rate $1/T_s$

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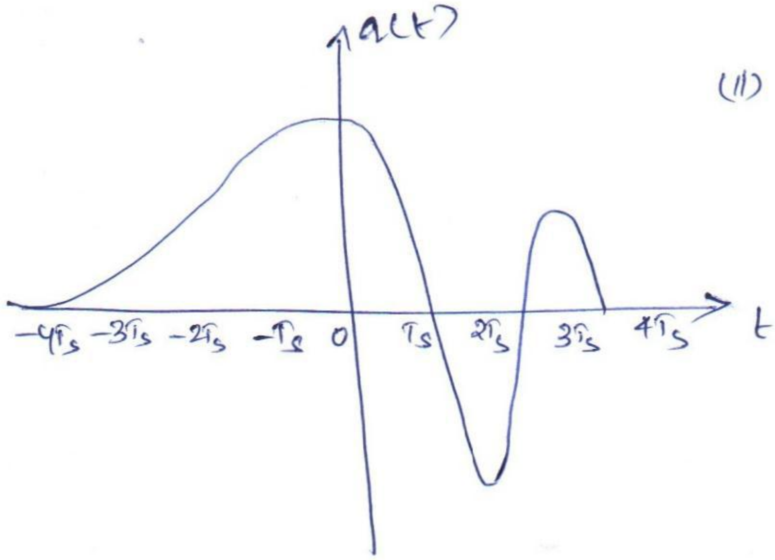
The channel, which is usually well modeled as a linear filter, distorts the pulse and thus causes intersymbol interference. For eg. in telephone channels, filters are used throughout the system to separate signals in different frequency ranges. The distorted signal is also corrupted by additive noise.



(ii) Effect of channel distortion on the pulse

At the receiving end of the communication system, the signal is first passed through a filter to eliminate the noise outside the frequency band.

The o/p of the filter reflect the presence of intersymbol interference and additive noise.

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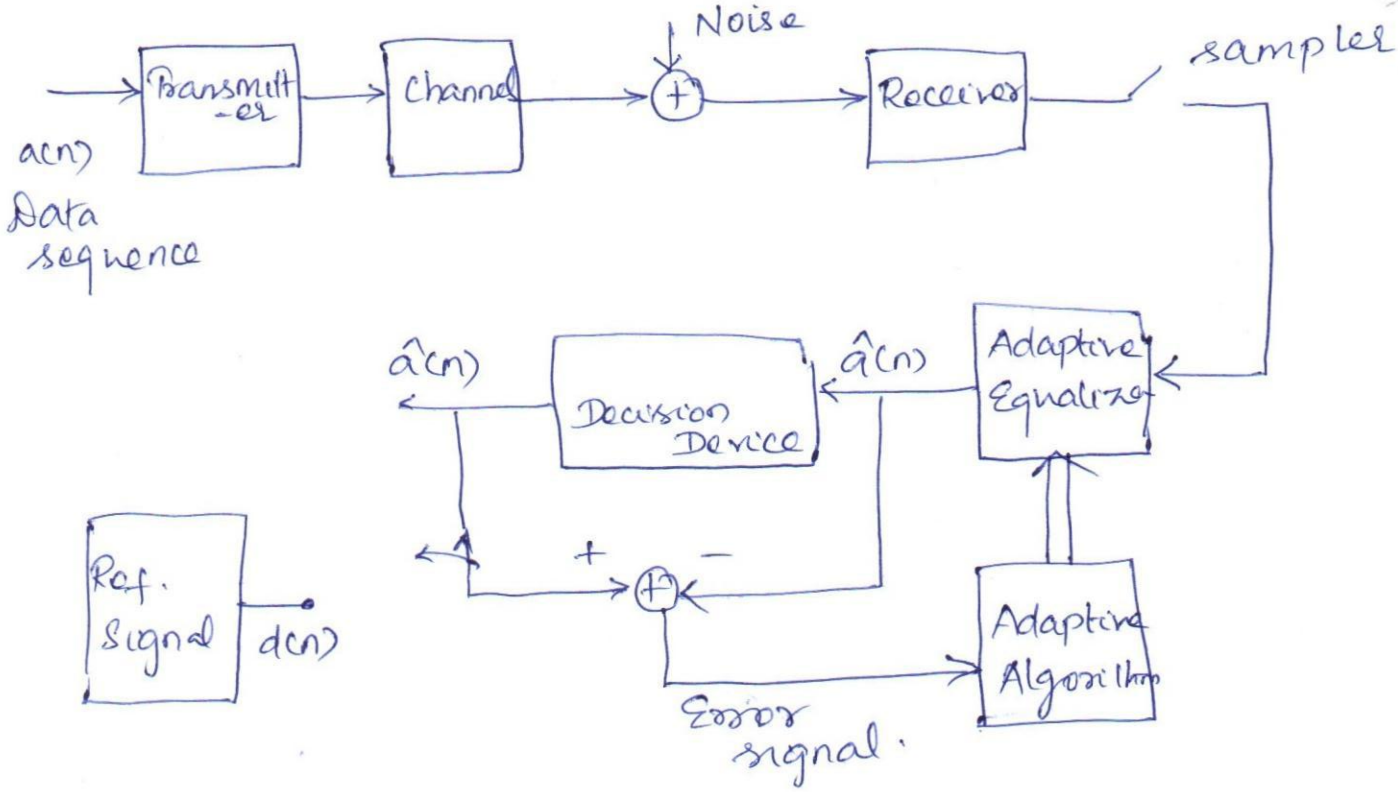

The sampled output at the receiver side is

$$x(nT_s) = \sum_{k=0}^{\infty} a(k)\, q(nT_s - kT_s) + w(nT_s).$$

where $w(t) \rightarrow$ additive noise

$q(t) \rightarrow$ distorted pulse at the output of the receiver filter.

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The channel vary slowly with time such that the intersymbol interference effects are time variant. The adaptive equalizer is an FIR filter with M adjustable coefficients h(n).

$$\hat{a}(n) = \sum_{k=0}^{M-1} h(k)\,x(n+\Delta-k).$$

$\Delta \rightarrow$ Delay in the signal.

$\hat{a}(n) \rightarrow$ estimate of the n^{th} information symbol.

Only After the training mode, the transmitter begins to transmit the input sequence a(n).

$$\text{Error} = \sum_{n=0}^{N} [d(n) - \hat{a}(n)]^2.$$

The coefficients are selected to minimize the error.

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During decision directed mode, the adaptive equalizer is adjusted continuously to track the correct sequence by finding the time variations in the channel.

The output of the adaptive equalizer is sent to the decision device receiver to obtain estimate. The estimate of the error signal is used to adjust the coefficients of the adaptive equaliser.

After the determination of appropriate coefficients of the adaptive filter, the system decodes the signal and produces a new signal, $\hat{a}(n)$.

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DIGITAL SIGNAL PROCESSORS.

DSP's are general purpose microprocessors designed specifically for digital signal processing applications and intensive DSP algorithms.

It can be divided into two categories

(i) General Purpose Digital Signal Processors

(ii) Special Purpose Digital Signal Processors.

General Purpose Digital Signal Processors:

They contain special architecture and instruction sets optimized for DSP operations.

Eg. Fixed Point Processors

(i) TMS320C5X

(ii) TMS 320C54X

(iii) Motorola DSP563X

Floating Point Processors

(i) TMS320C4X

(ii) TMS320C67XX

Special Purpose Digital Signal Processors:

It consists of specified hardware for FFT, PCM and filtering.

Eg FFT Processor (PDSP 16515A, TM-44)

Programmable FIR filter (UPDSP 16256).

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Selecting DSP :-

The factors that influence the selection of a digital signal processor. are

(i) Architectural features

(ii) Execution speed.

(iii) Type of arithmetic

(iv) Word length.

Applications of PDSP :-

They are (i) Communication systems

(ii) Audio signal processing

(iii) control and Data acquisition

(iv) Biometric Information Processing

(v) Image / Video Processing.

(vi) Patient Monitoring.

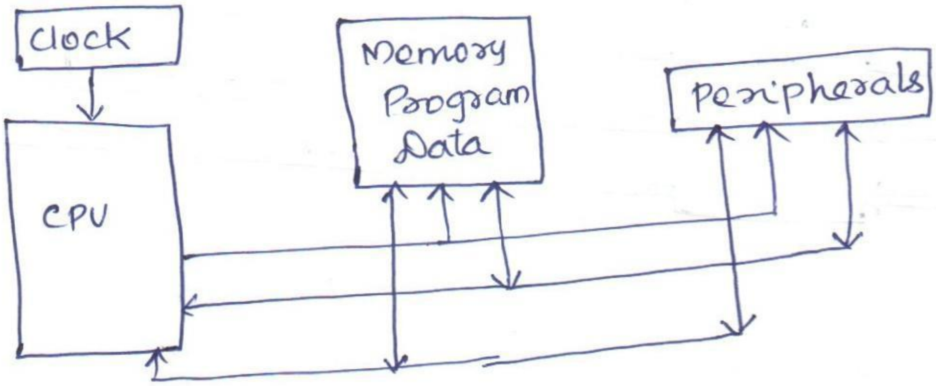
(vii) Music synthesis

(ix) Digital cellular Phones

(x) satellite communication.

Von Neumann Architecture :

* Mostly used in majority of microprocessors.
* In this architecture, the CPU can either read an instruction or read/write data from/to the memory.
* It uses only 1 bus system.
* The same bus carries all the information exchanged between the CPU and the peripherals.



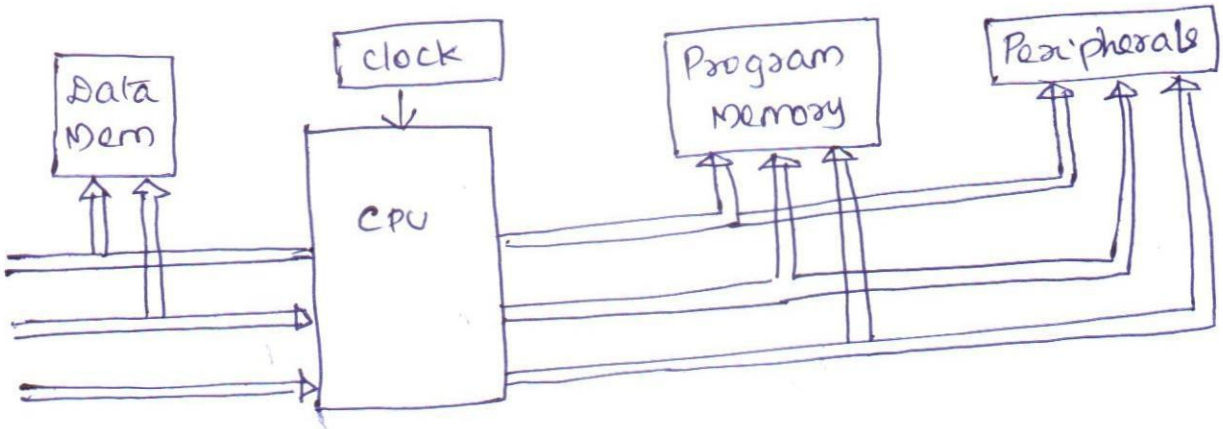
Harvard Architecture :-

In this architecture, there are separate memories for their instruction and data, requiring dedicated buses for each of them. Instructions and data can therefore be fetched simultaneously.

Most DSP processors use a modified Harvard architecture with two or three memory buses, allowing accessing to filter coefficients and input signals on the same cycle.

It can read an instruction code and at the same time, it can read/write the data.

It is less flexible. It needs two independent memory banks.



VLIW Architecture:

The Very Long Instruction Word (VLIW) processing increase the number of instructions that are processed per cycle.

It require multiple execution units and runs in parallel to carry out the instructions in a single cycle.

It combines many simple instructions into a single long instruction word that uses

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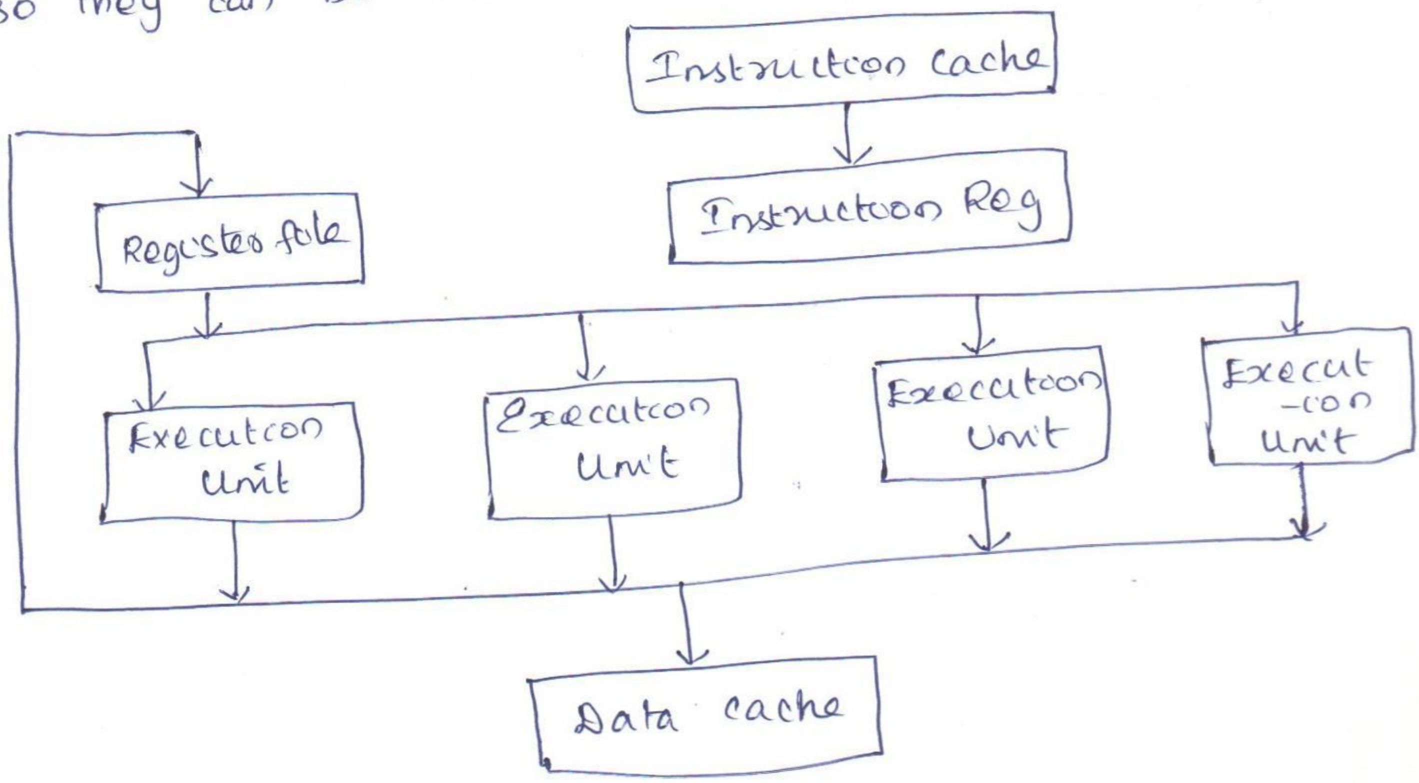
different registers.

The group might contain four instructions and the compiler ensures that those four instructions are not dependent on each other so they can be executed simultaneously.



Advantages of VLIW Architecture!

1. Increased performance
2. Better compiler targets
3. Potentially easier to program
4. Potentially scalable
5. Able to add more execution unit and allow more instructions to be placed into the VLIW instruction.

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Disadvantages of VLIW architecture:

1. Compiler complexity
2. Program must keep track of instruction scheduling
3. Increased memory use
4. High power consumption
5. Misleading MIPS ratings.

Architecture of TMS320C5X Processor :-

* The TMS320C5x is a 16 bit fixed point processor.
* It has advanced Harvard architecture.

The functional block diagram of TMS320C5x is divided into four sub blocks. They are

(i) Bus structure
(ii) Central Processing Unit
(iii) On chip Memory
(iv) On chip peripherals

I. Bus Structure :

* Separate bus for program and data provides high degree of parallelism
* More operations can be performed in a single machine cycle.
  Eg Multiply - Accumulate operation.

The architecture has four buses.

(i) Program Bus (PB) :
It carries the instruction code and immediate operands from program memory to CPU.

(iv) Program Address Bus : [PAB] :

It provides addresses to program memory for both reads and writes.

(iii) Data read Bus [DB]

It interconnects various elements of the CPU to data memory space.

(v) Data read Address Bus [DAB] :

It provides the address to access the data memory space.

II. Central Processing Unit :

It consists of the following elements.

(i) Central Arithmetic Logic Unit (CALU)
(ii) Parallel logic Unit (PLU)
(iii) Auxiliary register arithmetic Unit (ARAU)
(iv) Memory mapped registers
(v) Program Controller

1. Central Arithmetic Logic Unit :

The CALU consists of

* 16 bit × 16 bit Multiplier
* 32 bit Product register (PREG) holds the result of multiplication
* 32 bit Accumulator used for arithmetic and logical operations as a register.

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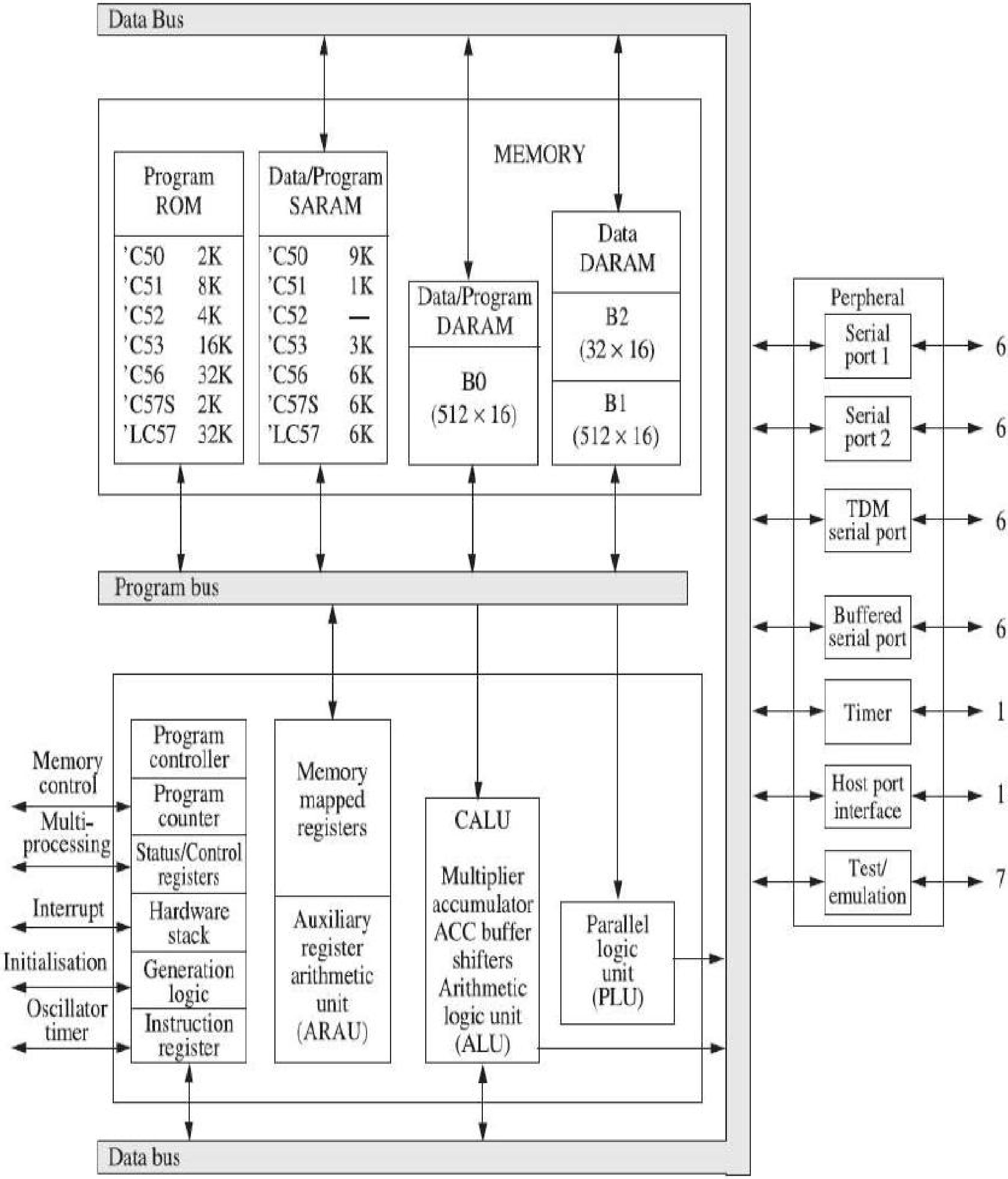
Architecture of TMS320C5X Processor:



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* Arithmetic Logic (ALU) performs the arithmetic and logical operations.
* The result is stored in the accumulator
* Accumulator Buffer → Temporary register.
* 0-16 bit Left Barrel Shifter and Right Barrel Shifter.

2. Parallel Logic Unit [PLU] :
   * It performs logical operations directly on data memory values without affecting the contents of the accumulator
   * It can directly set, clear, test or toggle bits in the status register, control register or any data memory location.

3. Auxiliary Register Arithmetic Unit [ARAU].
   * It consists of 8 nos of 16 bit auxiliary registers (AR_0 to AR_7)
   * 3 bit Auxiliary register pointer (ARP)
   * It is used for indirect addressing of the data memory

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* The 16 bit Index register is used by the ARAU to modify the address in the AR during indirect addressing

* ARCR - Auxiliary register Compare Register is a 16-bit register used for address boundary comparison.

3. Memory mapped register :

* It has 96 registers mapped into page 0 of the data memory space (00 - 5F h)

* It contains 28 CPU registers and 16 input/output port registers.

4. Program Controller :

* This contain logic circuits that decodes the instruction, manages the CPU pipeline, stores the status of CPU operations and decodes the conditional operations.

* It consists of

→ Program Counter

→ Status and Control registers

→ Hardware Stack

→ Address Generation Logic

→ Instruction register

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1. Program Counter :

→ It is a 16-bit counter which contains the address of program memory used to fetch instructions.

2. Status & Control Registers :

The processor has four status and control registers.

* Circular Buffer Control register
* Process mode status register
* Status registers ST0 & ST1.

III. On chip Memory in TMS320C5X Processor :

The C5X architecture has a total memory address range of 224 K words × 16 bit. The memory space is divided into four memory segments.

* 64 K word - Program Memory Space
* 64 K word - Local data Memory Space
* 64 K word - Input / output ports
* 32 K word - Global data Memory space.

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The on-chip Memory of C5x includes

* Program read only memory
* Data / Program single access RAM (SARAM)
* Data / Program Dual access RAM (DARAM)

ogram ROM:

The C5x DSP carry a 16 bit on-chip askable programmable ROM.

Pin MP/$\overline{MC}$ is high → Device starts its execution from off chip Memory

Pin MP/$\overline{MC}$ is low → Device executes from on chip Memory.

Data / Program Dual - Access RAM:

It is 1056 word × 16 bit memory.

It is divided into three memory blocks

(i) Block B_0 - 512 word data } Program or Data Memory

(ii) Block B_1 - 512 word data }

(iii) Block B_2 - 512 word data } Data Memory

Data / Program Single Access RAM:

The SARAM can be configured in three ays

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(i) Data memory only

(ii) Program memory only

(iii) Configured as both data and Program.

On-chip Peripherals:

It includes

(i) Clock Generator

(ii) Programmable wait state Generators

(iii) Hardware timer

(iv) Parallel I/O ports

(v) Host Port Interface (HPI)

(vi) Serial port

(vii) Buffered serial Port

(viii) TAM serial port

(ix) User Maskable interrupts.

Clock Generator:

It generate a low frequency clock than that of CPU [when PLL is selected].

Hardware Timer:

The timer is an on-chip down counter that can be used to periodically generate CPU interrupts.

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Software Programmable Wait state generators:

This logic is incorporated in C5X allowing wait state generation without any external hardware for interfacing with slower off chip memory and I/O devices.

Parallel I/O ports:

The C5X has 64 K parallel I/O ports.

Serial Port:

(i) General - Purpose serial port

(ii) TDM serial port

(iii) Buffered serial port.

User Maskable Interrupts:

The C5X has four external interrupts (INT1 - INT4) and five internal interrupts.

When an interrupt service routine (ISR) is executed, the contents of the program counter are saved on an 8-level hardware stack and the content of 11 specific registers are saved in 1 deep stack.

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Architecture of TMS320C67X Floating Point Processor:

Features:

- Advanced VLIW CPU of TMS 320C67X consists 32 general purpose register. Each register has 32-Bits.
- It has 8 functional units, each functional unit consists of two multiplier and 6 Arithmetic Logic units.
- It can execute 8 instructions per cycle. Highly effective RISC codes can be developed.
- Industry's first assembly optimizer for fast development and improved Parallelization
- Variable-width instructions: Flexibility of data types of 8/16/32-bit data support, providing efficient memory support
- It provides hardware support for single precision 32-Bits and double precision 64-Bits IEEE floating point operations.

TMS320C67X Devices come with

1. Program memory
2. Varying sizes of data memory
3. Peripherals
4. Direct memory access (DMA) controller
5. Power-down logic
6. External memory interface (EMIF)
7. Serial ports
8. Host ports

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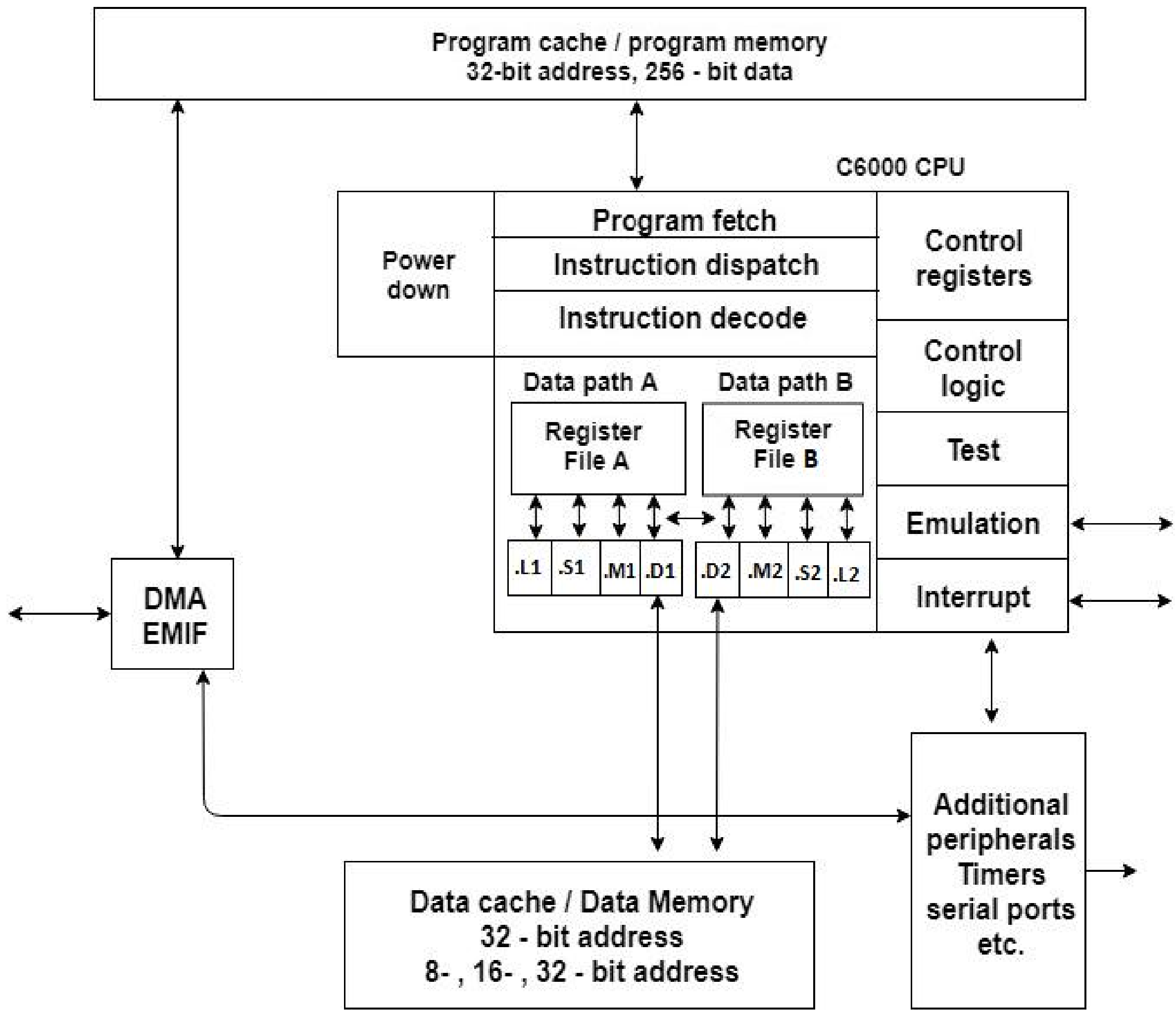

9.

Central processing unit (CPU)

The CPU contains:

- Program fetch unit.
- Instruction dispatch unit.
- Instruction decode unit.
- Two data paths, each with four functions units.
- 32 32-bit registers.
- Control logic.
- Test, emulation and interrupt logic.

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The program fetch, instruction dispatch and instruction decode unit can deliver up to eight 32 bit instructions to the functional unit every CPU clock cycle.

The processing of instruction occurs in each of the two data paths, each contains four functional units and 16, 32-bit general-purpose registers.

A control register file provides the means to configure and control various processor operation.

Components of Data Path

The components of data path consists of the following

- Two general-purpose register files (A and B)
  - The general-purpose registers can be used for data, data address pointers, or condition registers.
- Eight functional units (.L1, .L2, .S1, .S2, .M1, .M2, .D1, and .D2)
- Two 32–bit paths for loading data from memory to the register file
  - LD1 (LD1 LSB and LD1 MSB) for register file A
  - LD2 (LD2 LSB and LD2 MSB) for register file B
- Two 32–bit paths for storing data to memory from the register file
  - ST1 for register file A
  - ST2 for register file B
- Two data address paths (DA1 and DA2)
- Two register file data cross paths (1X and 2X).

Internal Memory

The c67x DSP has a 32 bit, byte addressable address space.

Internal memory is organized in separate data and prog spaces.

When off chip memory is used, these spaces are unified on most devices to a single memory space via the external; memory interface (EMIF).

Memory and peripheral options

A variety of memory and peripherals options are available for the C6000 platform.

- Large on chip RAM, up-to 7M bits
- Program cache.

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- 2 level cache.
- 32 bit external memory interface supports SDRAM, SBSRAM, SRAM, and other asynchronous memories for a board range of external memory requirement and max system performance.
- DMA controller transfers data between address ranges in the memory map without intervention by the CPU.
- EDMA controller performs the same functions as the DMA controller.
- HPI is a parallel port through which a host processor can directly access the CPU’s memory space.
- Expansion bus is a replacement for the HPI, as well as an expansion of the EMIF.
- McBSP is based on the standard serial port interface

Timers in the C67X devices are two 32-bit general-purpose timers used for these functions.

- Time event.
- Count event.
- Generate pulses.
- Interrupt the CPU.
- Send synchronization events to the DMA/EDMA controllers.

Power-down logic allows reduced clocking to reduce power consumption.

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ADDRESSING MODES:

The term addressing modes refers to the way in which the operand of an instruction is specified. The C5x supports the following six addressing modes.

1. Immediate addressing
2. Direct addressing
3. Indirect addressing.
4. Dedicated Register addressing
5. Memory - mapped register addressing
6. Circular Addressing.

Immediate Addressing:

In this mode, the operand (data) is specified in the instruction itself. It is used to handle either 16 bit constant data or (13, 9 & 8) constant data. It is indicated by the symbol #

Add #55h → adds 55h to the accumulator (Short Immediate addressing)

Add #1267h → adds 1267h to accumulator (Long Immediate addressing)

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Direct Addressing Mode :-

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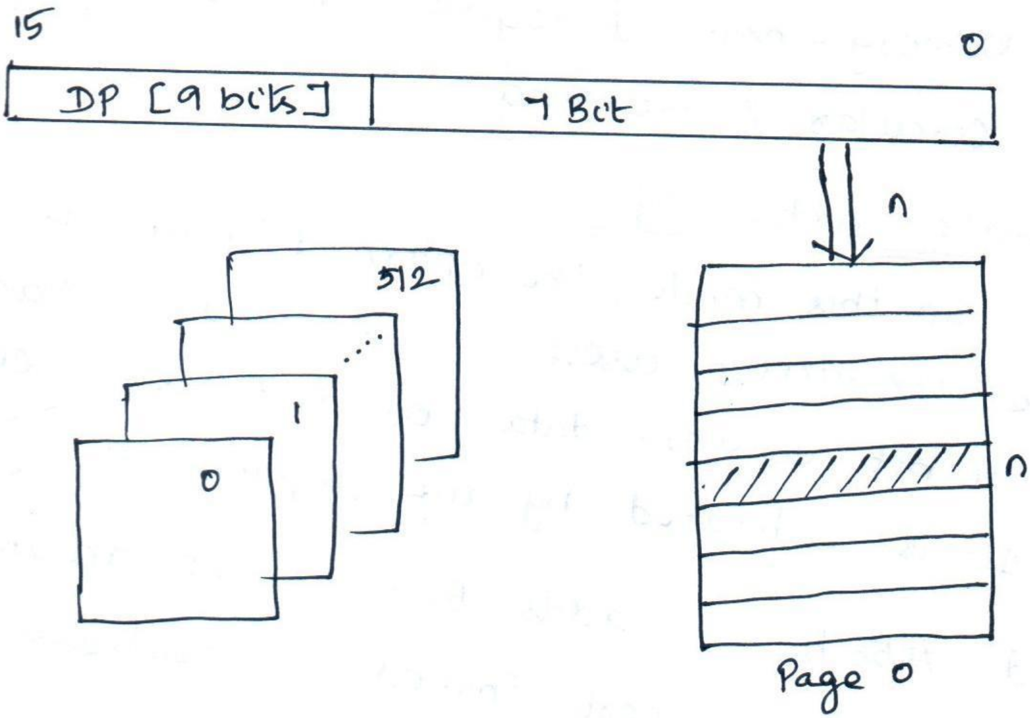
In this mode, the address specified in the instruction contains the data. The C5X processor has 512 pages each of 128 words long. Here, only lower oder 7 bits of the address are specified in the instruction and remaining upper bits (9) are taken from the Data Memory Page Pointer (DP). The DP is in status register (ST0).



Example ADDC, 2ch

↳ offset address (7 bit).

→ Add the data from the address given & carry with accumulator data.

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Indirect Addressing :-

In this mode, the address is specified in the auxiliary registers (AR0 – AR7). The AR register that is currently used is specified in the Auxiliary register Pointer (ARP).

Example :

1. LACC *, 0 → Load the content of data memory addressed by AR.

Let ARP → $\boxed{2}$

AR2 → $\boxed{1250\,H}$ (Data memory address)

Address 1250H → $\boxed{FABC}$ ACC → $\boxed{1023}$

After Execution

ACC → $\boxed{FABC}$

	Symbol	Value of AR after Execution
1.	*	AR unaltered
2.	*+	AR incremented by 1
3.	*–	AR decremented by 1
4.	*0+	AR incremented by the content of INDX
5.	*0–	AR decremented by the content of

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6. * BR0+ AR incremented by the content of INDX with reverse carry propagation

7. * BR0- AR decremented by the content of INDX with reverse carry propagation.

Example

LACC *+, 1

Let ARP → $\boxed{2}$

AR_2 → $\boxed{1250}$ Memory address

1250 → $\boxed{2345}$ Data.

Acc → $\boxed{0000}$ Data.

After Execution.

AR_2 → $\boxed{1251}$

ACC → $\boxed{468A}$ 2345 is left shifted by 1

Note

	2	3	4	5	
	0010	0011	0100	0101	left shift by 1
	0100	0110	1000	1010	
	4	6	8	A	

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Memory Mapped Register Addressing :-

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This mode of addressing is a special case of direct addressing in which only 7 bit page offset address is used and the default (9 bit) address is 000 H. Therefore the DP (data pointer) is need not to be loaded.

Example:

LAMM 16 H : Load the accumulator with the content of memory mapped register of address 0016 H.

SAMM 20H : Store the content of the accumulator to the address 0020 H.

Before Execution | After Execution

LAMM 16 H

Before Execution		After Execution	
ACC	2345	ACC	1205
0016	1205	0016	1205

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Register addressing :-

In this mode, the address of the data is specified in one of two special register

BMAR → Block Move address register

DBMR → Dynamic Bit Manipulation register.

Example:

BLDD BMAR, DAT 100

If BMAR contains the value of 300 H, then the content of data memory location 300 H is copied to data memory location 100 H.

Circular addressing :-

This mode allows the specified memory register buffer to be accessed sequentially.

It automatically goes to the beginning of the buffer when last location is accessed.

For this operation, there are five registers:

1. CBSR1 - Circular buffer 1 - start address.
2. CBSR2 - Circular buffer 2 - start address.

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3. CBER 1 - Circular Buffer 1 End register

4. CBER 2 - Circular Buffer 2 End register

5. CBCR - Circular Buffer Control register.

The 8 bit CBCR enables or disables the circular buffer operation.

Many algorithms such as convolution, correlation and FIR filters can use circular buffer.

Instructions of TMS320 5X Processor :-

The TMS320C5X processors instruction set consists of instructions that supports both numeric-intensive signal processing operations and general-purpose applications.

The instructions can be classified into following groups.

1. Arithmetic Instructions
2. Logical Instructions
3. Branch / Control Instructions
4. Load / Store Instructions
5. Block Move Instructions
6. Push and Pop Instructions.
7. Repeat Instructions
8. IN and OUT Instructions.

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1. Arithmetic Instructions:-
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(i) ADD, #23H:

Acc is added with immediate constant 23H.

(ii) ADD, #2345H, 2:

Data 2345H is left shifted by two positions before it is added to Accumulator.

(iii) ADCB: The contents of the accumulator Buffer (ACCB) and the value of the C bit are added to the contents of the ACC and the result is stored in Acc.

(iv) ADRK #K:

The 8 bit immediate constant value is added to the current auxiliary register (AR). The result is stored in the AR.

eg: ARRK #25H

(v) SBB:

The content of the accumulator buffer (ACCB) are subtracted from the contents of the ACC.

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(vi) SUB dma, [shift]
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Eg Sub 25H, 2 : The accumulator is subtracted with the content of data memory shifting it left by two position.

(vii) MAC pma, dma.

→ Multiply and accumulate

(viii) MPYU : Multiply unsigned numbers.

(ix) MPYA : Multiply and Add the product to Acc.

(x) MADD : Multiply and add the product to Accumulator, the address of the operand is given in BMAR

Logical Instructions :

(i) AND # 23H5H, 2

The accumulator content is 'AND' operation with the content of the memory locations.

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(ii) ANDB : The content of the accumulator are ANDed with the content of the ACCB.

OR instruction :-

(iii) OR * ARI.

(iv) ORB. → The content of the accumulator are 'OR'ed with the contents of the buffer

XOR instruction :

(v) XOR dma.

(vi) XOR #LK, [shift].

(vii) XORB

Shift Instructions :

(i) ROL : Rotate accumulator content once (left)

(ii) ROLB : Rotate the acc and accb left once

(iii) ROR : Rotate acc right once

(iv) RORB : Rotate acc and ACCB right once.

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Load / Store Instructions:

(i) LACB : Load the contents of Acc to AccB

(ii) LACC : Load the data memory value, with left shift to Acc.

(iii) LACL : Load the data memory value to ACCL and zeros to ACCH

(iv) LAMM : Load the contents of memory-mapped register to ACCL and zeros filled to ACCH.

(v) SACB : Store the contents of ACC to ACCB.

(vi) SACH : Store the accumulator content with left shift to data memory address

(vii) SAMM : Store accl in memory mapped register.

(viii) LAR : Load data value to AR register

(ix) SAR : Store the AR value in the data memory location.

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(x) LDP : Load data memory value to Data Pointer.

Block Move Instructions:

(i) BLDP → Block move data from data memory to program memory.

(ii) BLDD → Block move data from data memory to another memory

(iii) BLPD → Block move data from program memory to data memory.

Branch and Call Instructions:

The DSProcessor has both conditional and unconditional branch & call instructions.

(i) B → Branch unconditionally.

(ii) BACC → Branch unconditionally to the address given by Accumulator

(iii) BANZ → Branch unconditionally if AR register is not zero.

(iv) CALA → Call a subroutine using Indirect addressing

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(v) CALL → Call unconditionally.

(vi) CC → Call conditionally.

Push and Pop instructions:

(i) Push:

Pushes the values down one level in the seven lower locations of the stack. The contents of ACCL are copied to the top of the stack.

(ii) PUSHD: Pushes a data memory location to the top of the stack.

(iii) POP: Pop of stack to low accumulator

(iv) POPD: Pops the top of the stack to a data memory.

RET instructions:

(i) Ret: Return from subroutine

(ii) RETD: Delayed return from subroutine

Repeat Instructions:

RPT: Repeat Next Instruction.

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(ii) RPTB : Repeat Block.

(iii) RPTZ : Repeat preceded by clearing of accumulator and product register

IN and OUT instructions :

IN → Input data from port

The 16 bit value from an external I/O port is read into the data memory address.

OUT → Output data to port

A 16-bit value from the data memory address is written to the specified I/O Port.

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