MA3354 **DISCRETE MATHEMATICS** **L T P C**
3 1 0 4

COURSE OBJECTIVES:

- To extend student's logical and mathematical maturity and ability to deal with abstraction.
- To introduce most of the basic terminologies used in computer science courses and application of ideas to solve practical problems.
- To understand the basic concepts of combinatorics and graph theory.
- To familiarize the applications of algebraic structures.
- To understand the concepts and significance of lattices and boolean algebra which are widely used in computer science and engineering.

UNIT I LOGIC AND PROOFS 9+3

Propositional logic – Propositional equivalences - Predicates and quantifiers – Nested quantifiers – Rules of inference - Introduction to proofs – Proof methods and strategy.

UNIT II COMBINATORICS 9+3

Mathematical induction – Strong induction and well ordering – The basics of counting – The pigeonhole principle – Permutations and combinations – Recurrence relations – Solving linear recurrence relations – Generating functions – Inclusion and exclusion principle and its applications.

UNIT III GRAPHS 9+3

Graphs and graph models – Graph terminology and special types of graphs – Matrix representation of graphs and graph isomorphism – Connectivity – Euler and Hamilton paths.

UNIT IV ALGEBRAIC STRUCTURES 9+3

Algebraic systems – Semi groups and monoids - Groups – Subgroups – Homomorphism's – Normal subgroup and cosets – Lagrange's theorem – Definitions and examples of Rings and Fields.

UNIT V LATTICES AND BOOLEAN ALGEBRA 9+3

Partial ordering – Posets – Lattices as posets – Properties of lattices - Lattices as algebraic systems – Sub lattices – Direct product and homomorphism – Some special lattices – Boolean algebra – Sub Boolean Algebra – Boolean Homomorphism.

TOTAL: 60 PERIODS

COURSE OUTCOMES:

At the end of the course, students would :

CO1: Have knowledge of the concepts needed to test the logic of a program.
CO2: Have an understanding in identifying structures on many levels.
CO3: Be aware of a class of functions which transform a finite set into another finite set which relates to input and output functions in computer science.
CO4: Be aware of the counting principles.
CO5: Be exposed to concepts and properties of algebraic structures such as groups, rings and fields.

TEXT BOOKS:

1. Rosen. K.H., "Discrete Mathematics and its Applications", 7th Edition, Tata McGraw Hill Pub. Co. Ltd., New Delhi, Special Indian Edition, 2017.
2. Tremblay. J.P. and Manohar. R, "Discrete Mathematical Structures with Applications to Computer Science", Tata McGraw Hill Pub. Co. Ltd, New Delhi, 30th Reprint, 2011.

REFERENCES:

1. Grimaldi. R.P. "Discrete and Combinatorial Mathematics: An Applied Introduction", 5thEdition, Pearson Education Asia, Delhi, 2013.
2. Koshy. T. "Discrete Mathematics with Applications", Elsevier Publications, 2006.
3. Lipschutz. S. and Mark Lipson., "Discrete Mathematics", Schaum's Outlines, Tata McGraw Hill Pub. Co. Ltd., New Delhi, 3rd Edition, 2010.

DIGITAL PRINCIPLES AND COMPUTER ORGANIZATION

To analyze and design combinational circuits.
To analyze and design sequential circuits
To understand the basic structure and operation of a digital computer.
To study the design of data path unit, control unit for processor and to familiarize with the hazards.
To understand the concept of various memories and I/O interfacing.

UNIT-I

LOGIC AND PROOFS

* Logical Connectives

Five Basic Connectives

S. No	English language Usages	Logical Connectives	Types of Operator	Symbols
1.	and	Conjuction	binary	$\wedge$
2.	or	disjunction	binary	$\vee$
3.	not	negation (or) denial	unary	$\neg$ or $\sim$
4.	If . . then	Implication or Conditional	binary	$\rightarrow$
5.	If and only if	biconditional	binary	$\leftrightarrow$

* Modular [compound] [composite] statements

Def: New statements can be formed from atomic statements through the use of connectives such as "and", 'or' etc. the resulting statements are called modular or compound statements

Eg: Niranjan is a boy and sita is a girl

Note: Atomic statements do not contain connectives.

Def Compound Propositions

Many mathematical statements are constructed by combining one or more propositions, new propositions, called compound propositions, are formed from existing propositions using logical operators.

Def Truth Table

A table, giving the truth values of a compound statement interms of its compound parts is called a 'Truth Table'.

Def Negation ($\neg$ or $\sim$) [Not]

The negation of a statement is generally formed by introducing the word 'not' at a proper place in the statement

The truth table for the negation of a proposition

p	$\neg p$
T	F
F	T

Eg: 1 P: Today is Monday [True]
$\neg P$: Today is not Monday [False]

2. $P: x < y$
$\neg P: x \nless y$ or $x \geq y$

Def: Conjuction [$\wedge$] [AND]

The conjuction of two statements P and Q is statement $P \wedge Q$ which is read as "P and Q"

Truth table

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

E.g $P: 4+3 < 5$ [False]
$Q: -3 > -5$ [True]
$P \wedge Q: 4+3 < 5$ and $-3 > -5$ [False]

Def: Disjunction [$\vee$] [or]

The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P or Q".

Truth Table.

P	Q	PvR
T	T	T
T	F	T
F	T	T
F	F	F

Eg: P: 2 is a positive integer [True]
Q: $\sqrt{2}$ is a rational number [True]
PvQ : 2 is a positive Integer or $\sqrt{2}$ is a rational number [True]

* Conditional Statement: [If .. then] [→]

If P and Q are any two statements then the statement P → Q which is read as "If P then Q" is called a conditional statement.

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

Eg: P: I am hungry [T]
Q: I will eat [T]
P→Q : If I am hungry, then I will eat (T)

* Biconditional [equivalance] [$\leftrightarrow$] [If and only if] Statement

If P and Q are any two statements then the statement $P \leftrightarrow Q$ which is read as "p if and only if Q" and abbriviated as "p iff Q" is called biconditional statement.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Eg:1 P: you can take the flight (T)

Q: you buy a ticket (T)

$P \leftrightarrow Q$: You can take the flight iff you buy a ticket (T)

* Contra positive

If $P \leftrightarrow Q$ is all Implication, then the converse of $P \to Q$ is the Implication $Q \to R$ and the Contrapositive of $P \to Q$ is the Implication $\neg Q \to \neg P$

Eg:1 Give the converse and contrapositive of the implication "If it is raining, then

I get wet" [A/M 2014]

Sol P: It is raining

Q: I get wet

$Q \to P$: (Converse) If I get wet, then it is raining

$\neg Q \to \neg P$: (Contrapositive) If I do not get wet, then it is not raining

Tautology:

Def: A statement formula which is true always irrespective of the truth values of the individual variables is called a tautology.

Eg: $P \vee \neg P$ is a Tautology.

Contradiction:

Def: A statement formula which is always false is called a contradiction.

Eg: $P \wedge \neg P$ is a contradiction.

Contigency:-

Def: A statement formula which is neither Tautology nor contradiction is called Contigency.

Eg: $P \Leftrightarrow Q$ is a contigency

Logical Equivalence or Equivalence Rules

1.	Idempotent laws	$p \wedge p \Leftrightarrow p \quad p \vee p \Leftrightarrow p$
2.	Associative laws	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
3.	Commutative law	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
4.	De-Margan's law	$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
5.	Distributive laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
6.	Complement laws	$p \wedge \neg p \Leftrightarrow F$ $p \vee \neg p \Leftrightarrow T$
7.	Dominance Laws	$p \vee T \Leftrightarrow T$ $p \wedge T \Leftrightarrow F$
8.	Identity laws	$p \wedge T \Leftrightarrow p$ $p \vee F \Leftrightarrow p$
9.	Absorption laws	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$
10.	Double negation law	$\neg(\neg p) \Leftrightarrow p$
11.	Contra positive law	$p \to q \Leftrightarrow \neg q \to \neg p$
12.	Conditional as disjunction	$p \to q \Leftrightarrow \neg p \vee q$
13.	Biconditional as conditional	$p \to q \Leftrightarrow (p \to q) \wedge (q \to p)$

14. Exportational laws $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$ 4

Problems

1. Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

Sol

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$	Reasons
$\Rightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$	Distributive law
$\Rightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R$	Associative law
$\Rightarrow [(\neg P \wedge \neg Q) \vee (Q \vee P) \wedge R$	Distributive law
$\Rightarrow [\neg (P \vee Q) \vee (P \vee Q)] \wedge R$	De-morgan law
$\Rightarrow T \wedge R$	$(P \vee \neg P \Rightarrow T)$
$\Rightarrow R$	$(P \wedge T \Leftrightarrow P)$

$\therefore$ Given statement formula is a tautology.

2. Show that $[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology [N/D-2013, A/M-2015]

Sol

$\neg(\neg P \wedge (\neg Q \vee \neg R))$	Reasons
$\Rightarrow \neg(\neg P \wedge \neg(Q \wedge R))$	Demorgan's law
$\Rightarrow P \vee (Q \wedge R)$	De-morgan's law
$\Rightarrow (P \vee Q) \wedge (P \vee R)$	Distributive law —(1)
Consider	
$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$	
$\Rightarrow \neg(P \vee Q) \vee \neg(P \vee R)$	Demorgan's law

$\Rightarrow \neg((P \vee Q) \wedge (P \vee R))$. Demorgan's law — (2)

From (1) & (2) we get

$((P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$

$\Rightarrow [(P \vee Q) \wedge (P \vee R)]$

$\vee \neg [(P \vee Q) \wedge (P \vee R)]$ Idempotent law

$\Rightarrow T$ $(\because P \vee \neg P \Leftrightarrow T)$

3. Show that $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$ [M/J 2013]

Sol

$(P \to Q) \wedge (R \to Q)$	Reasons
$\Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$	Since $P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow (\neg P \wedge \neg R) \vee Q$	Distributive law
$\Leftrightarrow \neg(P \vee R) \vee Q$	De-Morgan's law
$\Leftrightarrow P \vee R \to Q$	Since $\neg P \vee Q \Leftrightarrow P \to Q$

4. Show that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$

$P \to (Q \to P)$	Reasons $\neg P \to (\neg P \vee Q)$ $(P \vee \neg P) \vee Q_T$
$\Leftrightarrow P \to [\neg Q \vee P]$	Since $Q \to R \Leftrightarrow \neg Q \vee P$
$\Leftrightarrow \neg P \vee (\neg Q \vee P)$	Since $P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow \neg P \vee (P \vee \neg Q)$	Commutative
$\Leftrightarrow (\neg P \vee P) \vee \neg Q$	Associative
$\Leftrightarrow T \vee \neg Q$	Negation

$\Leftrightarrow T$ Since $T \vee \neg Q \Leftrightarrow T$

5. Show that $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$

Sol

	Reasons
i) $\neg P \vee (\neg P \vee Q)$	Associative law
$\Leftrightarrow (\neg P \vee \neg P) \vee Q$	Idempotent law $P \vee P \Leftrightarrow P$
$\Leftrightarrow \neg P \vee Q$	

$\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q))$	Given
$\Leftrightarrow \neg(P \wedge Q) \to (\neg P \vee Q)$	by (i)
$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee Q)$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\Leftrightarrow (P \vee (\neg P \vee Q)) \wedge (Q \vee (\neg P \vee Q))$	Distributive law $(P \wedge Q) \vee R \Leftrightarrow (P \vee R) \wedge (Q \vee R)$
$\Leftrightarrow ((P \vee \neg P) \vee Q) \wedge (Q \vee (Q \vee \neg P))$	Associative law & Comm. law
$\Leftrightarrow (T \vee Q) \wedge ((Q \vee Q) \vee \neg P)$	negation law & Asso law
$\Leftrightarrow T \wedge (Q \vee \neg P)$	Domination law & Idempo
$\Leftrightarrow Q \vee \neg P$	Identity law
$\Leftrightarrow \neg P \vee Q$	Comm. law

* Principal Disjunctive normal form (PDNF)

Def A logical formula P is said to be in principal disjunctive form (PDNF) if it is equivalent to a sum of minterms only.

* principle conjunctive normal form (PCNF)

Def A logical formula P is said to be in principal conjunctive normal form (PCNF) if it is equivalent to a product of maxterms only.

1) Find the PDNF and PCNF of the formula
$P \vee (\neg P \to (Q \vee (\neg Q \to R)))$

Sol Let A denote the given formula
$A = P \vee (\neg P \to [Q \vee (\neg Q \to R)])$

$A = P \vee (\neg\neg P \vee (Q \vee (\neg Q \to R)))$ [By conversion law]

$\Rightarrow P \vee (P \vee Q \vee (\neg\neg Q \vee R))$ [By conversion law]

$\Rightarrow P \vee (P \vee Q \vee (Q \vee R))$ [By negation law]

$\Rightarrow P \vee (P \vee (Q \vee Q) \vee R)$ [By Asso. law]

$\Rightarrow P \vee (P \vee Q \vee R)$ [By idempotent law]

$\Rightarrow (P \vee P) \vee (Q \vee R)$

$\Rightarrow P \vee (Q \vee R)$

$\Rightarrow P \vee Q \vee R$

This is the PCNF of A, as it is a maxterm for P, Q, R

To Find the PDNF: ~~we proceed as below~~

we find the PCNF of $\neg A$, which is the product of maxterms not in A.

$\therefore \neg A = (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R)$
$\wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$

$\wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee \neg R)$

Now $A \equiv \neg(\neg A)$

$= \neg(\neg P \vee Q \vee R) \vee \neg(P \vee \neg Q \vee R) \vee \neg(P \vee Q \neg R)$

$\vee \neg(\neg P \vee \neg Q \vee R) \vee \neg(P \vee \neg Q \vee \neg R) \vee \neg(\neg P \vee Q \vee \neg R)$

$\vee \neg(\neg P \vee N \neg R)$

$A \equiv (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$

$\vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee$

$(P \wedge Q \wedge R)$

which is the PDNF.

② obtain the principal conjunctive normal form (PCNF) and principal disjunctive normal form (PDNF) of $(\neg P \to R) \wedge (Q \leftrightarrow P)$ by using equivalences (M/J 2016, A/M 2017)

sol

let A denote the given expression

PCNF of A is the product of maxterms in P, Q, R.

$A \equiv (\neg P \to R) \wedge (Q \leftrightarrow P)$

$\equiv (\neg\neg P \vee R) \wedge ((Q \to P) \wedge (P \to Q))$

$\equiv (P \vee R) \wedge ((\neg Q \vee P) \wedge (\neg P \vee Q))$

$\equiv (P \vee R \vee F) \wedge (\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)$

(by identity law)

$\equiv [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [(\neg Q \vee P) \vee (R \wedge \neg R)]$

$\wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$ (Complement law)

$\equiv (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R)$
$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

$\equiv (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$
$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$ [omitting repetition]

This is the product of maxterms in P, Q, R and so it is the PCNF.

To Find the PDNF: Now the PCNF of $\neg A$ is the product of maxterms not in A

$\therefore \neg A = (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R)$

$\therefore A = \neg(\neg A) = \neg(\neg P \vee \neg Q \vee R) \vee \neg(\neg P \vee \neg Q \vee \neg R)$
$\vee \neg(P \vee Q \vee \neg R)$
$= (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

which is the sum of minterms and so it is the PDNF.

3. Find the principal disjunctive normal form (PDNF) of the statement
$(q \vee (p \wedge r)) \wedge \neg((p \vee r) \wedge q)$ [N/D 2012]

Sol $[q \vee (p \wedge r)] \wedge [\neg((p \vee r) \wedge q)]$

$\Rightarrow [q \vee (p \wedge r)] \wedge (\neg(p \vee r) \vee \neg q)$ [De-morgan $\neg(p \wedge q) = \neg p \vee \neg q$ (")

$\Rightarrow [q \vee (p \wedge r)] \wedge [(\neg p \wedge \neg r) \vee \neg q]$

$\Rightarrow [q \wedge (\neg p \wedge \neg r)] \vee [q \wedge \neg q]$
$\vee [(p \wedge r) \wedge (\neg p \wedge \neg r)] \vee [(p \wedge r) \wedge \neg q]$ (Distr. law)

$\Rightarrow (\neg p \wedge q \wedge \neg r) \vee F \vee (F \wedge F) \vee (p \wedge r \wedge \neg q)$ $[\because p \wedge \neg p = F]$

$\Rightarrow (\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$ (by Idempotent rule & $p \vee F = p$)

(c) obtain the principle disjunctive normal form of $(P \wedge Q) \vee (\neg P \wedge R)$

i) using truth table (ii) without using truth table

Sol

i) using Truth table

P	Q	R	$\neg P$	$P \wedge Q$	$\neg P \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R)$	minterms
T	T	T	F	T	F	$\boxed{T}$	$P \wedge Q \wedge R$
T	T	F	F	T	F	$\boxed{T}$	$P \wedge Q \wedge \neg R$
T	F	T	F	F	F	F	—
T	F	F	F	F	F	F	—
F	T	T	T	F	T	$\boxed{T}$	$\neg P \wedge Q \wedge R$
F	T	F	T	F	F	F	—
F	F	T	T	F	T	$\boxed{T}$	$\neg P \wedge \neg Q \wedge R$
F	F	F	T	F	F	F	—

The PDNF is

$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

ii) without using truth table:-

$(P \wedge Q) \vee (\neg P \wedge R)$

$\Rightarrow [(P \wedge Q) \wedge T] \vee [(\neg P \wedge R) \wedge T]$ $(\because P \wedge T = P)$

$\Rightarrow [(P \wedge Q) \wedge (R \vee \neg R)] \vee [(\neg P \wedge R) \wedge (Q \vee \neg Q)]$

$(\because (P \vee \neg P = T)$

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q)$

(Distributive law)

$\Rightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

is the required PDNF.

* THE THEORY OF INFERENCE

Using rules of inference

1. Rule P: A premise may be introduced at any point in the derivation.
2. Rule T: A formula S may be introduced in a given derivation if S is tautology implied by any one or more of the proceeding formula in the derivation.
3. Rule CP: If we can derive S from R and a set of premise then we can derive $R \to S$ from the set of premises alone.

S. No	Tautological form	Rules of inference	Name
1.	$p \Rightarrow (p \vee q)$	$\frac{p}{p \vee q}$	Addition
2.	$Q \Rightarrow (p \vee q)$	$\frac{q}{p \vee q}$	
3.	$p \wedge q \Rightarrow p$	$\frac{p \wedge q}{p}$	Simplication
4.	$p \wedge q \Rightarrow q$	$\frac{p \wedge q}{q}$	
5.	$[p \wedge (p \rightarrow q)] \Rightarrow q$	p, $p \rightarrow q$ / q	Modus phones
6.	$[\neg q \wedge (p \rightarrow q)] \Rightarrow \neg p$	$\neg q$, $p \rightarrow q$ / $\neg q$	Modus tollens
7.	$(p \vee q) \wedge (\neg p) \Rightarrow q$	$p \vee q$, $\neg p$ / q	Disjunctive syllogism
8.	$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$	$p \rightarrow q$, $q \rightarrow r$ / $\therefore p \rightarrow r$	Hypothetical syllogism (or) (transitive rule)
9.	$[(pq) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$	$p \vee q$, $p \rightarrow r$, $q \rightarrow r$ / r	Dilemma
10.	$[(p \vee q) \wedge (\neg p \vee r) \Rightarrow (q \vee r)$	$p \vee q$, $\neg p \vee r$ / $q \vee r$	Resolution

Problems

1. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$

(N/D 2016)

Sol

S. NO	Statement	Reason
1.	$P \to M$	Rule P
2.	$\neg M$	Rule P
3.	$\neg P$	1, 2 Rule T modus tollens
4.	$P \vee Q$	Rule P
5.	Q	3, 4 Rule T, disjunctive syllogism
6.	$Q \to R$	Rule P
7.	R	5, 6 Rule T, modus ponens
8.	$R \wedge (P \vee Q)$	4, 7, Rule T, conjunction

2. Show that $R \to S$ can be derived from the premise: $P \to (Q \to S)$, $\neg R \vee P$ and Q.

[N/D 2015, M/J 2016]

Sol

S.No	Statement	Reasons
1.	R	Assumed premise
2.	$\neg R \vee P$	Rule P
3.	$R \to P$	Rule T $(P \to Q \Leftrightarrow \neg P \vee Q)$
4	P	1, 2 Rule T $(P, P \to Q \Rightarrow Q)$

5.	$P \to (Q \to S)$	Rule P
6.	$Q \to S$	1, 2, 5 Rule P $(P, P \to Q \Rightarrow Q)$
7.	Q	Rule P
8.	S	1, 2, 4, 7 Rule T
9.	$R \to S$	1, 2, 5, 7 Rule CP

3. Show that $A \to \neg D$ follows logically from the premises $A \to B \vee C$, $B \to \neg A$ and $D \to \neg C$ by using Conditional proof [N/D 2014]

S.NO	Statement	Reasons
1.	A	Assumed promises
2.	$A \to B \vee C$	Rule P
3.	$B \vee C$	1/2 Rule T
4.	$\neg B \to C$	1, 2 Rule T
5.	$B \to \neg A$	Rule P
6.	$A \to \neg B$	5, Rule T
7.	$A \to C$	Rule T (1, 2, 5)
8.	$D \to \neg C$	Rule P
9.	$C \to \neg D$	8, Rule T
10.	$A \to \neg D$	1, 2, 5, 8 Rule CP

4. Prove that the premises $a \to (b \to c)$; $d \to (b \wedge \neg c)$ and $a \wedge d$, are inconsistent [N/D 2010]

Sol

S.NO	Statement	Reason
1.	$a \wedge d$	Rule P
2.	a	1 & simplification
3.	d	1 & simplification
4.	$a \to (b \to c)$	Rule P
5.	$b \to c$	2, 4 Modus phonens
6.	$\neg b \vee c$	5, equivalence
7.	$d \to b \wedge \neg c$	Rule P
8.	$\neg(b \wedge \neg c) \to \neg d$	7, contrapositive
9.	$\neg b \vee c \to \neg d$	8, Demorgans law
10.	$\neg d$	6, 9, Modus phonens
11.	$d \wedge \neg d$	3, 10 conjunction
12.	F	11, negation law

5. Using indirect method proof, derive $p \to \neg s$ from the premises $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$ and p [N/D 2011]

Sol

We have to prove that the given premise $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$ and $p \to \neg s$ by indirect method.

For this we assume the contrary $\neg(p \to \neg s)$ as an additional premise and come to contradiction

10

But $\neg(p \to \neg s) \equiv \neg(\neg p \vee \neg s) \equiv p \wedge s$, by De morgan's law

So we use $p \wedge s$ as the additional premise

S.L NO	Statement	Reasons
1.	$p \to (q \vee r)$	Rule P
2.	p	Rule P
3.	$q \vee r$	Rule T, 1, 2 and modus phonens
4.	$p \wedge s$	Rule P
5.	s	Rule T
6.	$s \to \neg r$	Rule P
7.	$\neg r$	Rule T, 5, 6 modus phonens
8.	q	Rule T 3, 7 disjunctive syllogism
9.	$q \to \neg p$	Rule P
10.	$\neg p$	Rule T, 8, 9 modus phonens
11.	$p \wedge \neg p$	Rule T, 2, 10, conjunction
12	F	Rule T, 11, negation law

6. Show that $R \vee S$ is a valid conclusion from the premises $C \vee D$, $C \vee D \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$ [N/D 2007, 2008]

Sol

S.NO	Statement	Reasons.
1.	$(C \vee D) \to H$	Rule P
2.	$\neg H \to (A \wedge \neg B)$	Rule P

3.	$C \vee D \rightarrow (A \wedge \neg B)$	Rule T, 1,2 Hypothetical syllogism
4.	$(A \wedge \neg B) \rightarrow (R \vee S)$	Rule P
5.	$(C \vee D) \rightarrow (R \vee S)$	Rule T, 3, 4 Hypothetical syllogism
6.	$C \vee D$	Rule P
7.	$R \vee S$	Rule T, 5, 6 modus phonens

7. Prove that the premises $P \rightarrow Q$, $Q \rightarrow R$, $R \rightarrow S$, $S \rightarrow \neg R$ and $P \wedge S$ are inconsistent. [N/D 2014]

S.NO	Statement	Reasons
1.	$P \rightarrow Q$	Rule P
2.	$Q \rightarrow R$	Rule P
3.	$P \rightarrow R$	1,2 Rule T (chain rule)
4.	$S \rightarrow \neg R$	Rule P
5.	$R \rightarrow \neg S$	Rule T [contrapositive]
6.	$P \rightarrow \neg S$	Rule T 1,2,4 Chain rule
7.	$\neg P \vee \neg S$	Rule T
8.	$\neg (P \wedge S)$	Rule T (1,2,4) De-morgans
9.	$P \wedge S$	Rule P
10.	$(P \wedge S) \wedge \neg (P \wedge S)$	Rule T 1,2,9

which is nothing but false value.

$\therefore$ Given set of promises are inconsistent.

8. Prove that $\sqrt{2}$ is irrational by giving a proof using contradiction [N/D 2011, M/J 2013]

Sol Assume $\sqrt{2}$ is rational number

$\therefore \sqrt{2} = \frac{p}{q}$ for some integers p and q

Such that p and q have no common factors

$$\frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

Since p^2 is an even integer, p is all even integer

$\therefore p = 2m$ for some integer m.

$\therefore (2m)^2 = 2q^2 \Rightarrow 4m^2 = 2q^2$

$\Rightarrow q^2 = 2m^2$

Since q^2 is even, q is an even integer

$\therefore q = 2k$ for some integer k

Thus p and q are even. Hence they have a common factor 2. This contradicts the assumption p and q have no common factors. Thus our assumption $\sqrt{2}$ is rational is wrong.

Hence $\sqrt{2}$ is irrational.

9. Determine the validity of the following argument If 7 is less than 4, then 7 is not a prime number. 7 is not less than 4. Therefore 7 is a prime number. (M/J 2012)

Sol Let A : 7 is less than 4

B : 7 not a prime number

then, given premises are

1. $A \rightarrow \neg B$
2. $\neg A$

S.NO	Statement	Reason
1.	$\neg A$	Rule P
2.	$A \rightarrow \neg B$	Rule P
3.	$\neg(\neg B)$	1, 2 Rule T
4.	B	1, 2 Rule T

10. Show that the hypothesis. "It is not sunny this afternoon and it is colder than yesterday". "We will go swimming only if it is sunny". "If we donot go swimming then we will take a canoe trip" and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "we will be home by suset" [N/D 2012, N/D 2013]

Sol Let A : It is not sunny

B : It is colder than yesterday

C : We will go swimming

D : We will take a canoe trip

E : We will be home by sunset

The given premises are

(1) $\neg A \wedge B$ (2) $A \rightarrow C$ (3) $\neg C \rightarrow D$ (4) $D \rightarrow E$

Conclusion E :-

S.NO	Statement	Reason
1.	$\neg A \wedge B$	Rule P
2.	$\neg A$	Rule T
3.	$A \rightarrow C$	Rule P
4.	$\neg C$	1,2 Rule T
5.	$\neg C \rightarrow D$	Rule P
6.	D	1,2,5 Rule T
7.	$D \rightarrow E$	Rule P
8.	E	1,2,5,7 Rule T

** Quantifiers **

Quantifiers is one which is used to quantify the nature of variables there are two important quantifiers which are for "all" and for "some" where "Some" means "atleast one"

S.NO	Rule	Inference
1.	US	$\dfrac{\forall x P(x)}{P(c)}$ for some c
2.	ES	$\dfrac{\exists x P(x)}{P(c)}$ for a particular c
3.	UG	$\dfrac{P(c) \text{ for an arbitrary } c}{\forall x P(x)}$
4.	EG	$P(c)$ for some c / $\exists x P(x)$

problems

1. Show that

$(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$

[N/D 2016]

S.NO	Statement	Reason
1.	$(x)(P(x) \rightarrow Q(x))$	Rule P
2.	$P(y) \rightarrow Q(y)$	Rule US
3.	$x(Q(x) \rightarrow R(x))$	Rule P
4.	$Q(y) \rightarrow R(y)$	3, Rule US
5.	$P(y) \rightarrow R(y)$	1, 3, Rule T chain rule $[P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R]$
6.	$(x)(P(x) \rightarrow R(x))$	Rule UG

2. Use indirect method of proof to prove that

$(\forall x)(P(x) \vee Q(x)) \Rightarrow (\forall x)P(x) \vee (\exists x)Q(x)$

[A/M 2011, N/D 2011]

Sol We shall use the indirect method of proof.

Assume $\neg[(x)\, P(x) \vee (\exists x)\, Q(x)]$ as all additional premises

S. NO	Statement	Reason
1.	$\neg[(x)P(x) \vee (\exists x)Q(x)]$	Assumed Premises
2.	$(\exists x) \neg P(x) \wedge (x) \neg Q(x)$	Rule T
3.	$(\exists x) \neg P(x)$	Rule T
4.	$(\exists x) \neg Q(x)$	Rule T
5.	$\neg P(y)$	Rule ES
6.	$\neg Q(y)$	Rule US
7.	$\neg P(y) \wedge \neg Q(y)$	Rule T
8.	$\neg (P(y) \vee Q(y))$	Rule T
9.	$(x)(P(x) \vee Q(x))$	Rule P
10.	$P(y) \vee Q(y)$	Rule US
11.	$[P(y) \vee Q(y)] \wedge \neg[P(y) \vee Q(y)]$	Rule T

Which is nothing but false value. therefore by method for contradiction we have,

$$(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists (x))\, Q(x)$$

(4) write the symbolic form and negate the following statements:

(i) Every one who is healthy can do all kinds of work

(ii) Some people are not admired by everyone

(iii) Every one should help his neighbours or his neighbours will not help him [A/M-15]

Sol

i) Let $H(x)$ represent "x" is healthy

$W(x)$ represent "x" can do all kind of works.

Then statement in symbolic form is

$(x)\ (H(x) \rightarrow W(x))$

Negation of this expression is

$\neg((x)\ (H(x) \rightarrow W(x))$

$\Rightarrow \neg((x)\ (\neg H(x) \vee W(x)))$

$\Rightarrow (\exists(x)\ (H(x) \wedge \neg W(x)))$

Some one who is healthy and cannot do all kind of works.

ii) Let $A(x)$: x is admired. Then the given statement can be written as, for some x, it is not a case that x is admired by every one.

Symbolic form is $(\exists x)\ (\neg A(x))$

Negation of the above statement is

$\neg((\exists x) \neg A(x)) = (\forall x)\ A(x)$

All people are admired by every one.

iii) Statement 3, can be restated as

" for all x, x is person x should

help his neighbours or his neighbours will not helpline."

Let $H(x)$: x help neighbour

$P(x)$: x is person

In symbolic term

$(x)\ [(P(x) \to H(x)] \vee (H(x) \to \neg P(x))]$

Negation of the above statement is

$\exists x\ [(H(x) \to P(x)) \wedge (\neg P(x) \to \neg H(x))]$

Unit - II

COMBINATORICS

Mathematical Induction:

The Word Induction refers to the method of inferring a general statement from the validity of particular cases.

* Principles of Mathematical Induction:

Let $P(n)$ be a stmt or proposition for all positive integers 'n' then,

Step: 1 If $p(1)$ is true.

Step: 2 If $p(k+1)$ is true on "Assumption" then $p(k)$ is true.

Problems:

1. Prove by induction $1+2+3+\cdots+n = \frac{n(n+1)}{2}$; $n \geq 1$

Let $P(n)$ be ; $1+2+3+\cdots+n = \frac{n(n+1)}{2}$; $n \geq 1$

To prove $P(1)$ is true:

For $n=1$; We have ; $1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

$\Rightarrow$ P(1) is true.

Assume that P(K) is true for any positive integer 'K'

ie) $1+2+3+\cdots\cdots+K = \frac{K(K+1)}{2}$

To Prove : P(K+1) is true

$$P(K+1) = \frac{(K+1)(K+2)}{2}$$

$$[1+2+3+\cdots\cdots+K]+K+1 = \frac{K(K+1)}{2} + (K+1)$$

$$= \frac{K(K+1)+2(K+1)}{2}$$

$$= \frac{(K+1)(K+2)}{2}$$

$$= \frac{(K+1)[(K+1)+1]}{2}$$

Which is p(K+1)

ie) p(K+1) is true

$\therefore$ By the Principle of Mathematical Induction P(n) is true for all positive integers 'n'.

2. Show that $1^2+2^2+3^2+\cdots\cdots+n^2 \Rightarrow \frac{n(n+1)(2n+1)}{6}$; $n \geq 1$ by Mathematical Induction. [M/J 2012, M/J 2015, N/D 2016]

Soln: let P(n) be $1^2+2^2+3^2+\cdots\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

To Prove: P(1) is true

For $n=1$; $1^2 = \frac{1(1+1)(2+1)}{6} = \frac{2(3)}{6} = 1 \Rightarrow 1=1$

$\Rightarrow$ P(1) is true.

Assume that P(K) is true.

To Prove: P(K+1) is true. $\Rightarrow n = K+1$

$$\Rightarrow P(K+1) = \frac{(K+1)[(K+1)+1][2(K+1)+1]}{6} = \frac{(K+1)(K+2)(2K+3)}{6}$$

$$[1^2+2^2+\cdots+K^2]+(K+1)^2 = \frac{K(K+1)(2K+1)}{6} + (K+1)^2$$

$$= \frac{K(K+1)(2K+1)+6(K+1)^2}{6}$$

$$= \frac{(K+1)[2K^2+K+6K+6]}{6}$$

$$= \frac{(K+1)(K+2)(2K+3)}{6}$$

$$= P(K+1)$$

ie) P(K+1) is true; whenever P(K) is true.

By the Principle of Mathematical induction P(n) is true for all +ve integers 'n'.

3. Using M.I Show that $\sum_{r=0}^{n} 3^r = \frac{3^{n+1}-1}{2}$ [M/J 2016 ; A/M 2017]

Soln:

Let $P(n) \Rightarrow \sum_{r=0}^{n} 3^r = \frac{3^{n+1}-1}{2}$

To Prove: P(1) is true

let $P(n) \Rightarrow 3^0 + 3^1 + \cdots + 3^n = \frac{3^{n+1}-1}{2}$

Assume $P(0): 3^0 = \frac{3^{0+1}-1}{2} = 1 \Rightarrow$ true.

Assume $P(k): 3^0 + 3^1 + \cdots + 3^k = \frac{3^{k+1}-1}{2}$ is true

To Prove: P(k+1) is true

ie) to prove $P(k+1) = \frac{3^{k+2}-1}{2}$

$$3^0 + 3^1 + \cdots 3^k + 3^{k+1} = \frac{3^{k+1}-1}{2} + 3^{k+1} = \frac{3^{k+1}-1+2\cdot 3^{k+1}}{2}$$

$$= \frac{3\cdot 3^{k+1}-1}{2} = \frac{3^{k+2}-1}{2}$$

$\therefore$ P(k+1) is true

$\therefore$ By Mathematical induction, we have

$P(n): \sum_{r=0}^{n} 3^r = \frac{3^{n+1}-1}{2}$ is true, for $n \geq 0$.

4. Prove by Mathematical induction that $2^n > n \ \forall n \in N$. (or) $n < 2^n \ \forall n \in N$. [N/D 2012]

Soln:

let P(n) be $n < 2^n$

To Prove P(1) is true

$1 < 2^1 \Rightarrow 1 < 2 \Rightarrow$ P(1) is true.

Assume that $P(k)$ is true.

$\Rightarrow k < 2^k$

To Prove : $P(k+1)$ is true

ie) to Prove $P(k+1) = 2^{k+1}$

$k < 2^k$

$\Rightarrow (k+1) < 2^k + 1 \Rightarrow k+1 < 2^k + 2^k \quad (\because 1 \le 2^k)$

$\Rightarrow k+1 < 2(2^k)$

$\Rightarrow (k+1) < 2^{k+1} \Rightarrow P(k+1)$

$\Rightarrow P(k+1)$ is true

$\Rightarrow P(k)$ is true.

5. Prove by induction that a finite set with 'n' elements has exactly 2^n subsets.

(or)

Prove that the no. of Subsets of set having 'n' elements is 2^n. [M/J - 2014]

Soln:

let 'A' be a set with 'n' elements.

let $P(n)$ denote the proposition " the no. of subsets of a Set 'A' is 2^n ".

We've to prove $P(n)$ is true $\forall n \geq 0$.

let $n_0 = 0$

$\therefore A = \phi$; so A has exactly $2^0 = 1$ subset, which is true, $\because \phi$ is the only subset. $\Rightarrow P(0)$ is true.

Assume P(K) is true, K>0

A set with K elements has 2^k subsets is true.

To Prove P(K+1) is true.

To Prove a set A with (K+1) elements has 2^{k+1} subsets is true. let $a \in A$, then $B = A - \{a\}$ is a set with K elements. ∴ the no. of subsets of B is 2^K by induction hypothesis. ∵ Every subset of A either contains a (or) does not contain a, the total no. of subset of A is

$$2^K + 2^K = 2 \cdot 2^k = 2^{K+1}$$

∴ P(K+1) is true.

∴ By Principle of induction P(n) is true $\forall n \geq 0$

⇒ the no. of subsets of a set with 'n' element is 2^n.

6. Use Mathematical Induction to Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\cdots + \frac{1}{\sqrt{n}} > \sqrt{n};\ n \geq 2$$ [N/D 2011, 2016]

Soln:

let P(n): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots\cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

∴ P(2): $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \geq \sqrt{2}$

$\Rightarrow 1 + \frac{1}{\sqrt{2}} \geq \sqrt{2} \quad \Rightarrow 1 + \frac{\sqrt{2}}{2} \geq \sqrt{2}$ which is true

Now assume $P(K)$ is true $\forall K > 2$

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{K}} > \sqrt{K} \text{ is true} \quad \to (1)$$

To Prove $P(K+1)$ is true.

ie) to prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K+1}$ is true. $\to (2)$

Adding $\frac{1}{\sqrt{K+1}}$ on both sides of (1) we get,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K} + \frac{1}{\sqrt{K+1}}$$

Applying (2) in the above eqn: we get,

$$\sqrt{K} + \frac{1}{\sqrt{K+1}} > \sqrt{K+1}$$

ie) to prove that $\sqrt{K(K+1)} + 1 > K+1$

ie) $\sqrt{K^2+K} > K \Rightarrow K^2+K > K^2$

$\Rightarrow K > 0 \Rightarrow$ It is true

$\therefore K > 2$

$\therefore P(K)$ is true $\Rightarrow P(K+1)$ is true.

$\therefore$ By Mathematical Induction Principle $P(n)$ is true $\forall n \geq 2$.

7. Prove by Mathematical Induction to Prove that $3^n + 7^n - 2$ is divisible by 8 $\forall n \geq 1$ [M/J 2007, 2008]

Soln:

let $P(n)$ be the Proposition $3^n + 7^n - 2$ is divisible by 8.

To Prove $P(n)$ is true for all $n \geq 1$.

let $n_0 = 1$

$\Rightarrow P(1) = 3+7-2 = 8$; which is divisible by 8

$\therefore P(1)$ is true.

Assume $P(K)$ is true for K ; $K>1$

ie) $3^k+7^k-2=8$ is divisible by 8

$\Rightarrow 3^k+7^k-2 = 8x$; where x is integer $\rightarrow (1)$

To Prove $P(k+1)$ is true,

ie) to prove that $3^{k+1}+7^{k+1}-2$ is divisible by 8.

Consider $3^{k+1}+7^{k+1}-2$

$$= 3 \cdot 3^k + 7^{k+1} - 2 = 3[8x+2-7^k] + 7^{k+1} - 2 \text{ (From (1))}$$

$$= 24x + 6 - 3(7^k) + 7^{k+1} - 2$$

$$= 24x + 4 + 7^k(7-3)$$

$$= 24x + 4(7^k+1)$$

$\because 7^k$ is odd for all k, 7^k+1 is even.

$\therefore 7^k+1 = 8y$; (y is an integer)

$\therefore 3^{k+1}+7^{k+1}-2 = 24x+8y = 8(3x+y)$

$\therefore \Rightarrow 3^{k+1}+7^{k+1}-2$ is divisible by 8.

$\therefore P(k+1)$ is true

$\therefore$ By the 1st principle of induction $P(n)$ is true $\forall n \geq 1$

ie) 3^n+7^n-2 is divisible by 8 $\forall n \geq 1$ by strong induction and Well ordering.

8. Prove by Mathematical Induction that $6^{n+2} + 7^{2n+1}$ is divisible ~~di~~ by 43 ∀ positive integer 'n'.

Soln:

Let $P(n) : 6^{n+2} + 7^{2n+1}$ is divisible by 43

To Prove $P(1)$ is true,

$\Rightarrow 6^3 + 7^3 = 216 + 343 = 559 = 43(13)$

is divisible by 43

$\Rightarrow P(1)$ is true.

Assume that $P(k)$ is true

$\therefore 6^{k+2} + 7^{2k+1}$ is divisible by 43 is true.

$\Rightarrow 6^{k+2} + 7^{2k+1} = 43(r)$; where 'r' is a +ve integer

To Prove that $P(k+1)$ is true

$\Rightarrow 6^{k+3} + 7^{2k+3}$ is divisible by 43.

$$\therefore 6^{k+3} + 7^{2k+3} = 6^{k+3} + 7^{2k+1} \cdot 7^2$$

$$= 6^{k+3} + 7^2 [43r - 6^{k+2}]$$

$$= 6^{k+3} + 49 [43r - 6^{k+2}]$$

$$= 6^{k+2}(6-49) + 49 \cdot 43r$$

$$= -43 \cdot 6^{k+2} + 43 \cdot 49r$$

$6^{k+3} + 7^{2k+3} = 43[49r - 6^{k+2}]$ is divisible by 43

$\therefore P(k+1)$ is true

$\therefore$ By Mathematical induction $P(n)$ is divisible by 43 which is true.

9. Use Mathematical induction to show that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for all non-negative integers 'n'

Soln:

let $P(n): 2+2^2+2^3+\cdots+2^n = 2^{n+1}-1-1$

To Prove $P(1)$ is true

$$P(1) = 2^1 = 2^{1+1}-2 = 2$$

$\therefore P(1)$ is true

Assume that $P(k)$ is true

$$P(k): 2+2^2+2^3+\cdots+2^k = 2^{k+1}-2$$

To Prove that $P(k+1)$ is true.

$$P(k+1): 2+2^2+2^3+\cdots+2^k+2^{k+1}$$

$$\Rightarrow 2^{k+1}-2+2^{k+1}$$

$$\Rightarrow 2(2^{k+1})-2$$

$$P(k+1) = 2^{k+2}-2$$

$\therefore P(k+1)$ is true.

$\therefore$ By Mathematical Induction $P(n)$ is true.

10. Using mathematical Induction Prove that

$$1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Formula:

* $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

* $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

* $|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$

1. Find the no. of integers between 1 to 100 that are divisible by (i) 2, 3, 5 (or) 7 (ii) 2, 3, 5 but not by 7.

Soln

(i) Let A, B, C and D denote the number of +ve integers between 1 to 100 which are divisible by 2, 3, 5, 7.

$|A| = \left|\frac{100}{2}\right| = 50$ $\qquad$ $|D| = \left|\frac{100}{7}\right| = 14$

$|B| = \left|\frac{100}{3}\right| = 33$ $\qquad$ $|A \cap B| = \left|\frac{100}{2 \times 3}\right| = 16$

$|C| = \left|\frac{100}{5}\right| = 20$ $\qquad$ $|A \cap C| = \left|\frac{100}{2 \times 5}\right| = 10$

$|A \cap D| = \left\lfloor \frac{100}{2 \times 7} \right\rfloor = 7$ $\quad |B \cap C| = \left\lfloor \frac{100}{3 \times 5} \right\rfloor = 6$

$|B \cap D| = \left\lfloor \frac{100}{3 \times 7} \right\rfloor = 4$ $\quad |C \cap D| = \left\lfloor \frac{100}{5 \times 7} \right\rfloor = 2$

$|A \cap B \cap C| = \left\lfloor \frac{100}{2 \times 3 \times 5} \right\rfloor = 3$ $\quad |A \cap C \cap D| = \left\lfloor \frac{100}{2 \times 5 \times 7} \right\rfloor = 1$

$|A \cap B \cap D| = \left\lfloor \frac{100}{2 \times 3 \times 7} \right\rfloor = 2$ $\quad |B \cap C \cap D| = \left\lfloor \frac{100}{3 \times 5 \times 7} \right\rfloor = 0$

$|A \cap B \cap C \cap D| = \left\lfloor \frac{100}{2 \times 3 \times 5 \times 7} \right\rfloor = 0$

By Principle of Inclusion - Exclusion:

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= (50+33+20+14) - (16+10+7+6+4+2) + (3+2+1+0) - 0$$

$$= 117 - 45 + 6$$

$|A \cup B \cup C \cup D| = 78.$

ii) The no. of integers b/w 1 to 100 that are divisible by 2, 3, 5 but not by 7.

$$= |A \cap B \cap C| - |A \cap B \cap C \cap D|$$
$$= 3 - 0$$
$$= 3.$$

2. Determine the no. of positive integers n, $1 \le n \le 1000$, that are not divisible by 2, 3, 5 but divisible by 7.

Soln:

Let A, B, C and D denote the no. of positive integers between 1 – 1000 that are not divisible by 2, 3, 5 & divisible by 7.

$$\therefore \quad |D| = \left|\frac{1000}{7}\right| = 142.8 = 142$$

$$|A \cap B \cap C \cap D| = \left|\frac{1000}{2 \times 3 \times 5 \times 7}\right| = \left|\frac{1000}{210}\right| = 4.7 = 4$$

The nos. b/w 1 – 1000 that are divisible by 7 but not by 2, 3, 5 $\Big\}$ $= |D| - |A \cap B \cap C \cap D|$

$= 142 - 4$

$= 138.$

3. Find the no. of integers b/w 1 to 250 that are not divisible by any of the integers 2, 3, 5 and 7.

Soln

Let A denote the integer from 1 to 250 divisible by 2.

B denote the integer from 1 to 250 divisible by 3.

C denote the integer from 1 to 250 divisible by 5

D denote the integer from 1 to 250 divisible by 7

$$|A| = \left|\frac{250}{2}\right| = 125$$

$|B| = \left|\frac{250}{3}\right| = 83$

$|C| = \left|\frac{250}{5}\right| = 50$

$|D| = \left|\frac{250}{7}\right| = 35$

The no. of integer b/w 1 to 250 that are divisible by 2 & 3 $\Big\}$ $= |A \cap B| = \left|\frac{250}{2\times 3}\right| = 41$

The no of integee b/w 1 to 250 that are divisible by 2 & 5 $\Big\}$ $= |A \cap C| = \left|\frac{250}{2\times 5}\right| = 25$

$|A \cap D| = \left|\frac{250}{2\times 7}\right| = 17$

$|B \cap C| = \left|\frac{250}{3\times 5}\right| = 16$

$|B \cap D| = \left|\frac{250}{3\times 7}\right| = 11$

$|C \cap D| = \left|\frac{250}{5\times 7}\right| = 7$

The no. of integers divisible by 2, 3, 5 $\Big\}$ $= |A \cap B \cap C| = \left|\frac{250}{2\times 3\times 5}\right| = 8$

$|A \cap B \cap D| = \left|\frac{250}{2\times 3\times 7}\right| = 5$ $\qquad$ $|B \cap C \cap D| = \left|\frac{250}{3\times 7\times 5}\right| = 2$

$|A \cap C \cap D| = \left|\frac{250}{2\times 5\times 7}\right| = 3$ $\qquad$ $|A \cap B \cap C \cap D| = \left|\frac{250}{2\times 3\times 5\times 7}\right| = 1$

By Principle of Inclusion- Exclusion:

$$|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D| - |A\cap B| - |A\cap C| - |A\cap D| - |B\cap C| - |B\cap D| - |C\cap D| + |A\cap B\cap C| + |A\cap C\cap D| + |B\cap C\cap D| + |A\cap B\cap D| - |A\cap B\cap C\cap D|$$

$$= (125+83+50+35) - (41+25+17+16+11+7) + (8+5+3+2) - 1$$

$$= 293 - 117 + 18 - 1$$

$$= 193$$

$\therefore$ Nos. of integers not divisible by any of 2, 3, 5, 7 $= \text{Total} - |A\cup B\cup C\cup D|$

$$= 250 - 193$$

$$= 57.$$

4. Determine 'n' such that $1 \le n \le 100$ which are not divisible by 5 or by 7.

Soln:

let A denote the no. n, $1\le n\le 100$ which is divisible by 5

B denote the no. n, $1\le n\le 100$ which is divisible by 7.

$$|A| = \left\lfloor\frac{100}{5}\right\rfloor = 20$$

$$|B| = \left\lfloor\frac{100}{7}\right\rfloor = 14$$

$$|A\cap B| = \left\lfloor\frac{100}{5\times 7}\right\rfloor = 2$$

By Principle of Inclusion - Exclusion.

$$|A\cup B| = |A|+|B| - |A\cap B|$$

$$= 20+14-2$$

$$= 32.$$

The no. 'n' $1 \le n \le 100$ which is not divisible by either 5 or 7 is $= 100-32 = 68$.

5. In a Survey of 100 Students it was found that 30 Studied Mathematics, 54 Studied Statistics, 25 Studied Operations Research, 1 Studied all 3 subjects, 20 Studied maths & Statistics, 3 Studied mathematics & operations Research, 15 Studied Statistics & operations Research.

i) How many Students Studied none of these subjects?

ii) How many Studied only Mathematics?

Soln:

let A denotes students who studied mathematics.

B denotes Students who Studied Statistics

C denotes Students who studied operations Research

$|A| = 30$; $|B| = 54$; $|C| = 25$; $|A\cap B| = 20$; $|A\cap C| = 3$;

$|B\cap C| = 15$; $|A\cap B\cap C| = 1$

By Principle of Inclusion - Exclusion,

$$|A\cup B\cup C| = |A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= (30+54+25) - (20+3+15) + 1$$

$\therefore |A \cup B \cup C| = 110 - 38$

$\Rightarrow |A \cup B \cup C| = 72$

$\therefore$ Students who studied none of these Subjects $\Big\} = 100 - 72 = 28.$

No. of Students who studied only mathematics $\Big\} = n(A \cap B) - n(A \cap B \cap C)$

$= 20 - 1$

$= 19.$

Permutations & Combinations

Defn of Permutation:

Each different arrangements which can be made by taking some or all at a time is called a Permutation.

The no. of Permutations of 'n' things taken 'r' at a time is denoted by nP_r

1. In how many ways can letters of the word 'INDIA' be arranged?

Soln

The word Contains 5 letters of which 2 are I's

$\therefore$ The no. of Possible ways is $5P_2 \Rightarrow \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$

$= 60$ ways.

2. Find the no. of distinct permutations that can be formed from all the letters of each word (i) RADAR ii) UNUSUAL

Soln:

i) The word 'RADAR' contains 5 letters of which 2 A's and 2 R's are there

$\therefore$ The no. of Possible words $= \dfrac{5!}{2!\,2!} = \dfrac{5\times\cancel{4}\times3\times2\times\cancel{1}}{\cancel{2}\times\cancel{1}\times\cancel{2}\times\cancel{1}}$

$= 30$ ways.

ii) The word 'UNUSUAL' contains 7 letters of which 3 U's are there

$\therefore$ The no. of possible words $= \dfrac{7!}{3!} = \dfrac{7\times6\times5\times4\times\cancel{3!}}{\cancel{3!}}$

$\therefore$ No. of distinct Permutations $= 840$ ways.

3. A box contains 6 white balls and 5 Red balls. Find the no. of ways that 4 balls can be drawn from the box if

i) It can be any color?

ii) Two white & Two red?

iii) All of Same color?

Soln:

i) 4 balls of any color can be chosen from $(6+5) = 11$ balls in $^{11}C_4$ ways.

$$= \frac{11\times10\times9\times8}{4\times3\times2\times1}$$

$= 330$ ways.

ii) 2 white balls can be chosen in $6C_2$ ways.

2 Red balls can be chosen in $5C_2$ ways.

No. of ways selecting 4 balls = 2 white and 2 Red

$$= 6C_2 + 5C_2$$

$$= \frac{6\times5}{2\times1} + \frac{5\times4}{2\times1}$$

$$= 15 + 10$$

$= 25$ Ways.

iii) $\left.\begin{array}{l}\text{No. of ways selecting 4 balls}\\ \text{and all of same color}\end{array}\right\} = 6C_4 + 5C_4$

$$= \frac{6\times5\times4\times3}{1\times2\times3\times4} + \frac{5\times4\times3\times2}{1\times2\times3\times4}$$

$$= 15 + 5$$

$= 20$ Ways.

4. How many positive integers 'n' can be formed using the digits 3, 4, 4, 5, 5, 6, 7 if 'n' has to exceed 500,0000?

Soln:

In order that 'n' may be exceeds 50,00,000 the first place will be occupied by either 5 or 6 or 7.

If 5 occupies the first place, then the remaining 6 places are to be occupied by the digit 3, 4, 4, 5, 6, 7 such a no. is possible in $\frac{6!}{2!}$ ways $= 360$ ways

If 6 occupies the first place, then the remaining 6 places are to be occupied by 3, 4, 4, 5, 5, 7 which can be done in

$$\frac{6!}{2!\,2!} \text{ ways} = 180 \text{ ways}$$

If 7 occupies the first place, then the remaining 6 places can be occupied by 3, 4, 4, 5, 5, 6

$$\frac{6!}{2!\,2!} \text{ ways} = 180 \text{ ways.}$$

$$\therefore \left.\begin{array}{l}\text{Total no. of numbers} \\ \text{exceeds } 50{,}00{,}000\end{array}\right\} = 360 + 180 + 180$$

$$= 720 \text{ ways.}$$

5. A question paper has 3 parts, Part A, Part B and C having 12, 4, 4 Questions respectively. A Student has to answer 10 questions from Part A and 5 Questions from Part B and Part C put together selecting atleast 2 from each one of these two parts. In how many ways the selection of questions can be done.

Soln:

The Student can answer 15 questions in the following ways

either 1) 10 questions from part A, 3 questions from Part B and 2 questions from Part C.

Or 2) 10 questions from Part A, 2 questions from Part B and 3 questions from Part C.

The above 2 cases can be done in

$= (12C_{10} \times 4C_2 \times 4C_3) \times (12C_{10} \times 4C_3 \times 4C_2)$ ways

$= 2\,[12C_{10} * 4C_2 \times 4C_3]$

$= 2\,[66 \times 6 \times 4]$

$= 3168$ ways.

6. Prove that $nP_r = (n-r+1) \times nP_{r-1}$

Soln:

$$nP_r = \frac{n!}{(n-r)!}$$

$$\therefore nP_{r-1} = \frac{n!}{[n-(r-1)]!}$$

$$n! = n(n-1)!$$

$$\therefore (n-r+1)! = (n-r+1)(n-r)!$$

$$\Rightarrow n(n-r+1)P_{r-1} = (n-r+1) * \frac{n!}{[n-(r-1)]!}$$

$$= \frac{(n-r+1)\,n!}{(n-r+1)!} = \frac{(n-r+1)\,n!}{(n-r+1)(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$$= nP_r$$

7. What is the value of 'r' if $5P_r = 60$

Soln:

$$5P_r = 60 = 5\times4\times3$$

$$= 5P_3$$

$$\therefore \boxed{r=3}$$

8. Find the value of 'n' if $nP_3 = 5nP_2$

Soln:

$$nP_3 = 5nP_2$$

$$n(n-1)(n-2) = 5n(n-1)$$

$$n-2 = 5$$

$$\therefore \boxed{n=7}$$

9. Find 'n' if $nP_{13} : (n+1)P_{12} = \frac{3}{4}$

Soln:

$$nP_{13} = \frac{n!}{(n-13)!}$$

$$(n+1)P_{12} = \frac{(n+1)!}{(n+1-12)!} = \frac{(n+1)!}{(n-11)!}$$

$$\therefore \frac{nP_{13}}{(n+1)P_{12}} = \frac{n!}{(n-13)!} \times \frac{(n-11)!}{(n+1)!} = \frac{3}{4}$$

$$\frac{\cancel{n!} \times (n-11)(n-12)\cancel{(n-13)!}}{\cancel{(n-13)!}\,(n+1)\times\cancel{n!}} = \frac{3}{4}$$

$$\frac{(n-11)(n-12)}{n+1} = \frac{3}{4}$$

$$4[(n-11)(n-12)] = 3(n+1)$$

$$4[n^2-12n-11n+132] = 3(n+1)$$

$$4n^2-92n+528-3n-3=0$$

$$4n^2-95n+525=0$$

$$(n-15)(4n-35)=0$$

$$n=15 \quad \text{or} \quad n=\frac{35}{4}$$

$$\therefore \boxed{n=15}$$

10. How many bit strings of length 10 contain

i) Exactly 4 1's

ii) Atmost 4 1's

iii) Atleast 4 1's

iv) An equal no. of 0's and 1's.

Soln

i) A bit string of length 10 can be considered to have 10 positions. These 10 positions should be filled with 4 1's and 6 0's.

$\therefore$ No. of required bit string $= \dfrac{10!}{4!\,6!} = 210$ ways.

ii) The 10 position should be filled with

a) 0 1's & 10 0's

b) 1 1's & 9 0's

c) 2 1's & 8 0's

d) 3 1's & 7 0's

e) 4 1's & 6 0's

$\therefore$ Required no. of bit strings :

$$= \frac{10!}{0!\,10!} + \frac{10!}{1!\,9!} + \frac{10!}{2!\,8!} + \frac{10!}{3!\,7!} + \frac{10!}{4!\,6!}$$

= 386 ways.

iii) The 10 Positions are filled with

a) 4 1's & 6 0's (or)

b) 5 1's & 5 0's (or)

c) 4 1's & 6 0's (or)

$\vdots$

$\therefore$ Required no. of Strings

$$= \frac{10!}{4!\,6!} + \frac{10!}{5!\,5!} + \frac{10!}{6!\,4!} + \frac{10!}{7!\,3!} + \frac{10!}{8!\,2!}$$

$$+ \frac{10!}{9!\,1!} + \frac{10!}{10!\,0!}$$

= 848 ways.

iv) The 10 Positions are equally filled

$\therefore$ Required no. of Strings

$$= \frac{10!}{5!\,5!}$$

= 252 ways.

Recurrence Relations:

Defn:

Let $\{a_n\}$ be a sequence of real numbers with 'a_n' as the n^{th} term. A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses 'a_n' in terms of one or more of the earlier terms.

Characteristic Roots:

A linear homogeneous recurrence relations with constant coefficients.

Defn:

A Recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots\cdots + c_k a_{n-k} \longrightarrow (1)$$

where $c_1, c_2 \cdots\cdots c_k$ are real numbers and $c_k \neq 0$ is called a linear homogeneous recurrence relation of degree 'k' with constant coefficients.

The eqn: (1) is called a linear homogeneous difference eqn: of order 'k'.

The degree (order) $= n-(n-k)$

Case(i):

If r_1 & r_2 are real and different $a_n = Ar_1^n + Br_2^n$, where A and B are arbitrary constants.

Case(ii):

If r_1 and r_2 are real and equal

$$a_n = (A+Bn)r^n.$$

Case(iii):

If r_1 and r_2 are complex

$$a_n = r^n\left[A\cos n\theta + B\sin n\theta\right]$$ where $r = \sqrt{\alpha^2+\beta^2}$, $\tan\theta = \frac{\beta}{r}$.

Problems:

1. Solve $a_n = 3a_{n-1} + 4a_{n-2}$; $n \geq 2$; $a_0 = 0$; $a_1 = 5$

Soln:

Gn $a_n = 3a_{n-1} + 4a_{n-2}$; $n \geq 2$; $a_0 = 0$; $a_1 = 5$

$\Rightarrow a_n - 3a_{n-1} - 4a_{n-2} = 0 \longrightarrow (1)$

$\because$ $n-(n-2) = 2$, it is a second order relation.

$\therefore$ The characteristic eqn: is

$$r^2 - 3r - 4 = 0$$

$$\Rightarrow (r-4)(r+1) = 0$$

$$\Rightarrow r = 4 \;;\; r = -1$$

$$\Rightarrow r_1 \neq r_2$$

$\therefore$ The general soln is $a_n = A(4)^n + B(-1)^n$.

To find the values of A & B

using $a_0 = 0$; $a_1 = 5$

Put $n=0$ $\therefore$ $a_0 = A + B = 0 \rightarrow$ (2)

$n=1$ $\therefore$ $a_1 = 4A - B = 5 \rightarrow$ (3)

solving (2) & (3) we get $A = 1$; $B = -1$

$\therefore a_n = 4^n - (-1)^n$; $n \geq 0$

2. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$; $n \geq 2$

$a_0 = 2$; $a_1 = 3$

Soln

Gn $a_n = 6a_{n-1} - 9a_{n-2}$; $n \geq 2$; $a_0 = 2$; $a_1 = 3$

$\Rightarrow a_n - 6a_{n-1} + 9a_{n-2} = 0$

$\because n(n-2) = 2$; it is of order '2'

The characteristic eqn: is

$r^2 - 6r + 9 = 0$

$\Rightarrow (r-3)^2 = 0$

$\Rightarrow r = 3, 3$

$\Rightarrow r_1 = r_2$

$\therefore$ The general soln is $a_n = (A + Bn)3^n$.

We shall get the values of A & B by using $a_0 = 2$; $a_1 = 3$

Put $n=0 \Rightarrow a_0 = A = 2$

$n=1 \Rightarrow a_1 = (A+B)3 \Rightarrow 3A+3B = 3$

$6+3B=3$

$B=-1$

$\therefore$ The general Soln is $a_n = (2-n)3^n \; ; \; n \geq 0.$

3. Solve: $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial conditions $a_0 = 2 \; ; \; a_1 = 5 \; ; \; a_2 = 15.$

Soln

Gn $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \; ; \; a_0 = 2 ; a_1 = 5 ; a_2 = 15$

$\Rightarrow a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

$\because n-(n-3) = 3$ it is of Order 3.

The characteristic eqn: is $r^3 - 6r^2 + 11r - 6 = 0$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$r^2 - 5r + 6 = 0$

$\Rightarrow (r-2)(r-3) = 0$

$\Rightarrow r = 2, 3.$

$\therefore$ The roots are $r = 1, 2, 3$

$\therefore$ The general Soln is $a_n = A(1)^n + B(2)^n + C(3)^n.$ $n \geq 0$

$\Rightarrow a_n = A + B(2)^n + C(3)^n$; $a_0 = 2$; $a_1 = 5$; $a_2 = 15$.

Put $n=0$;

$a_0 = A+B+C \Rightarrow A+B+C = 2 \longrightarrow (1)$

Put $n=1$;

$a_1 = A+2B+3C \Rightarrow A+2B+3C = 5 \longrightarrow (2)$

Put $n=2$;

$a_2 = A+4B+9C \Rightarrow A+4B+9C = 15 \longrightarrow (3)$

$(2) - (1) \Rightarrow B+2C = 3 \rightarrow (4)$

$(3) - (2) \Rightarrow 2B+6C = 10 \Rightarrow B+3C = 5 \rightarrow (5)$

Solving 4 & 5, we get $C = 2$

$\Rightarrow B = -1$; $A = 1$.

$\therefore$ The general soln is $a_n = 1 - 2^n + 2(3)^n$; $n \geq 0$.

4. Solve $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$; $a_0 = 1$; $a_1 = 2$; $a_2 = 4$

Soln

Gn $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$

$\because n-(n-3) = 3$; It is of order 3.

The characteristic eqn: is $r^3 + 6r^2 + 12r + 8 = 0$

$(r+2)^3 = 0$

$r = -2, -2, -2$

$\therefore$ The general Soln is $a_n = (A + Bn + Cn^2)(-2)^n$; $n \geq 0$.

Put $n=0$; $a_0 = A \Rightarrow A = 1 \rightarrow (1)$

$n=1$; $a_1 = (A+B+C)(-2) \Rightarrow (A+B+C)(-2) = 2$

$B + C = -2 \rightarrow (2)$

$n=2$; $a_2 = (A+2B+4C)(4) \Rightarrow (A+2B+4C)4 = 4$

$2B + 4C = 0 \rightarrow (3)$

$\therefore$ solving (1), (2) & (3)

we get $B = -4$; $C = 2$.

$\therefore$ $a_n = (1 - 4n + 2n^2)(-2)^n$; $n \geq 0$.

5. Solve: $a_n + 3a_{n-1} - 4a_{n-2} = 0$; $n \geq 2$; $a_0 = 3$; $a_1 = -2$

Soln:

Given $a_n + 3a_{n-1} - 4a_{n-2} = 0$

$\because$ $n - (n-2) = 2$, it is of order 2.

$\therefore$ The characteristic eqn: $r^2 + 3r - 4 = 0$

$(r+4)(r-1) = 0$

$r = -4 ; 1$

$\therefore$ The general Soln is $a_n = A(-4)^n + B(1)^n$.

Put $n=0$; $a_0 = A + B \Rightarrow A + B = 3 \rightarrow (1)$

$n=1$; $a_1 = -4A + B \Rightarrow -4A + B = -2$

$\Rightarrow 4A - B = 2 \rightarrow (2)$

(1) + (2) $5A = 5 \Rightarrow \boxed{A = 1}$

$\Rightarrow B = 2$

$\therefore$ The general Soln is $a_n = (-4)^n + 2(1)^n$; $n \geq 0$.

Non-Homogeneous Linear Recurrence Relations with Constant Coefficients.

Defn:

A recurrence relation of the form

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(x) \longrightarrow (1)$$

where $c_0, c_1, c_2 \ldots c_k$ are constants with $C_0 \neq 0$; $C_k \neq 0$ is called non-homogeneous linear recurrence relations with constants.

The recurrence relation $C_0 a_n + C_1 a_{n-1} + \cdots C_k a_{n-k} = 0$ $\rightarrow (2)$ is called the associated homogeneous recurrence relation.

The Soln of (1) depends on soln of (2)

Let $a^{(h)}$ be the general soln of (2)

$\therefore$ The general Solution of (1) is $a_n = a_n^{(h)} + a_n^{(P)}$

1. Solve the recurrence relation $a_n - 2a_{n-1} = 2^n$; $a_0 = 2$

Soln:

Given: $a_n - 2a_{n-1} = 2^n$

The homogeneous recurrence relation is $a_n - 2a_{n-1} = 0$

$\because n - (n-1) = 1$ first order eqn:

$\therefore$ The characteristic eqn. is $r-2=0$

$\Rightarrow r=2$

$\therefore$ The soln is $a_n^{(h)} = C \cdot 2^n$.

Given $f(n) = 2^n$.

$\therefore$ $a_n = An2^n$ is the Particular Solution

$$a_n - 2a_{n-1} = 2^n$$

$$An2^n - 2A(n-1)2^{n-1} = 2^n$$

$$\cancel{2^n}\left[An - A(n-1)\right] = \cancel{2^n}$$

$$A(n-n+1) = 1$$

$$A = 1$$

$$\therefore a_n^{(P)} = n2^n$$

$\therefore$ The general solution is $a_n = a_n^{(h)} + a_n^{(P)}$

$$\Rightarrow a_n = C \cdot 2^n + n \cdot 2^n \rightarrow (1)$$

given $a_0 = 2$.

Put $n=0 \Rightarrow a_0 = C \Rightarrow C = 2$.

$\therefore$ The general soln is $a_n = 2 \cdot 2^n + n \cdot 2^n$

$$\Rightarrow a_n = (n+2)2^n, \quad n \geq 0$$

$f(n)$	Trial fn:
1. b^n (if b is not a root of the eqn:)	Ab^n
2. Polynomial $P(n)$ of degree m	$A_0 + A_1 n + A_2 n^2 \cdots A_m n^m$
3. $c^n P(n)$ [if 'c' is not a root of the eqn:]	$c^n [A_0 + A_1 n + A_2 n^2 \cdots A_m n^m]$
4. b^n [if 'b' is a root of the eqn: with multiplicity 's']	$A n^s b^n$
5. $c^n P(n)$ [if 'c' is a root of the eqn: with multiplicity 't']	$n^t [A_0 + A_1 n + \cdots A_m n^m] c^n$

2. Find the general Soln of $a_n - 5a_{n-1} + 6a_{n-2} = 4^n$; $n \geq 2$

Soln

Gn $a_n - 5a_{n-1} + 6a_{n-2} = 4^n \longrightarrow (1)$

The homogeneous recurrence relation is $a_n - 5a_{n-1} + 6a_{n-2} = 0$

$\because$ $n-(n-2) = 2$ (order

$\therefore$ The characteristic eqn: is $r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0 \Rightarrow r = 2, 3$

$\therefore$ The Solution of homogeneous eqn: $a_n^{(h)} = A \cdot 2^n + B \cdot 3^n$.

Given $f(n) = 4^n$, 4 is not a root of the characteristic eqn:

$\therefore$ The Particular Solution is $a_n^{(P)} = C \cdot 4^n$.

Sub in (1)

$$C \cdot 4^n - 5C \cdot 4^{n-1} + 6C \cdot 4^{n-2} = 4^n$$

$$4^{n-2} \cdot C\,[16 - 20 + 6] = 4^n$$

$$\Rightarrow 2C = 16$$

$$\Rightarrow C = 8$$

$$\therefore a_n^{(P)} = 8 \cdot 4^n$$

$\therefore$ The general solution is $a_n = a_n^{(h)} + a_n^{(P)}$

$$\Rightarrow a_n = A \cdot 2^n + B \cdot 3^n + 8 \cdot 4^n.$$

1. If $a_n = 3 \cdot 2^n$; $n \geq 1$ Find Recurrence Relation.

Soln

$a_n = 3 \cdot 2^n$ (gn)

Now $a_{n-1} = 3 \cdot 2^{n-1}$

$$= 3 \cdot \frac{2^n}{2}$$

$$a_{n-1} = \frac{a_n}{2}$$

$$\therefore a_n = 2(a_{n-1})$$

$\Rightarrow a_n = 2a_{n-1}$ for $n \geq 1$ with $a_0 = 3$

2. Find the recurrence relation satisfying $y_n = A\cdot 3^n + B(-2)^n$.

Soln:

Given : $y_n = A\cdot 3^n + B(-2)^n$

$$y_{n+1} = A\cdot 3^{n+1} + B(-2)^{n+1}$$

$$= 3\cdot A\cdot 3^n - 2B(-2)^n$$

$$y_{n+2} = A\cdot 3^{n+2} + B(-2)^{n+2}$$

$$= 9\cdot A3^n + 4B(-2^n)$$

$$y_{n+2} - y_{n+1} - 6y_n = 9A3^n + 4B2^n - 3A3^n + 2B(-2)^n - 6A3^n - 6B(-2)^n$$

$$\Rightarrow y_{n+2} - y_{n+1} - 6y_n = 0.$$

Generating Functions:

Defn:

The generating function of the sequence $a_0, a_1, a_2 \cdots a_n$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $G(x)$ is called the generating function.

1. Solve the recurrence relation using generating fn/:

$a_n = 4a_{n-1} - 4a_{n-2} + 4^n$; $n \geq 2$ given that $a_0 = 2$; $a_1 = 8$

Soln

let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given that $a_n - 4a_{n-1} + 4a_{n-2} = 4^n$; $n \geq 2$ $\hookrightarrow$ (1)

x by x^n

$\Rightarrow a_n x^n - 4x^n a_{n-1} + 4x^n a_{n-2} = 4^n x^n$

$$\sum_{n=2}^{\infty} a_n x^n - 4\sum_{n=2}^{\infty} x^n a_{n-1} + 4\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} 4^n x^n.$$

$$(a_2 x^2 + a_3 x^3 + \cdots) - 4(a_1 x^2 + a_2 x^3 + \cdots) + 4(a_0 x^2 + a_1 x^3 + \cdots) = \sum_{n=2}^{\infty} (4x)^n$$

$$(a_0 + a_1 x + a_2 x^2 + \cdots - a_0 - a_1 x) - 4x(a_0 + a_1 x + a_2 x^2 + \cdots - a_0) + 4x^2(a_0 + a_1 x + a_2 x^2 + \cdots) = (4x)^2 + (4x)^3 + \cdots$$

$$[G(x) - a_0 - a_1 x] - 4x[G(x) - a_0] + 4x^2[G(x)] = \frac{(4x)^2}{1-4x}$$

$\because$ Sum of infinite geometric progression $= \frac{a}{1-r}$

$a = 4x$; $r = 4x$.

$$\Rightarrow G(x)[1 - 4x + 4x^2] - a_0 - a_1 x + 4a_0 x = \frac{16x^2}{1-4x}$$

$$\Rightarrow G(x)[1 - 4x + 4x^2] - 2 - 8x + 4(2x) = \frac{16x^2}{1-4x}$$

$$G(x)[1-2x]^2 = \frac{16x^2}{1-4x} + 2 = \frac{16x^2+2(1-4x)}{1-4x}$$

$$= \frac{16x^2+2-8x}{1-4x}$$

$$\Rightarrow G(x) = \frac{16x^2+2-8x}{(1-2x)^2(1-4x)} = \frac{(16x^2-8x+1)+1}{(1-2x)^2(1-4x)}$$

$$G(x) = \frac{(1-4x)^2+1}{(1-2x)^2(1-4x)}$$

let $\dfrac{(1-4x)^2+1}{(1-2x)^2(1-4x)} = \dfrac{A}{1-2x} + \dfrac{B}{(1-2x)^2} + \dfrac{C}{1-4x}$

$$(1-4x)^2+1 = A(1-2x)(1-4x) + B(1-4x) + C(1-2x)^2$$

Put $x=\frac{1}{4}$; $1 = C\left(1-\frac{1}{2}\right)^2 \Rightarrow \frac{1}{4}C \Rightarrow C=4$

$x=\frac{1}{2}$; $(-1)^2+1 = B(1-2) \Rightarrow -B=2 \Rightarrow B=-2$

Now equating coefficients of x^2 we get,

$$16 = 8A+4C$$

$$\Rightarrow A=0.$$

$$\therefore G(x) = \frac{-2}{(1-2x)^2} + \frac{4}{1-4x}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = -2(1-2x)^{-2} + 4(1-4x)^{-1}$$

$$= -2\left[1+2(2x)+3(2x)^2+\cdots\cdots(n+1)(2x)^n+\cdots\right]$$

$$+4\left[1+4x+(4x)^2+\cdots\cdots+(4x)^n+\cdots\right]$$

Equating the coefficients of x^n we get,

$$a_n = -2(n+1)\cdot 2^n + 4\cdot 4^n$$

$$a_n = 4^{n+1} - (n+1)2^{n+1} \quad \forall n > 2.$$

2. Solve $a_n = 4a_{n-1}$; $n \geq 1$; $a_0 = 2$ by generating fn/1

Soln:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Given $a_n = 4a_{n-1}$

Multiply by x^n $\Rightarrow a_n x^n = 4a_{n-1} x^n$

$$\Rightarrow a_n x^n = 4x a_{n-1} x^{n-1}$$

$$\therefore \sum_{n=1}^{\infty} a_n x^n = 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$a_0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = 2 + 4x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$G(x) = 2 + 4x\, G(x)$$

$$G(x)[1-4x] = 2$$

$$G(x) = \frac{2}{1-4x} = 2(1-4x)^{-1}$$

$$G(x) = 2\left[1 + 4x + (4x)^2 + \cdots (4x)^n\right]$$

$$a_0 + a_1 x + a_2 x^2 + \cdots = 2 + 2\cdot 4x + 2\cdot 4^2 x^2 + \cdots 2\cdot 4^n x^n$$

$$\therefore \boxed{a_n = 2\cdot 4^n} \quad \forall n \geq 0$$

3. Using generating fn: solve the recurrence relation to the Fibonacci sequence $a_n = a_{n-1} + a_{n-2}$; $n \geq 2$; $a_0 = 1$; $a_1 = 1$

Soln:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Given $a_n = a_{n-1} + a_{n-2}$

$$\Rightarrow \quad a_n - a_{n-1} - a_{n-2} = 0.$$

$\times$ by x^n $\Rightarrow$ $a_n x^n - a_{n-1} x^n - a_{n-2} x^n = 0$

$$\sum_{n=2}^{\infty} a_n x^n - x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$[G(x) - a_0 - a_1 x] - x[G(x) - a_0] - x^2 G(x) = 0$$

$$G(x)[1 - x - x^2] = a_0 + a_1 x - a_0 x$$

$$G(x)[1 - x - x^2] = 1$$

$$G(x) = \frac{1}{1 - x - x^2}$$

$$= \frac{1}{\left[1 - \frac{1+\sqrt{5}}{2} x\right]\left[1 - \frac{1-\sqrt{5}}{2} x\right]}$$

$$\frac{1}{Dr} = \frac{A}{\left[1 - \frac{1+\sqrt{5}}{2} x\right]} + \frac{B}{\left[1 - \frac{1-\sqrt{5}}{2} x\right]}$$

$$1 = A\left[1 - \frac{1-\sqrt{5}}{2} x\right] + B\left[1 - \frac{1+\sqrt{5}}{2} x\right] \rightarrow (1)$$

Put $x = 0$ (2) $\Rightarrow$ $A + B = 1$ $\Rightarrow$ $\boxed{A = 1 - B}$

Put $x = \frac{2}{1-\sqrt{5}}$

$$② \Rightarrow 1 = B\left[1 - \frac{1+\sqrt{5}}{1-\sqrt{5}}\right]$$

$$\Rightarrow 1 = B\left[\frac{1-\sqrt{5}-1-\sqrt{5}}{1-\sqrt{5}}\right] \Rightarrow 1 = B\left[\frac{-2\sqrt{5}}{1-\sqrt{5}}\right]$$

$$\therefore B = \frac{1-\sqrt{5}}{-2\sqrt{5}} \Rightarrow A = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$\therefore G(x) = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]\left[1-\left(\frac{1+\sqrt{5}}{2}\right)x\right]^{-1} - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]\left[1-\frac{1-\sqrt{5}}{2}x\right]^{-1}$$

$$= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]\left[1+\left(\frac{1+\sqrt{5}}{2}x\right)+\left(\frac{1+\sqrt{5}}{2}x\right)^2+\cdots\right]$$

$$-\frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]\left[1+\left(\frac{1-\sqrt{5}}{2}x\right)+\left(\frac{1-\sqrt{5}}{2}x\right)^2+\cdots\right]$$

a_n = Coefficient of x^n in $G(x)$

$$a_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^{n+1} - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^{n+1}$$

4. Find the Sequence whose generating fn. is $\frac{6-29x}{30x^2-11x+1}$ using partial fraction.

Soln:

Given $G(x) = \frac{6-29x}{30x^2-11x+1}$

$$= \frac{6-29x}{(1-5x)(1-6x)}$$

$$\frac{6-29x}{(1-5x)(1-6x)} = \frac{A}{1-5x} + \frac{B}{1-6x}$$

$$6-29x = A(1-6x) + B(1-5x)$$

Put $x=\frac{1}{6} \Rightarrow 6-\frac{29}{6} = B\left[1-\frac{5}{6}\right]$

$$\frac{7}{6} = \frac{B}{6} \Rightarrow \boxed{B=7}$$

Put $x=\frac{1}{5} \Rightarrow 6-\frac{29}{5} = A\left[1-\frac{6}{5}\right]$

$$+\frac{1}{5} = -\frac{A}{5} \Rightarrow \boxed{A=-1}$$

$$\therefore \quad G(x) = \frac{-1}{1-5x} + \frac{7}{1-6x}$$

$$= -1\left[1-5x\right]^{-1} + 7\left[1-6x\right]$$

$$= -1\left[1+5x+(5x)^2+\cdots\right] + 7\left[1+6x+(6x)^2+\cdots\right]$$

$$= -\sum_{n=0}^{\infty}(5x)^n + 7\sum_{n=0}^{\infty}(6x)^n.$$

$\therefore$ equating the coefficients of x^n.

$$a_n = -(5)^n + 7(6)^n.$$

5. Using generating fn/: Solve: $y_{n+2} - 5y_{n+1} + 6y_n = 0$; $n \geq 0$

$y_0 = 1$; $y_1 = 1$

Soln

Given $a_{n+2} - 5a_{n+1} + 6a_n = 0$.

$*$ by x^n; $a_{n+2}x^n - 5a_{n+1}x^n + 6a_n x^n = 0$

$$\sum_{n=0}^{\infty} \frac{1}{x^2} a_{n+2} x^{n+2} - \frac{5}{x}\sum_{n=0}^{\infty} a_{n+1} x^{n+1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \frac{1}{x^2}\left[G(x) - a_0 - a_1 x\right] - \frac{5}{x}\left[G(x) - a_0\right] + 6G(x) = 0$$

$$\frac{1}{x^2}\left[G(x) - 1 - x\right] - \frac{5}{x}\left[G(x) - 1\right] + 6G(x) = 0$$

$$G(x)\left[\frac{1}{x^2} - \frac{5}{x} + 6\right] - \frac{1}{x^2} - \frac{1}{x} + \frac{5}{x} = 0$$

$$G(x)\left[1 - 5x + 6x^2\right] = 1 - 4x$$

$$G(x) = \frac{1-4x}{1-5x+6x^2} = \frac{1-4x}{6x^2-5x+1}$$

$$\frac{1-4x}{(3x-1)(2x-1)} = \frac{A}{3x-1} + \frac{B}{2x-1} = A(2x-1) + B(3x-1)$$

Put $x = 1/3 \Rightarrow 1 - \frac{4}{3} = A\left(\frac{2}{3} - 1\right)$

$$\Rightarrow -1/3 = -A/3 \Rightarrow \boxed{A = 1}$$

$x = 1/2 \Rightarrow 1 - \frac{4}{2} = B\left(\frac{3}{2} - 1\right)$

$$\Rightarrow -1 = \frac{B}{2} \Rightarrow \boxed{B = -2}$$

$$\Rightarrow G(x) = \frac{1}{3x-1} - \frac{2}{2x-1} \Rightarrow (3x-1)^{-1} - 2(2x-1)^{-1}$$

$$= -\left[1 + (3x) + (3x)^2 + \cdots\right] + 2\left[1 + 2x + (2x)^2 - \cdots\right]$$

equating coeff of x^n we get

$$a_n = -(3)^n + 2\cdot(2)^n.$$

Procedure for recurrence relation using generating fn:

Step:1 Rewrite the given recurrence relation as an eqn: with '0' on RHS.

Step:2 Multiply the eqn: obtained in step:1 by x^n and sum it from 1 to ∞ (or 0 to ∞) or (2 to ∞)

Step:3 Put $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and write $G(x)$ as a fn: of x.

Step:4 Decompose $G(x)$ into Partial fraction.

Step:5 Express $G(x)$ as a sum of familiar series

Step:6 Express a_n as the coefficient of x^n in $G(x)$.

6. Using generating function, solve the recurrence relation $a_{n+2} - 8a_{n+1} + 15a_n = 0$; given that $a_0 = 2$; $a_1 = 8$.

Soln:

Given $a_{n+2} - 8a_{n+1} + 15a_n = 0$

$$\Rightarrow a_{n+2} x^{n*} - 8a_{n+1} x^n + 15 a_n x^n = 0.$$

$$\Rightarrow \frac{1}{x^2} a_{n+2} x^{n+2} - \frac{8}{x} a_{n+1} x^{n+1} + 15 a_n x^n = 0$$

$$\Rightarrow \frac{1}{x^2} \sum_{n=0}^{\infty} a_{n+2} x^{n+2} - \frac{8}{x} \sum_{n=0}^{\infty} a_{n+1} x^{n+1} + 15 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \frac{1}{x^2}\left[G(x) - a_0 - a_1 x\right] - \frac{8}{x}\left[G(x) - a_0\right] + 15\, G(x) = 0$$

$$\frac{1}{x^2}\left[G(x) - 2 - 8x\right] - \frac{8}{x}\left[G(x) - 2\right] + 15\, G(x) = 0$$

$\times$ by x^2 $\quad G(x) - 2 - 8x - 8x\,G(x) + 16x + 15x^2 G(x) = 0$

$$G(x)\left[1-8x+15x^2\right] = 2-8x.$$

$$G(x) = \frac{2-8x}{1-8x+15x^2} = \frac{2-8x}{(1-3x)(1-5x)} = \frac{A}{1-3x} + \frac{B}{1-5x}$$

$$G(x) = A(1-5x) + B(1-3x) = 2-8x$$

Put $x = 1/5$; $2-\frac{8}{5} = B(1-3/5) \Rightarrow \frac{2}{5} = \frac{2}{5}B \Rightarrow \boxed{B=1}$

Put $x = 1/3$; $2-\frac{8}{3} = A(1-5/3) \Rightarrow -\frac{2}{3} = -\frac{2}{3}A \Rightarrow \boxed{A=1}$

$$G(x) = \frac{1}{1-3x} + \frac{1}{1-5x}$$

$$= (1-3x)^{-1} + (1-5x)^{-1}$$

$\therefore a_n \Rightarrow$ Coeff of x^n

$$a_n = 3^n + 5^n.$$

7. Identify the sequence $\frac{5+2x}{1-4x^2}$ as a generating fn:

Soln:

Given $G(x) = \frac{5+2x}{1-4x^2} = \frac{5+2x}{(1+2x)(1-2x)} = \frac{A}{1+2x} + \frac{B}{1-2x}$

$$5+2x = A(1-2x) + B(1+2x)$$

Put $x = 1/2$; $5+1 = 2B \Rightarrow B = 3$

$x = -1/2$; $5-1 = 2A \Rightarrow A = 2$

$$G(x) = \frac{2}{1+2x} + \frac{3}{1-2x} \Rightarrow 2(1+2x)^{-1} + 3(1-2x)^{-1}$$

Coeff of x^n is

$\therefore a_n = 2(-2)^n + 3(2)^n.$

PIGEON HOLE PRINCIPLE

If $(n+1)$ pigeon occupies 'n' holes then atleast one hole has more than 1 pigeon.

Proof: Assume $(n+1)$ pigeon occupies 'n' holes

Claim: Atleast one hole has more than one pigeon.

suppose not ie) Atleast one hole has not more than one pigeon. $\therefore$ each and every hole has exactly one pigeon.

$\because$ there are 'n' holes $\Rightarrow$ we have totally 'n' pigeon. which is a contradiction to our assumption that there are $(n+1)$ pigeon.

$\therefore$ atleast one hole has more than 1 pigeon.

Generalized Pigeon Hole Principle:

If 'm' pigeon occupies 'n' holes $(m>n)$, then atleast one hole has more than $\left\lceil \frac{m-1}{n} \right\rceil + 1$ pigeon.

Proof:

Assume 'm' pigeon occupy 'n' holes $(m>n)$

Claim:

Atleast one hole has more than $\left\lceil \frac{m-1}{n} \right\rceil + 1$ pigeon.

suppose not; ie) Atleast one hole has not more than $\left\lceil \frac{m-1}{n} \right\rceil + 1$ pigeon.

Each & every hole has exactly $\left\lceil \frac{m-1}{n} \right\rceil + 1$ pigeon

$\because$ we have n holes, totally there are $n\left[\left(\frac{m-1}{n}\right)+1\right]$ pigeon.

which is a contradiction

$\therefore$ atleast one hole has more than $\left\lceil \frac{m-1}{n} \right\rceil + 1$ pigeon.

1. Show that among 100 people, atleast 9 of them were born in the same month.

Soln

No. of Pigeon = m = No. of People = 100

No. of Holes = n = No. of Months = 12

$\therefore$ By Generalised Pigeon Hole Principle

$$\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{100-1}{12} \right\rceil + 1 = 9$$, were born in same month.

2. Show that if seven colours are used to paint 50 bicycles atleast 8 bicycles will be the same colour.

Soln

No. of Pigeon = m = No. of bicycle = 50

No. of Holes = n = No. of colours = 7

$\therefore$ By Generalized Pigeon Hole Principle,

$$\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{50-1}{7} \right\rceil + 1 = 8$$ bicycles will have same colour.

3. Show that if 25 dictionaries in a library contain a total of 40325 pages, then one of the dictionaries must have atleast 1614 pages.

Soln

No. of Pages = m = No. of Pigeon = 40325

No. of dictionaries = n = No. of Holes = 25

$\therefore$ By Generalized Pigeon Hole Principle

$$\left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{40325-1}{25} \right\rceil + 1 = 1614$$ Pages.

4. Prove that in any group of 6 people, there must be atleast 3 mutual friends (or) atleast 3 mutual enemies.

Soln:

Let these 6 People be A, B, C, D, E and F. Fix A. The remaining 5 people can be accommodated into 2 groups.

1. Friends of A and
2. Enemies of A.

$\therefore$ By Generalized pigeon hole Principle, atleast one of the group must contain,

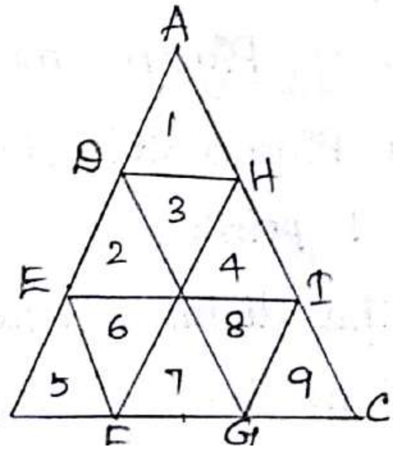
$$\left[\frac{m-1}{n}\right]+1 = \left[\frac{5-1}{2}\right]+1 = 3 \text{ People.}$$

Case i): If any two of these 3 people (B, C, D) are friends, then these two together with A form 3 mutual friends.

Case ii): If no 2 of these 3 people are friends, then these 3 people (B, C, D) are mutual enemies.

5. If we select 10 points in the interior of an equilateral triangle of side 1, show that there must be atleast 2 points whose distance apart is less than 1/3.

Soln:



Let ABC be the given equilateral triangle.

Let D & E are the points of trisection of the side AB, F & G are the points of trisection of the side BC ; H & I are the points of trisection of the side AC. ∴ the triangle ABC divided into 9 equilateral triangles each of side 1/3.

No. of interior Points = m = No. of Pigeon = 10

No. of interior triangle = n = No. of Holes = 9

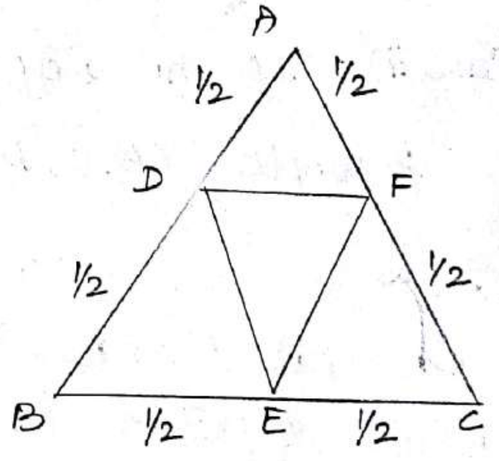
∴ By Generalized Pigeon Hole Principle,

$$\left[\frac{m-1}{n}\right]+1 = \left[\frac{10-1}{9}\right]+1 = 2 \text{ interior points.}$$

∵ each triangles of length 1/3, the distance b/w any 2 interior points of any sub triangle cannot exceeds 1/3.

6. Prove that in an equilateral triangle whose sides are of length 1 unit, if any 5 points are chosen then atleast 2 of them lies in a triangle whose sides apart is less than 1/2

Soln



Let D, E and F are mid-points of the side AB, BC, AC. ∴ The triangle ABC divided into 4 equilateral triangles each of side 1/2.

No. of Pigeon = m = 5 ; No. of Holes = n = 4.

∴ By Pigeon Hole Principle, atleast one triangle has more than 1 point.

∴ The distance b/w 2 interior points of any subtriangle is less than 1/2.

Additional Problems.

1. Show that n^3+2n is divisible by 3.

Soln:

Let $P(n)$: n^3+2n is divisible by 3.

To Prove $P(1)$ is true:

$P(1) = 1^3+2\cdot(1) = 3$ is divisible by 3.

$\Rightarrow P(1)$ is true.

Assume that $P(k)$ is true.

$P(k)$: k^3+2k is divisible by 3.

To Prove that $P(k+1)$ is true.

$$P(k+1) = (k+1)^3+2(k+1)$$
$$= k^3+3k^2+3k+1+2k+2$$
$$= (k^3+2k)+3(k^2+k+1)$$

$\Rightarrow k^3+2k$ is divisible by 3

$3(k^2+k+1)$ is divisible by 3

$\therefore P(k+1) = (k^3+2k)+3(k^2+k+1)$ is divisible by 3.

$\therefore P(k+1)$ is true.

$\therefore$ By Principle of Mathematical Induction $P(n)$ is true.

2. Show that $a^n - b^n$ is divisible by $a-b$.

Soln:

Let $P(n)$: $a^n - b^n$ is divisible by $a-b$.

To Prove that $P(1)$ is true.

$P(1) = a^1 - b^1 = a-b$ which is divisible by $a-b$.

$\Rightarrow P(1)$ is true.

Assume that $P(k)$ is true

$P(k) = a^k - b^k$ is divisible by $a-b$

$$a^k - b^k = m(a-b)$$

$$a^k = b^k + m(a-b)$$

To Prove that $P(k+1)$ is true.

$$P(k+1) = a^{k+1} - b^{k+1}$$

$$= a^k . a - b^k . b$$

$$= [b^k + m(a-b)]a - b^k . b$$

$$= am(a-b) + ab^k - bb^k$$

$$= am(a-b) + b^k(a-b)$$

$= (a-b)[am + b^k]$ is divisible by $a-b$

$P(k+1)$ is divisible by $(a-b)$ is true.

$\Rightarrow P(k+1)$ is true.

$\therefore$ By the Principle of Mathematical Induction $P(n)$ is true.

3. Show that $2^n < n!$ $\forall n \geq 4$

Soln:

Let $P(n): 2^n < n!$

To Prove $P(4)$ is true.

$2^4 < 4!$ is true

Assume that $P(k)$ is true

$\Rightarrow P(k): 2^k < k!$ is true $\longrightarrow ①$

To Prove that $P(k+1)$ is true

$① \Rightarrow 2^k < k!$

$\Rightarrow 2 \cdot 2^k < 2 \cdot k! \Rightarrow 2^{k+1} < (k+1) \cdot k!$ $(\because 2 < (k+1) \; \forall k \geq 4)$

$= (k+1)!$

$\Rightarrow 2^{k+1} < (k+1)!$

$\Rightarrow P(k+1)$ is true

$\therefore$ By the principle of Mathematical induction $P(n)$ is true.

4. Find an explicit formula for the fibonacci sequence.

Soln:

Fibonacci Sequence satisfies the recurrence relation.

$$f_n = f_{n-1} + f_{n-2}$$

$$\Rightarrow f_n - f_{n-1} - f_{n-2} = 0$$

and also satisfies the initial conditions $f_0 = 0$; $f_1 = 1$

$\therefore$ The characteristic eqn: is $r^2 - r - 1 = 0$

$\therefore \quad r = \dfrac{1 \pm \sqrt{5}}{2}$

$$\therefore \quad f_n = A\left[\frac{1+\sqrt{5}}{2}\right]^n + B\left[\frac{1-\sqrt{5}}{2}\right]^n \rightarrow (1)$$

Given $f_0 = 0$

Put $n=0$ in (1) ; $f_0 = A\left(\dfrac{1+\sqrt{5}}{2}\right)^0 + B\left(\dfrac{1-\sqrt{5}}{2}\right)^0$

$$\Rightarrow \quad A + B = 0 \rightarrow (2)$$

Given $f_1 = 1$

Put $n=1$ in (1) ; $f_1 = A\left(\dfrac{1+\sqrt{5}}{2}\right)^1 + B\left(\dfrac{1-\sqrt{5}}{2}\right)^1$

$$\Rightarrow \quad 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \rightarrow (3)$$

$(2) \times \left(\dfrac{1+\sqrt{5}}{2}\right) \Rightarrow \quad A\left(\dfrac{1+\sqrt{5}}{2}\right) + B\left(\dfrac{1+\sqrt{5}}{2}\right) = 0$

$(3) \Rightarrow \quad A\left(\dfrac{1+\sqrt{5}}{2}\right) + B\left(\dfrac{1-\sqrt{5}}{2}\right) = 1$

$$B\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = -1$$

$$\Rightarrow \boxed{B = -1/\sqrt{5}} \Rightarrow \boxed{A = 1/\sqrt{5}}$$

$$\therefore \quad f_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

5. Solve the recurrence relation $a_{n+1} - a_n = 3n^2 - n$; $n \geq 0$ & $a_0 = 3$.

Soln

The given non-homogeneous eqn: can be written as

$$a_{n+1} - a_n - 3n^2 + n = 0$$

The associated homogeneous eqn: is

$$a_{n+1} - a_n = 0$$

∴ The Characteristic eqn: is $r - 1 = 0$

$$\Rightarrow r = 1$$

∴ The general Soln $a_n^{(h)} = A(1)^n = A$.

To find the Particular Solution:

Since the right hand side of the recurrence relation is $3n^2 - n$, the solution is of the form,

$$\boxed{a_n = an^3 + bn^2 + cn}$$

Using the above in the recurrence relation the eqn: becomes

$$[a(n+1)^3 + b(n+1)^2 + c(n+1)] - (an^3 + bn^2 + cn) = 3n^2 - n$$

$$a(n^3 + 3n^2 + 3n + 1) + b(n^2 + 2n + 1) + c(n+1) - (an^3 + bn^2 + cn) = 3n^2 - n$$

$$n^3(a - a) + n^2(3a + b - b) + n(3a + 2b + c - c) + (a + b + c) = 3n^2 - n$$

Equating the coefficients, we get,

$$3a = 3 \rightarrow ① \Rightarrow a = 1$$

$$3a + 2b = -1 \rightarrow ②$$

$$a + b + c = 0 \rightarrow ③ \Rightarrow c = 1$$

solving the above

$$3 + 2b = -1 \Rightarrow b = -2$$

$\therefore$ Particular Solution is $a_n^{(P)} = n^3 - 2n^2 + n = n(n-1)^2$

$\therefore$ The general solution $a_n = a_n^{(h)} + a_n^{(P)}$

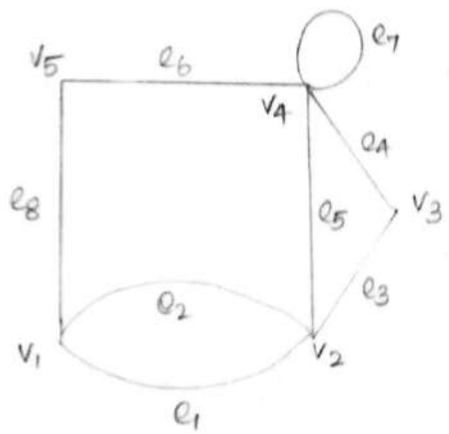
$$\Rightarrow a_n = A(1)^n + n(n-1)^2$$

Unit - III

GRAPHS

Graph:

A graph $G = (V, E, \phi)$ consists of a non-empty set $V = \{v_1, v_2 \ldots\}$ called the set of nodes (Points, Vertices) of the graph, $E = \{e_1, e_2 \ldots\}$ is said to be the set of edges of the graph, and ϕ is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V.



Self Loop:

If there is an edge from v_i to v_i then that edge is called Self Loop (or) Simply Loop.

Parallel Edges:

If two edges have same end points then the edges are called parallel edges.

Incident:

If the vertex v_i is an end vertex of some edge e_k the e_k is said to be incident with v_i.

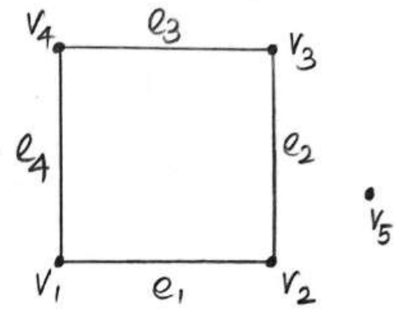
Adjacent edges and Vertices:

Two edges are said to be adjacent if they are incident on a common vertex. [e_6 & e_8 are adjacent

Two vertices v_i and v_j are said to adjacent if $v_i v_j$ is an edge of the graph [v_1 & v_5 are adjacent vertices.

Simple Graph:

A graph which has neither self loops nor parallel edges is called a simple graph.



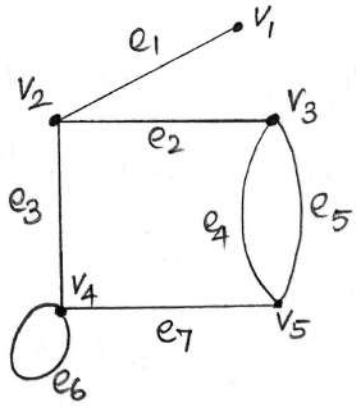
Isolated vertex:

A vertex having no edge incident on it is called an isolated vertex. It is obvious that for an Isolated vertex degree is zero. [v_5 is an isolated vertex]

Pendentant Vertex:

If the degree of any vertex is one, then that vertex is called pendent vertex

eg:



$V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

and

$e_1 = \langle v_1, v_2 \rangle$ (or) $\langle v_2, v_1 \rangle$

$e_2 = \langle v_2, v_3 \rangle$ (or) $\langle v_3, v_2 \rangle$

$e_3 = \langle v_2, v_4 \rangle$ (or) $\langle v_4, v_2 \rangle$

$e_6 = \langle v_4, v_4 \rangle$

$(v_1, v_2), (v_2, v_3), (v_2, v_4)$ (v_3, v_5) are adjacent

$(v_1, v_3), (v_3, v_4)$ are not adjacent.

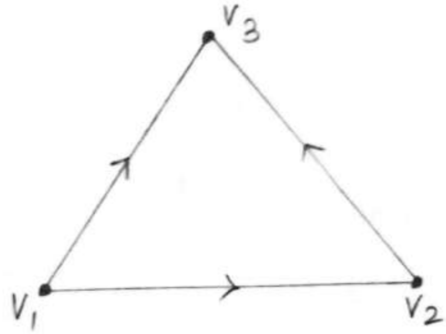
Directed Edges:

In a graph $G = (V, E)$ an edge which is associated with an ordered pair of $V \times V$ is called a directed edge of G.

If an edge which is associated with an unordered pair of nodes is called an undirected edge

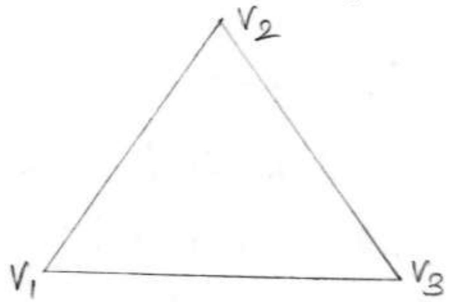
Digraph:

A graph in which every edge is directed edge is called a digraph or directed graph.



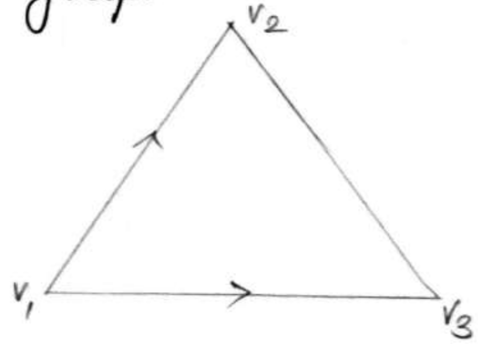
Undirected Graph:

A graph in which every edge is undirected is called an undirected graph.

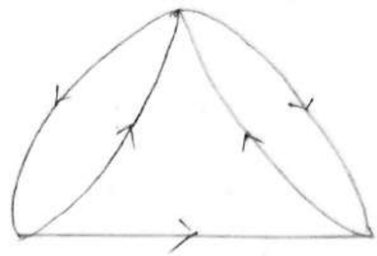


Mixed Graph:

If some edges are directed and some are undirected in a graph, the graph is called mixed graph.

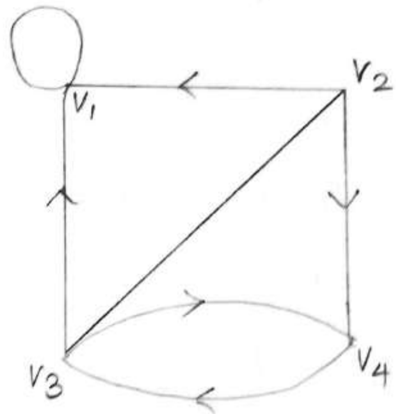


Multigraph:



A graph which contains some Parallel edges are called a <u>Multigraph</u>.

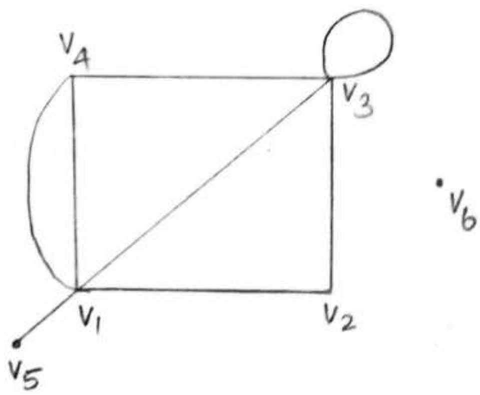
Pseudogeaph:



A graph in which loops and Parallel edges are allowed is called a pseudogeaph.

Graph Terminology

Degree of a Vertex:



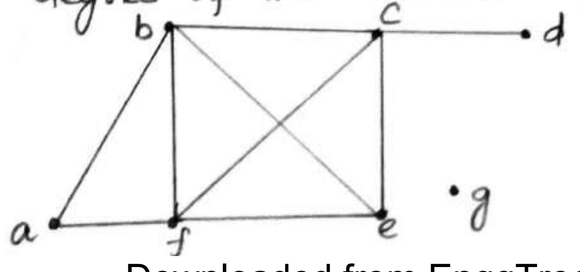
The no. of edges incident at the Vertex v_i is called the degree of the Vertex with self loops Counted twice and it is denoted by $d(v_i)$

eg: $d(v_1) = 5$ $\quad$ $d(v_4) = 3$

$d(v_2) = 2$ $\quad$ $d(v_5) = 1$

$d(v_3) = 5$ $\quad$ $d(v_6) = 0$

1. Find the degree of the vertices of the undirected graph.



$d(a) = 2$ $\quad d(d) = 1$ $\quad d(g) = 0$

$d(b) = 4$ $\quad d(e) = 3$

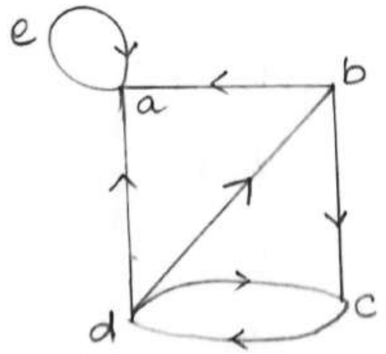
$d(c) = 4$ $\quad d(f) = 4$

In-degree and Out-degree of a directed graph:

In a directed graph, the in-degree of a vertex V, denoted by $deg^-(V)$ and defined number of edges with V as their terminal vertex.

The out-degree of V, denoted by $deg^+(V)$, is the no. of edges with V as their initial vertex.

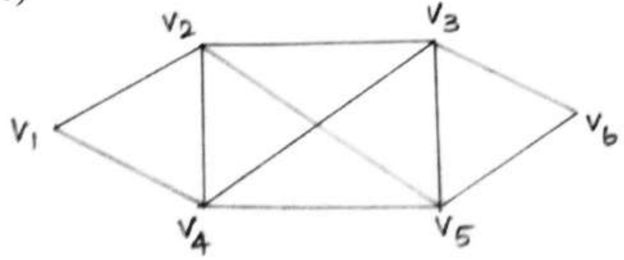
eg:



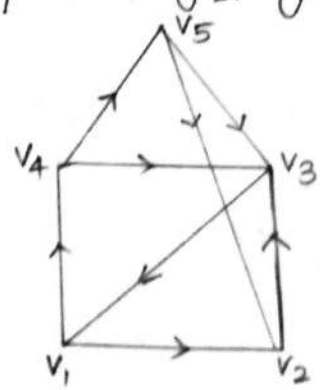
In-degree	Out-degree	Total degree
$deg^-(a) = 3$	$deg^+(a) = 1$	$deg(a) = 4$
$deg^-(b) = 1$	$deg^+(b) = 2$	$deg(b) = 3$
$deg^-(c) = 2$	$deg^+(c) = 1$	$deg(c) = 3$
$deg^-(d) = 1$	$deg^+(d) = 3$	$deg(d) = 4$

1. Find the degree of each vertices of the gn. graph:

a)



b)



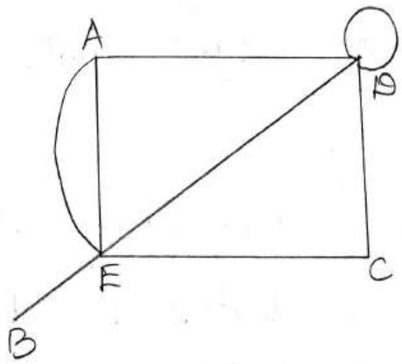
a) It is an undirected graph.

$d(v_1) = d(v_6) = 2$; $d(v_2) = 4 = d(v_3) = d(v_4) = d(v_5)$

b)

In-deg	Out-deg	Total deg
$deg^-(v_1) = 1$	$deg^+(v_1) = 2$	$deg(v_1) = 3$
$deg^-(v_2) = 2$	$deg^+(v_2) = 1$	$deg(v_2) = 3$
$deg^-(v_3) = 3$	$deg^+(v_3) = 1$	$deg(v_3) = 4$
$deg^-(v_4) = 1$	$deg^+(v_4) = 2$	$deg(v_4) = 3$
$deg^-(v_5) = 1$	$deg^+(v_5) = 2$	$deg(v_5) = 3$

2. Draw the graph with 5 vertices A, B, C, D, E $\ni$: $deg(A) = 3$; B is an odd vertex ; $deg(C) = 2$; D & E are adjacent.



Soln:

$d(E) = 5$

$d(C) = 2$

$d(D) = 5$

$d(A) = 3$

$d(B) = 1$

Theorem: 1 Handshaking Theorem:

Let $G = (V, E)$ be an undirected graph with 'e' edges then $\sum_{v \in V} deg(v) = 2e$

The sum of degrees of all vertices of an undirected graph is twice the no. of edges of the graph and hence it is even.

Proof:

$\because$ Every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the 'e' edges contribute (2e) to the sum of the degree of vertices.

$$\therefore \sum deg(v) = 2e.$$

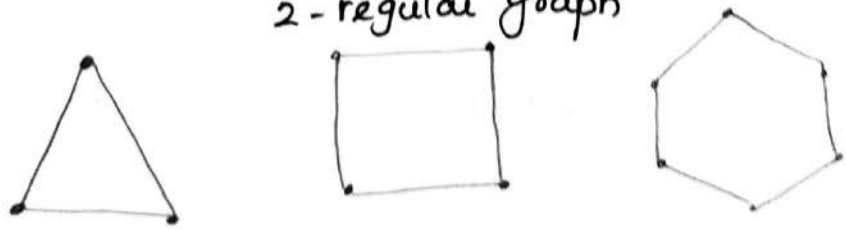
Special Types of Graphs:

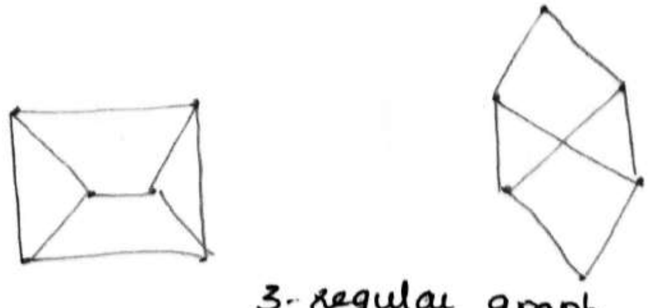
Regular Graph:

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.

If every vertex in a regular graph has degree K, then the graph is called K-regular.

2-regular graph

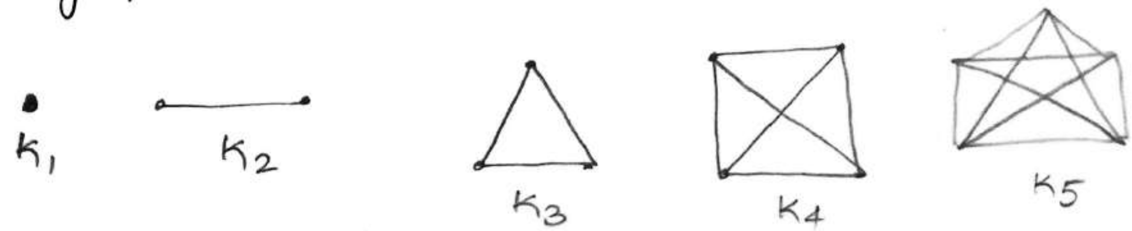




3-regular graph.

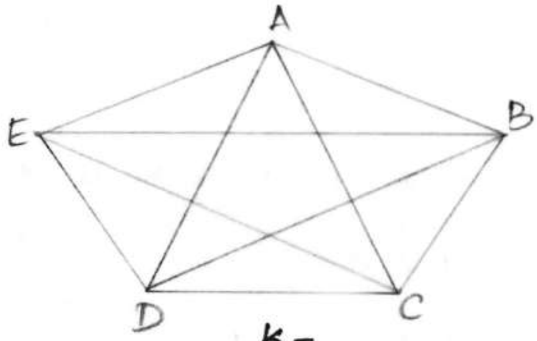
Complete graph:

In a graph, if there exist an edge b/w every pair of vertices then such a graph is called a complete graph.



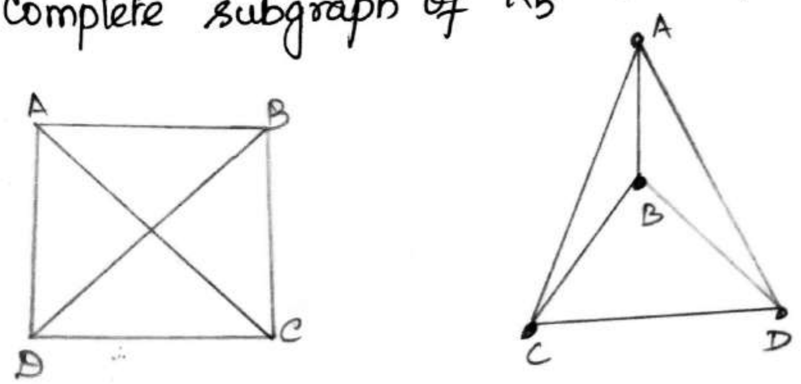
1. Draw the complete K_5 graph with vertices A, B, C, D, E. Draw all complete sub graph of K_5 with 4 vertices.

In a graph, if there exist an edge b/w every pair of vertices, then such graph is called a complete graph.



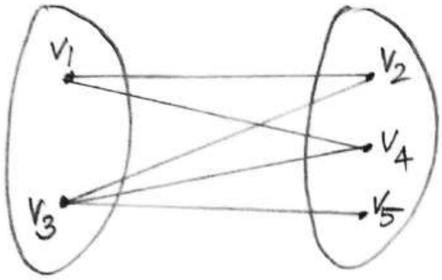
K_5

Now, complete subgraph of K_5 with 4 vertices are



Bipartite Graph:

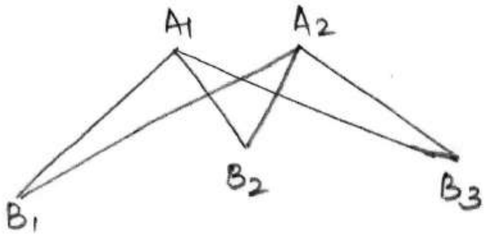
A Graph G is said to be bipartite if its vertex set $V(G)$ can be partitioned into 2 disjoint non-empty sets V_1 & V_2, $V_1 \cup V_2 = V(G)$, $\ni$: every edge in $E(G)$ has one end vertex in V_1 and another end vertex in V_2.



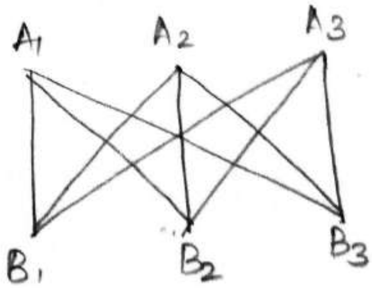
Complete Bipartite Graph:

A bipartite graph G, with the bipartition V_1 & V_2 is called complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2. Every vertex in V_2 is adjacent to every vertex in V_1.

A complete bipartite graph with 'm' and 'n' vertices in the bipartition is denoted by $K_{m,n}$.

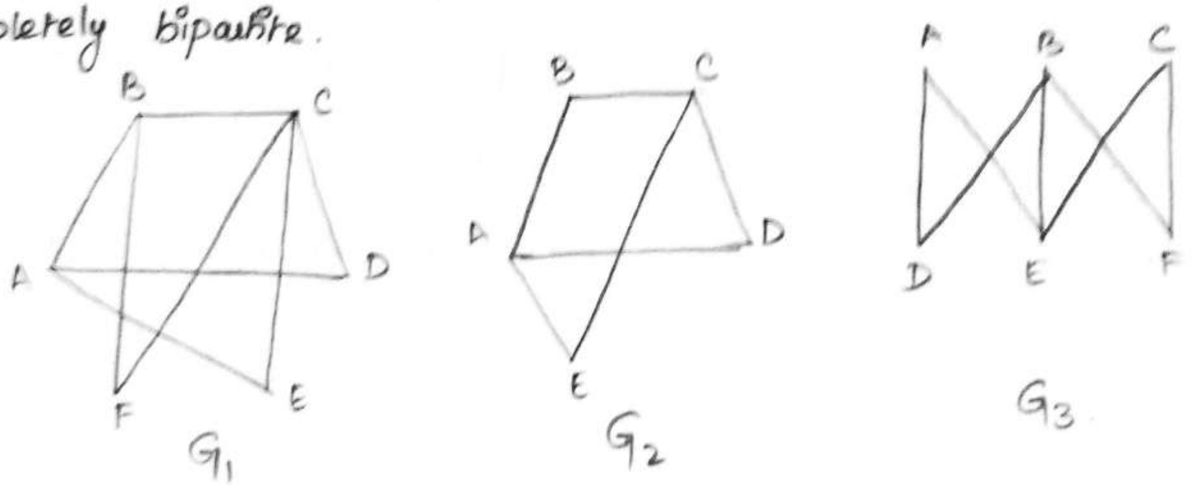


$K_{2,3}$ graph



$K_{3,3}$ graph

1. Determine which of the following graphs are bipartite & which are not. If a graph is bipartite, state if it is completely bipartite.



Soln

(i) In G_1 : The vertices D, E, F are not connected by edges, $\therefore V_1 = \{D, E, F\}$; $V_2 = \{A, B, C\}$

The vertices V_1 are connected by edges to the vertices of V_2, but V_2 are not.

$\therefore$ G_1 is not a Bipartite.

ii) In G_2 : $V_1 = \{A, C\}$; $V_2 = \{B, D, E\}$

the condition required for bipartite graph are satisfied. $\Rightarrow$ G_2 is bipartite.

Both A & C are adjacent to B, D, E.

$\Rightarrow$ G_2 is a Complete Graph.

iii) In G_3 : $V_1 = \{A, B, C\}$; $V_2 = \{D, E, F)$

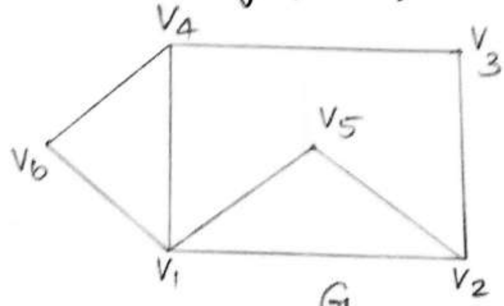
$\Rightarrow$ G_3 is bipartite.

A, F ; C, D are not connected.

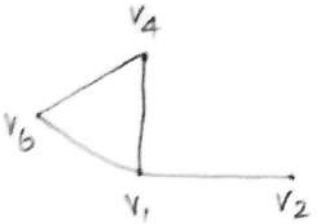
$\therefore$ G_3 is not a Complete bipartite graph.

Subgraph:

A graph $H = (V_1, E_1)$ is called a subgraph of $G = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$.

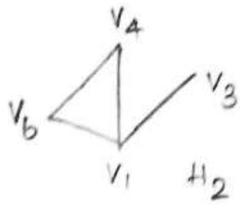


G



H_1

Subgraph of G



H_2

Not a Subgraph of G.

Adjacency Matrix of a Simple Graph:

Let $G = (V, E)$ be a simple graph with n-vertices $\{v_1, v_2, \ldots\ldots v_n\}$. Its adjacency matrix is denoted by $A = [a_{ij}]$ and defined by

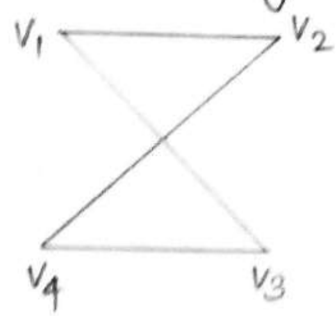
$$A = [a_{ij}] = \begin{cases} 1 & ; \text{ if there is an edge b/w } v_i \text{ and } v_j \\ 0 & ; \text{ Otherwise.} \end{cases}$$

Note:

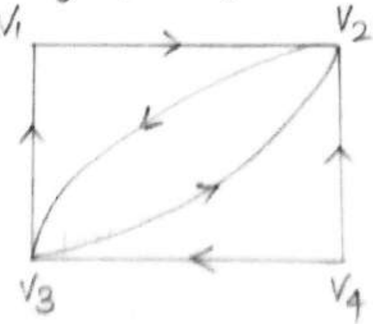
The adjacency Matrix of a simple graph is Symmetric, ie) $a_{ij} = a_{ji}$.

1. Find the adjacency matrix of the graphs given below.

a)



b)



Adjacency Matrix:

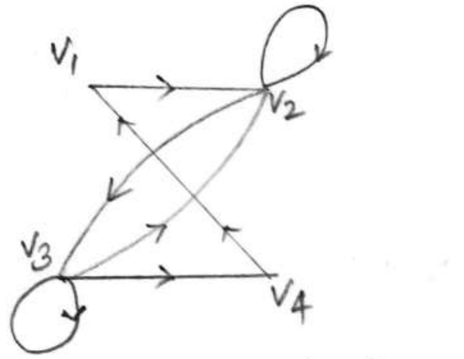
a) $A = [a_{ij}]$

$$= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 01 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b) $A = [a_{ij}]$ =

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2. Find adjacency matrix of the graphs. Hence find the degree of each Vertex.



Adjacency Matrix

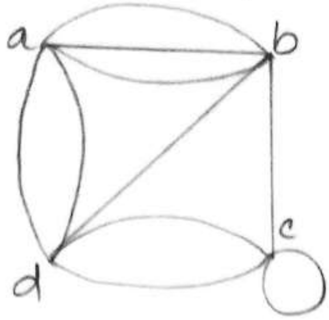
$$A = [a_{ij}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$d(v_1) = 1$ $\quad d(v_3) = 3$

$d(v_2) = 2$ $\quad d(v_4) = 1$

3. Obtain the adjacency Matrix to represent the pseudograph.

Adjacency Matrix



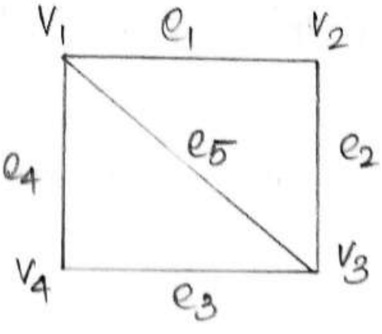
$$A = [a_{ij}] = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Incidence Matrices:

Let $G=(V,E)$ be an undirected graph with n-vertices $\{v_1, v_2 \ldots\ldots v_n\}$ and m edges $\{e_1, e_2 \ldots\ldots e_m\}$. Then the $(n \times m)$ matrix is $B=[b_{ij}]$ where,

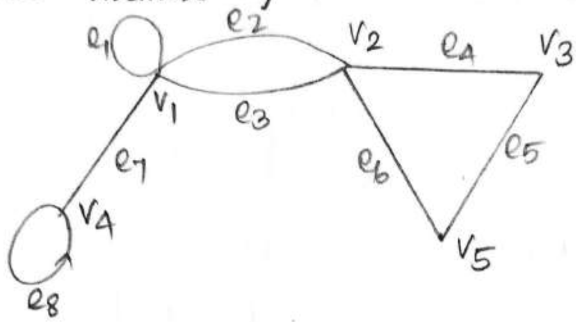
$$b_{ij} = \begin{cases} 1 & ; \text{when edge } e_j \text{ incident on } v_i \\ 0 & ; \text{otherwise.} \end{cases}$$

1. Find the incidence matrix of the following graph.



$$B = [b_{ij}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2. Find the incidence matrix of



$$B = [b_{ij}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 01 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

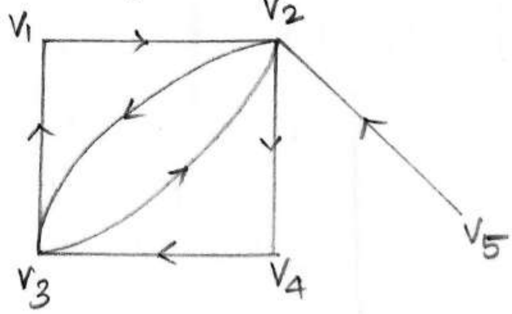
Path Matrix:

If $G = (V, E)$ be a simple digraph in which $|V| = n$ and the nodes of G are assumed to be ordered.

$$P_{ij} = \begin{cases} 1 & ; \text{ If there exists a path from } V_i \text{ to } V_j. \\ 0 & ; \text{ otherwise.} \end{cases}$$

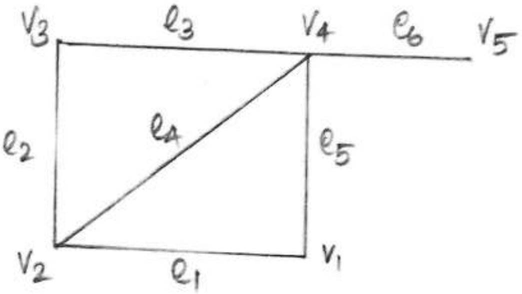
is called the path matrix (reachability matrix) of the graph G.

1. Find the Path Matrix of



$$B = [b_{ij}^{\circ\circ}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2. Find the Path matrix $P(v_2, v_4)$ for the following Graph G.

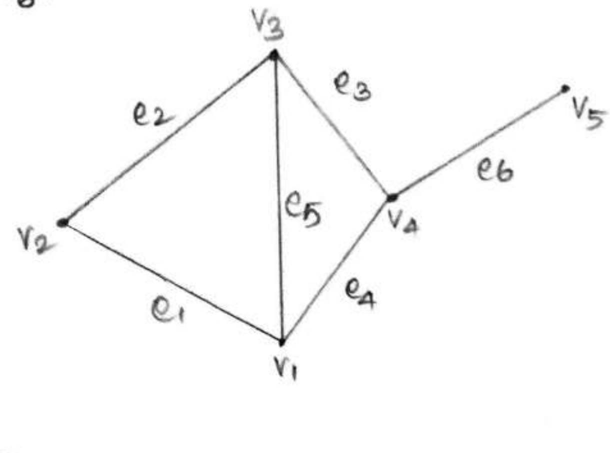


There are 3 different paths from V_2 to V_4.

ie) $\{e_4\}$; $\{e_1, e_5\}$; $\{e_2, e_3\}$

$$P(v_2, v_4) = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

3. Find the adjacency matrix. Hence find the deg. of vertices v_1, v_3 and v_6.



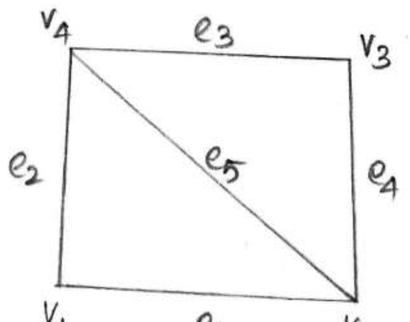
$$A = [a_{ij}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$deg(v_1)$ = sum of the entries in 1^{st} row = 3

$d(v_3) = 3$

$deg(v_6) = 0$

4. Find the adjacency matrix, Hence find degree at each vertex. Also find A^2 & A^3.



Adjacency Matrix is

$$A = [a_{ij}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$deg(v_1) = 2$; $deg(v_2) = 3$; $deg(v_3) = 2$; $deg(v_4) = 3$.

$A^2 = A \times A$.

$$= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

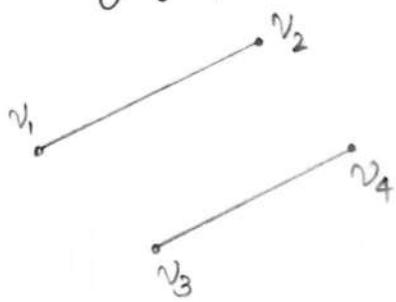
$A^3 = A^2 \times A$

$$= \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{bmatrix}$$

$\Rightarrow A^2$ & A^3 are symmetric Matrices.

5. Find the adjacency matrix of the following graph G. Find A^2, A^3 and $Y = A + A^2 + A^3 + A^4$.



Soln:

The adjacency Matrix is

$$A = [a_{ij}] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A^4 = A^3 * A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

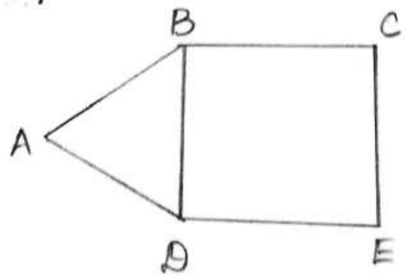
$$Y = A + A^2 + A^3 + A^4$$

$$Y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

A^2 and A^3 are Symmetric Matrices.

6. Find all the simple paths from A to E and all cycles with respect to vertex A of the given graph.



Simple Paths from A to E are

i) $A \to B \to C \to E$

ii) $A \to B \to D \to E$

iii) $A \to D \to E$

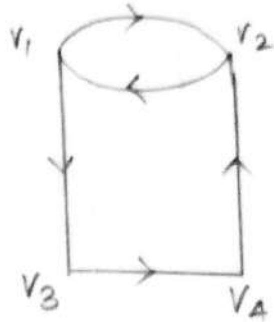
iv) $A \to D \to B \to C \to E$

The cycles are

i) $A \to B \to C \to E \to D \to A$

ii) $A \to D \to E \to C \to B \to A$

7. Consider the no. of possible elementry paths of length 3 from V_1 to V_2.



Soln

The adjacency matrix of the graph G is

$$A = [a_{ij}] = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^2 = A * A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

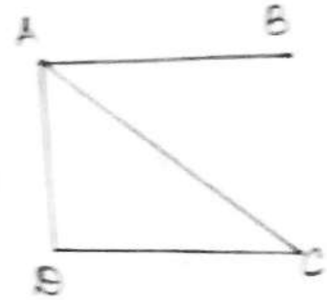
$$A^3 = A^2 * A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$(1,2)^{th}$ entry of A^3 is 2. $\therefore$ There are 2 elementry Paths of length 3 from V_1 to V_2

i) $V_1 \to V_3 \to V_4 \to V_2$

ii) $V_1 \to V_2 \to V_1 \to V_2$.

8. For the graph given below find all possible paths of length 4 from vertex B to D.



The adjacency matrix is

$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A^2 = A \times A = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 3 & 0 & 1 & 1 \\ 4 & 1 & 2 & 3 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

$$A^4 = A^3 \times A = \begin{bmatrix} 11 & 2 & 6 & 6 \\ 2 & 3 & 4 & 4 \\ 6 & 4 & 7 & 6 \\ 6 & 4 & 6 & 7 \end{bmatrix}$$

The entry of $(2,4)$ in A^4 is 4.

$\therefore$ Four paths of length 4 from B to D is

i) $B \to A \to B \to A \to D$

ii) $B \to A \to D \to C \to D$

iii) $B \to A \to D \to A \to D$

iv) $B \to A \to C \to A \to D$

Graph Isomorphism:

Two graphs G_1 & G_2 are said to be isomorphic to each other, if there exists a 1-1 correspondence between the vertex sets which preserves adjacency of the vertices.

Note:

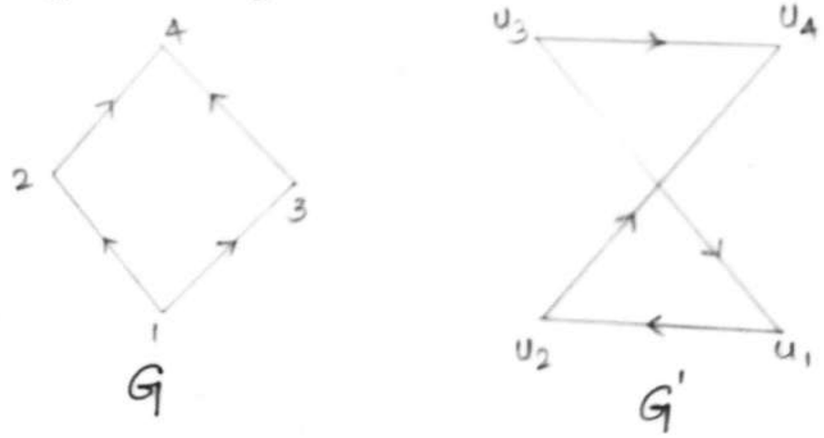
If G_1 and G_2 are isomorphic then G_1 and G_2 have

i) Same no. of vertices.

ii) same no. of Edges.

iii) An equal no. of vertices with a given degree.

1. Check the given 2 graphs G and G' are Isomorphic or not.



Soln:

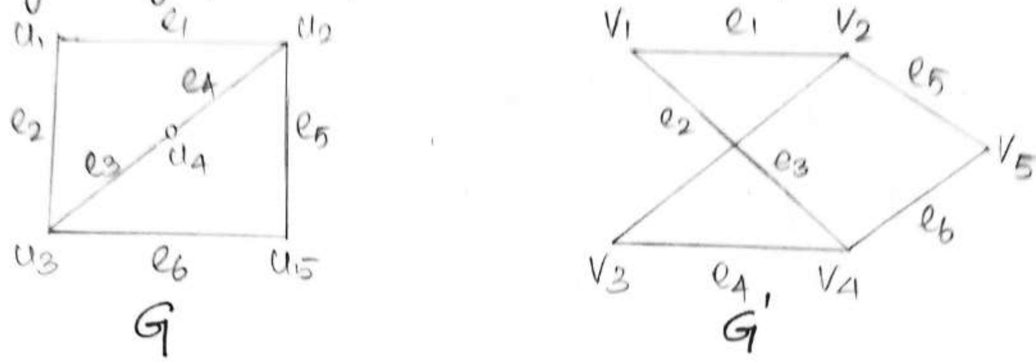
Both G and G' have same no. of vertices (namely 4) and same no. of edges (namely 4). under the Mapping

$1 \to u_3$ $2 \to u_1$ $3 \to u_4$ $4 \to u_2$	The edges $(1,3),(1,2),(2,4)$ and $(3,4)$ are mapped into (u_3,u_4) (u_3,u_1) (u_1,u_2) and (u_4,u_2)

$\therefore$ Adjacency of vertex sets are satisfied

$\therefore$ G & G' are isomorphic.

2. Check the given graphs G and G' are isomorphic or not.



Soln:

The no. of vertices (5) and no. of edges (6) are same.

The degree sequence are same. ∵ in G we have the vertices u_2 and u_3 of degree 3. They must be mapped to the vertices V_2 and V_4 in G'.

Define a mapping:

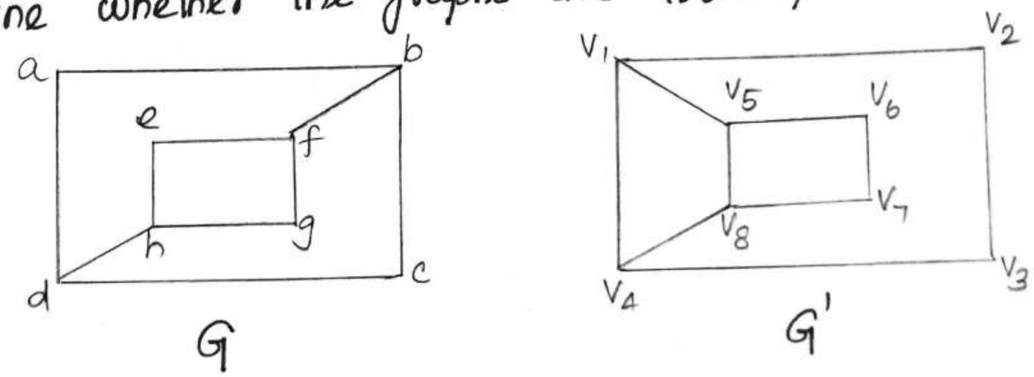
$u_1 \rightarrow v_1$; $u_3 \rightarrow v_2$; $u_5 \rightarrow v_3$; $u_2 \rightarrow v_4$ and $u_4 \rightarrow v_5$

∴ The edges are mapped to 1-1.

∴ there is a 1-1 correspondence between the vertices and edges.

∴ The graph G and G' are isomorphic.

3. Determine whether the graphs are isomorphic or not.



Soln

The graph G and G' have 8 vertices and 10 edges.

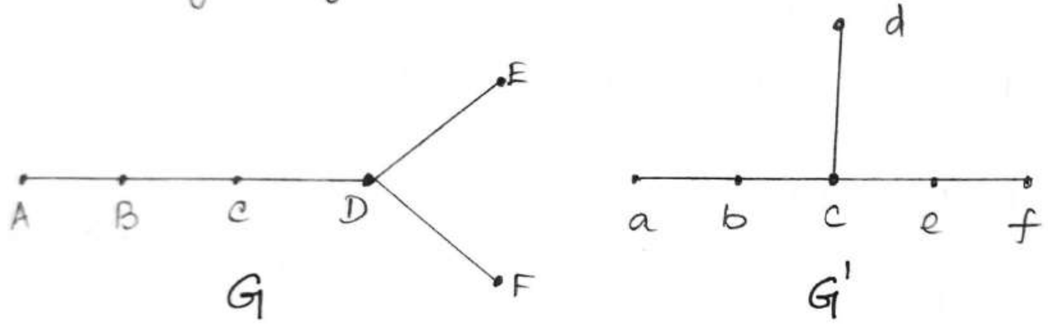
In G deg(a) = 2

In G' deg(v_2) = deg(v_3) = deg(v_6) = deg(v_7) = 2.

∴ a in G must correspond to either v_2, v_3, v_6, v_7 in G'

∴ G and G' are not isomorphic.

4. Check the given graphs are isomorphic or not.



Soln

The graph G & G' have

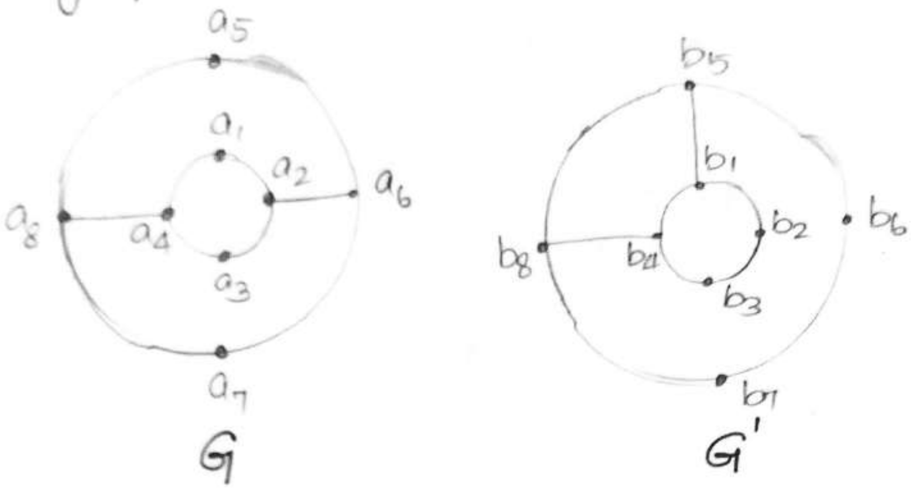
i) 6 vertices and 5 edges.

ii) 3 vertices of degree 1 ; 2 vertices of degree 2 ;
1 vertices of degree 3.

In G the vertex D is adjacent to 2 Pendent Vertices (E & F)
but in G' there is no vertex which is adjacent to 2 pendent vertices.

∴ They are not isomorphic.

5. Are the graphs isomorphic.



In G, the vertices a_2, a_4, a_6 and a_8 each of degree 3 is adjacent to exactly one vertex of degree 3.

In G' b_1, b_4, b_5 and b_8 are of degree 3. But these vertices are adjacent to more than one vertex of degree 3.

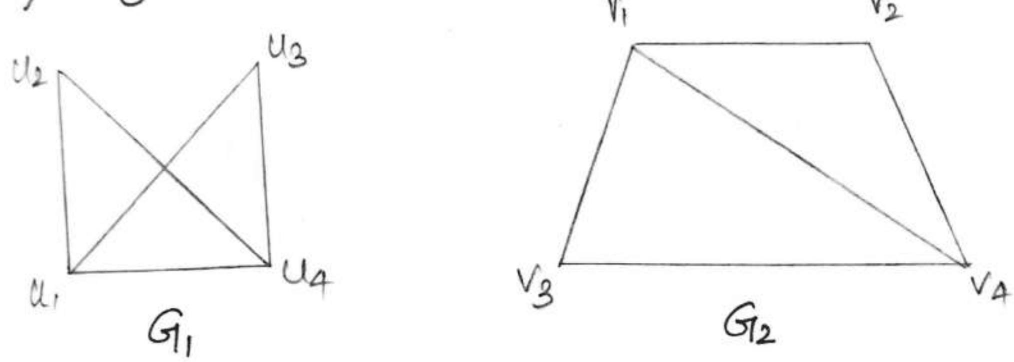
$\therefore$ G and G' are not isomorphic

Isomorphism & Adjacency:

* Two graphs are isomorphic, if and only if their vertices are labelled in such a way that the corresponding adjacency matrices are equal.

* Two simple graphs G_1 and G_2 are isomorphic $\Leftrightarrow$ their adjacency matrices A_1 & A_2 are related by $A_1 = P^{-1} A_2 P$, where P is a permutation matrix.

1. Test the isomorphism of the graphs by considering their adjacency matrices.



Soln:

Let A_1 & A_2 be the adjacency matrices of G_1 & G_2 respectively.

$$A_1 = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Interchanging C3 and C4

$$\sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 01 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

2. Are the simple graphs with the following adjacency matrices isomorphic?

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} ; \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Soln

let $A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad C_1 \leftrightarrow C_4.$$

$\therefore$ A_1 & A_2 are ~~not~~ similar.

$\therefore$ The Graphs are isomorphic.

3. The adjacency matrices of two pairs of graph. Examine the isomorphism of G and H by find a permutation matrix.

$$A_G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} ; \quad A_H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Soln:

We know that the graphs G_1 and G_2 are isomorphic iff their adjacency matrix A_1 & A_2 are related by $A_1 = P^{-1} A_2 P$; where P is a permutation Matrix.

$$A_G = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad C_1 \leftrightarrow C_3.$$

$$= A_2$$

$\Rightarrow$ A_1 is similar to A_2

$\Rightarrow$ The graphs are isomorphic.

Path:

A path is a graph in sequence $V_1, V_2, V_3 \dots V_k$ of vertices each adjacent to the next.

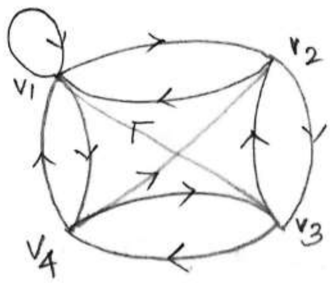
Length of the Path:

The no. of edges appearing in the sequence of a path is called length of the path.

Cycle (or) Circuit:

A path which originates and ends in the same node is called a cycle (or) circuit.

1. Consider



then some of the path originating in V_1 & ending in V_3 are

$P_1 = [\langle V_1, V_2 \rangle, \langle V_2, V_3 \rangle]$

$P_2 = [\langle V_1, V_4 \rangle, \langle V_4, V_3 \rangle]$

$P_3 = [\langle V_1, V_2 \rangle, \langle V_2, V_4 \rangle, \langle V_4, V_3 \rangle]$

$P_4 = [\langle V_1, V_2 \rangle, \langle V_2, V_4 \rangle, \langle V_4, V_1 \rangle, \langle V_1, V_2 \rangle, \langle V_2, V_3 \rangle]$

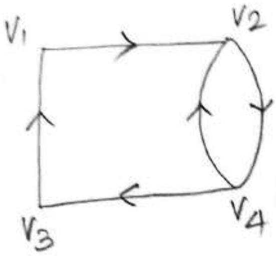
Reachable:

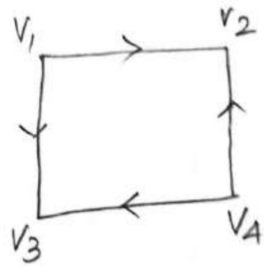
A node V of a simple digraph is said to be reachable from the node u of the same graph, if there exist a Path from u to V.

Connected Graph:

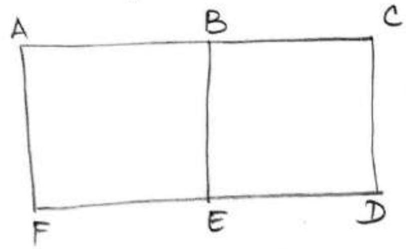
An directed graph is said to be connected if any pair of nodes are reachable from one another.

A graph which is not connected is called disconnected graph.



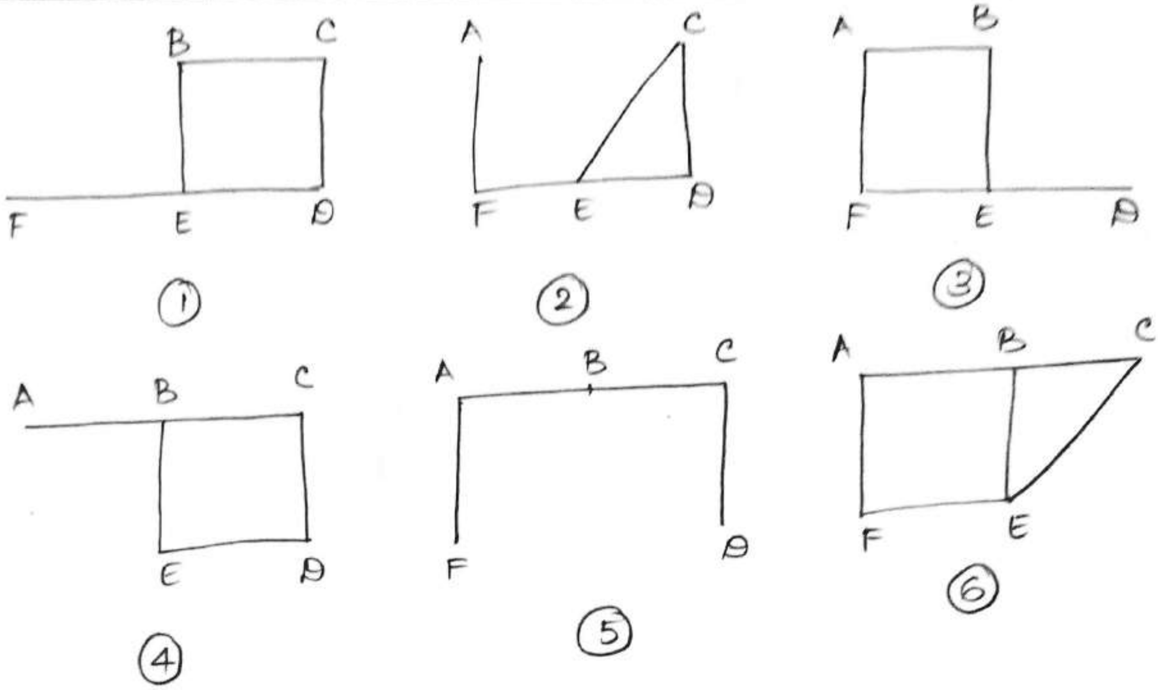


1. Find all the connected subgraph obtained from the graph obtained from the given graph by deleting each vertex. List out the paths from A to F.



Soln

The connected subgraph of the given graph is



1. In (1) & (4) there is no path from A to F
2. In (2), simple Path from $A \rightarrow F$
3. In (3), $A \rightarrow F$ and $A \rightarrow B \rightarrow E \rightarrow F$
4. In (5), $A \rightarrow F$
5. In (6), i) $A \rightarrow F$; ii) $A \rightarrow B \rightarrow E \rightarrow F$
   iii) $A \rightarrow B \rightarrow C \rightarrow E \rightarrow F$.

Unilaterally Connected:

A simple digraph is said to be unilaterally connected, if for any pair of nodes of the graph atleast one of the nodes of the pair is reachable from the other node.

Strongly Connected:

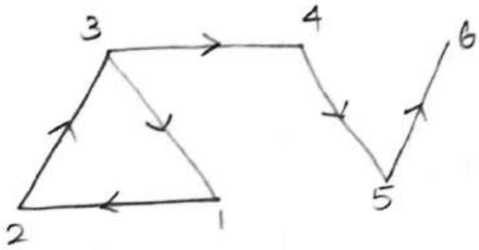
A simple digraph is said to be strongly connected, if for any pair of nodes of the graph both the nodes of the pair are reachable from one another.

Weakly Connected:

A simple digraph is weakly connected if it is connected as an undirected graph in which the direction of the edges is neglected.

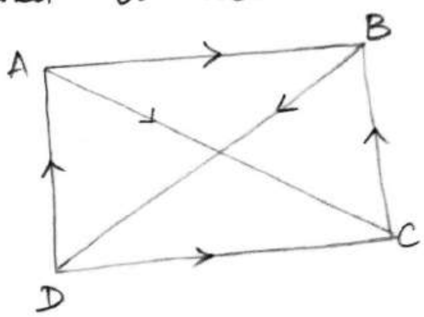
Strong Component:

A simple digraph which has a maximal strongly connected subgraph is called strong Component.



{1, 2, 3}, {4}, {5}, {6} are strong component.

1. Check the graph is strongly connected, weakly connected, unilaterally connected or not.



Soln:

Path for the vertices (A, B)

i) $A \rightarrow B$

ii) $B \rightarrow D \rightarrow A$

Path for the vertices (A, D) is

i) $A \rightarrow B \rightarrow D$

ii) $D \rightarrow A$

Path for the vertices (A, C) is

i) $A \rightarrow C$

ii) $C \rightarrow B \rightarrow D \rightarrow A$

Path for the vertices (B, C) is

i) $B \rightarrow D \rightarrow C$

ii) $C \rightarrow B$

Path for the vertices (B, D) is

i) $B \rightarrow D$

ii) $D \rightarrow A \rightarrow B$

Path for the vertices (C, D) is

i) $C \rightarrow B \rightarrow D$

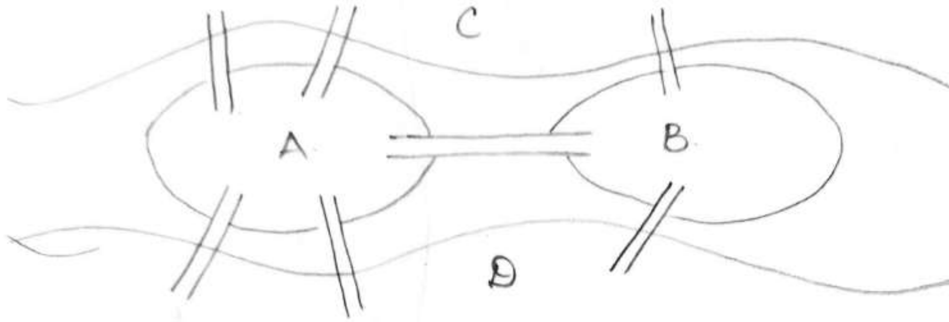
ii) $D \rightarrow C$

$\therefore$ there is path from each of the possible pairs of vertices of A, B, C, D the graph is strongly Connected.

$\because$ G is strongly connected it is both weakly and unilaterally connected.

Euler Graph & Hamilton Graph.

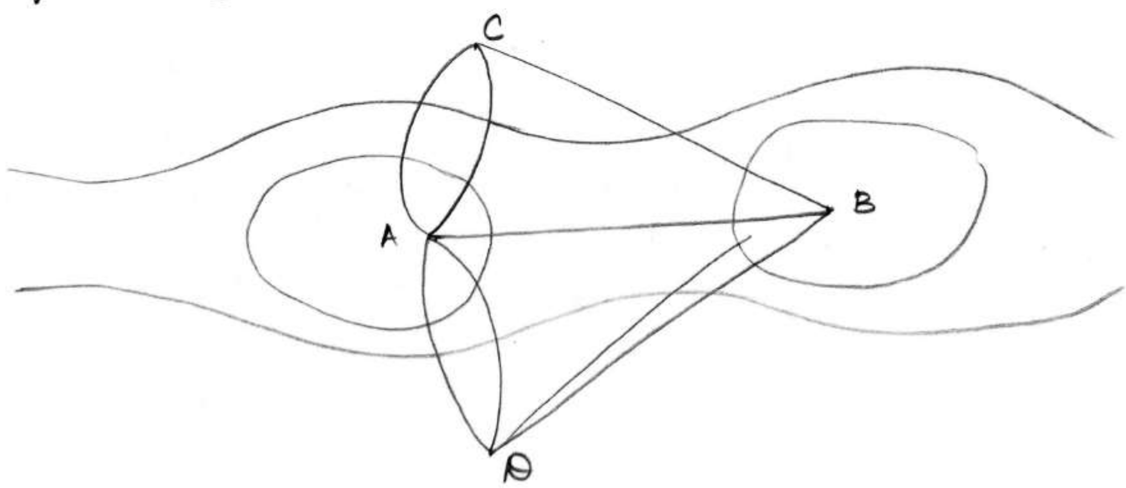
Konisberg Bridge Problem:



There are 2 islands A and B formed by a river. They are connected to each other and to the river banks C & D by means of 7 Bridges.

The problem is to start from any one of the 4 land areas A, B, C, D walk across each bridge exactly once and return to the starting point.

When the situation is represented by a graph, with vertices representing the land areas and the edges representing the bridges,



The problem is to find whethere there is an Eulerian Circuit or cycle [travel along every edge once]

Here we cannot find an Eulerian circuit. Hence Konisberg bridge problem has no solution.

Euler Graph:

Euleian Path:

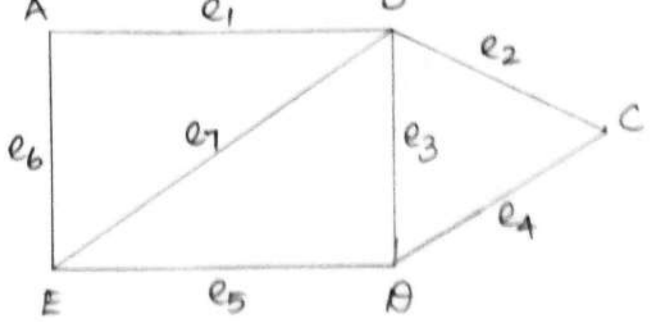
A Path of a graph G is called an Euleian Path, if it Contains each edge of the graph exactly once.

Euleian Circuit or Euleian Cycle:

A circuit or cycle of a graph G is called an Eulerian Circuit or cycle if it includes each edge of G exactly once.

Euleian Graph or Euler Graph:

Any graph containing an Euleian circuit or cycle is called an Euleian graph.

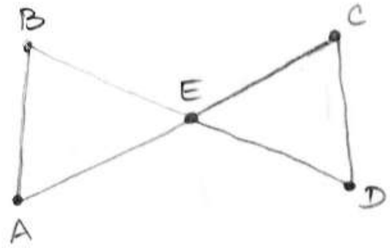


Then the Euler Path between E and D, namely

E — D — C — B — A — E — B — D

The above path consist of edges $e_5\ e_4\ e_2\ e_1\ e_6\ e_7\ e_3$ exactly once.

1. Check the given graph is Euler or not.



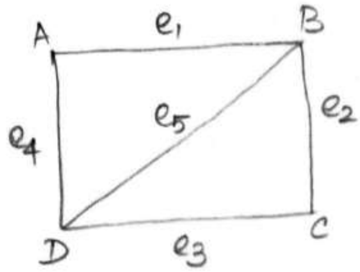
Soln:

Consider the cycle $A \rightarrow E \rightarrow C \rightarrow D \rightarrow E \rightarrow B \rightarrow A$.

∵ it includes each of the edges exactly once, the above cycle is an Eulerian cycle.

∴ the graph contains Eulerian cycle, it is a Euler Graph.

2. Find all the possible Eulerian path of the given graph. Is it Euler graph.



Soln:

Possible Euler Paths are:

1. $B \xrightarrow{e_5} D \xrightarrow{e_3} C \xrightarrow{e_2} B \xrightarrow{e_1} A \xrightarrow{e_4} D$
2. $B \xrightarrow{e_2} C \xrightarrow{e_3} D \xrightarrow{e_4} A \xrightarrow{e_1} B \xrightarrow{e_5} D$
3. $B \xrightarrow{e_1} A \xrightarrow{e_4} D \xrightarrow{e_3} C \xrightarrow{e_2} B \xrightarrow{e_5} D$
4. $D \xrightarrow{e_3} C \xrightarrow{e_2} B \xrightarrow{e_1} A \xrightarrow{e_4} D \xrightarrow{e_5} B$

5. $D \xrightarrow{e_5} B \xrightarrow{e_2} C \xrightarrow{e_3} D \xrightarrow{e_4} A \xrightarrow{e_1} B$

6. $D \xrightarrow{e_4} A \xrightarrow{e_1} B \xrightarrow{e_2} C \xrightarrow{e_3} D \xrightarrow{e_5} B$

Here we cannot find eulerian cycle

∴ the graph is not a Euler Graph.

Hamilton Graph:

Hamilton Path:

A path of a graph G is called a Hamilton Path, if it includes each vertex of G exactly once.

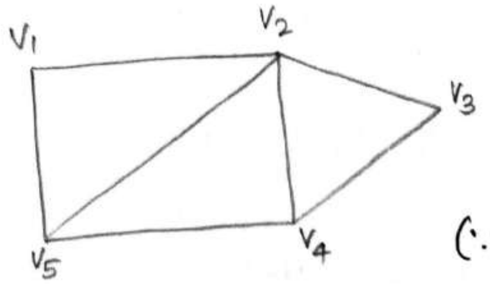
Hamilton Circuit or Cycle:

A circuit or cycle of a graph G is called a Hamilton circuit, if it includes each vertex of G exactly once, except the starting and ending vertices.

Hamiltonian Graph:

Any graph containing a Hamiltonian circuit or cycle is called a Hamiltonian graph.

eg:

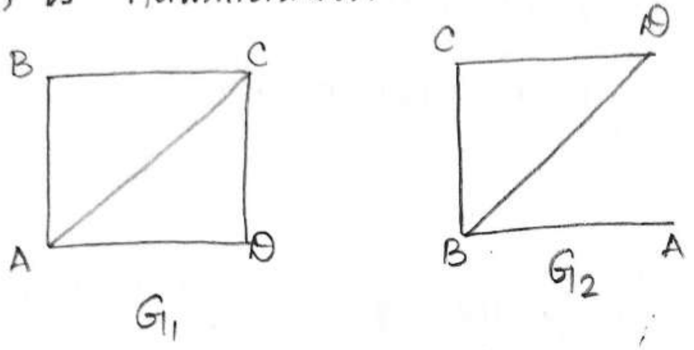


$V_1 - V_2 - V_3 - V_4 - V_5$ is a Hamiltonian Path.

(∵ All vertices appears exactly once)

$V_4 - V_3 - V_2 - V_1 - V_5 - V_4$ is a Hamiltonian cycle.

1. Find Hamilton Path and Hamiltonian ~~Graph~~ cycle, if it exist in each of the graphs given below. Also identify which Graph is Hamiltonian.



Soln:

For G_1, the Possible Hamiltonian paths are

1) A — B — C — D
2) A — D — C — B
3) B — C — D — A
4) B — A — D — C
5) C — D — A — B
6) C — B — A — D
7) D — A — B — C
8) D — C — B — A

The Possible Hamiltonian cycles are

1) A — B — C — D — A
2) A — D — C — B — A
3) B — C — D — A — B
4) B — A — D — C — B
5) C — D — A — B — C
6) C — B — A — D — C
7) D — A — B — C — D
8) D — C — B — A — D

⇒ G_1 contains Hamiltonian cycle.

⇒ G_1 is a Hamiltonian graph.

G_2 Contains Hamiltonian Paths namely,

1) A — B — C — D

2) A — B — D — C

3) D — C — B — A

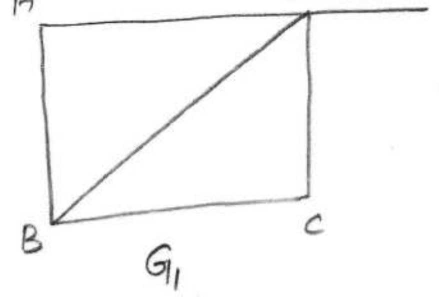
We cannot find Hamiltonian cycle in G_2.

∴ G_2 is not a Hamiltonian Graph.

Properties:

1. A Hamiltonian Circuit contains a Hamiltonian path, but a graph containing a Hamiltonian path need not have a Hamiltonian Cycle.
2. By deleting any one edge from Hamiltonian cycle, we can get Hamiltonian Path.
3. A graph may contain more than one Hamiltonian Cycle.
4. A Complete graph K_n, will always have a Hamiltonian cycle when $n \geq 3$.

2. Check the given graph is Hamiltonian or not.



In G_1, For the Vertex E degree is 1

∴ There is no Hamiltonian cycle in G_1

∴ G_1 is not a Hamiltonian Graph.

3. Give an example of a graph which is

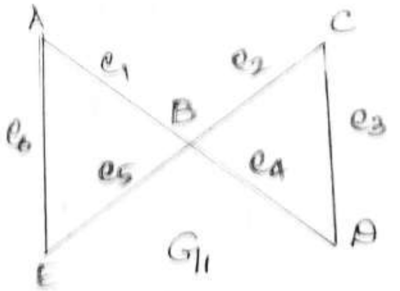
i) Euleian but not Hamiltonian

ii) Hamiltonian but not Euleian

iii) Both Euleian and Hamiltonian

iv) Non Euleian and Non Hamiltonian.

i) Euleian Graph but not a Hamiltonian Graph.



G_1 Contains Euleian cycle

$A \xrightarrow{e_1} B \xrightarrow{e_2} C \xrightarrow{e_3} D \xrightarrow{e_4} B \xrightarrow{e_5} E \xrightarrow{e_6} A$

($\because$ All the edges occurs exactly once)

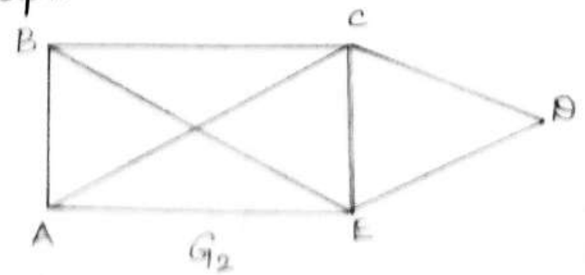
$\therefore$ G_1 is an Eulerian Graph

We cannot find Hamiltonian cycle as the vertex B is repeated twice.

$\therefore$ G_1 is not a Hamiltonian Graph.

$\therefore$ G_1 is Euleian but not Hamiltonian.

ii) Hamiltonian Graph but not an Euleian Graph.



$\because$ G_2 Contains the Hamilton Cycle

A — B — C — D — E — A

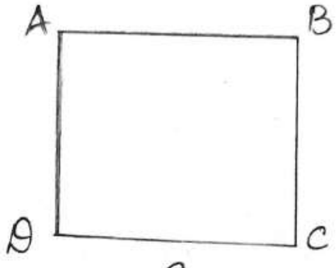
All the Vertices occurs exactly once.

$\therefore$ G_2 is a Hamilton Graph.

$\because$ the degree at the vertices are not even

$\Rightarrow$ G_2 is not an Eulerian Graph.

iii) Both Eulerian and Hamiltonian.



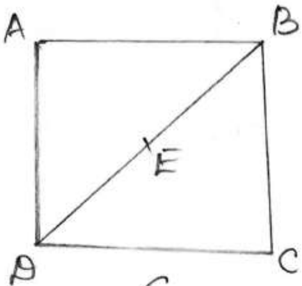
G_3

In G_3, A — B — C — D — A

$\because$ the cycle contains all the edges, G_3 is Eulerian.

$\because$ the cycle contains all the vertices exactly once,
G_3 is Hamiltonian.

iv) Neither Eulerian nor Hamiltonian.



G_4

In G_4 : $\deg(B) = \deg(D) = 3$

$\because$ degree of B and D are not even. G_4 is not an Eulerian graph.

No cycle with all the vertices at exactly once,
G_4 is not a Hamiltonian Graph.

$\therefore$ G_4 is neither Euler nor Hamiltonian Graph.

4. Show that complete graph on 'n' vertices K_n with $n \geq 3$ has Hamilton cycle. Obtain all the two edge disjoint Hamilton cycles in K_7.

Soln

Let 'u' be any vertex of K_n.

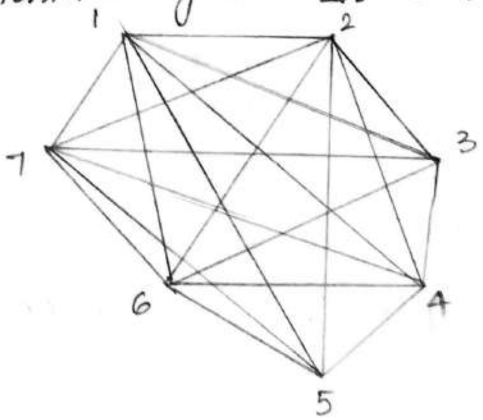
$\because$ K_n is a complete graph with 'n' vertices, any two vertices are joined. So we start with 'u' and visit vertices in any order exactly once and back to 'u'.

Hence there is an Hamiltonian cycle in K_n and $\therefore$ K_n is Hamiltonian. The two edge disjoint Hamiltonian cycles in K_7 are

i) 1 — 2 — 3 — 4 — 5 — 6 — 7 — 1

ii) 1 — 3 — 6 — 2 — 4 — 7 — 5 — 1

5. Show that K_7 has Hamiltonian Graph. How many edge disjoint Hamiltonian cycles are there in K_7?. List all the edge disjoint Hamiltonian cycles. Is it Euleian graph?



K_7 has 2-edge disjoint Hamiltonian cycles.

The edge disjoint Hamiltonian cycles are

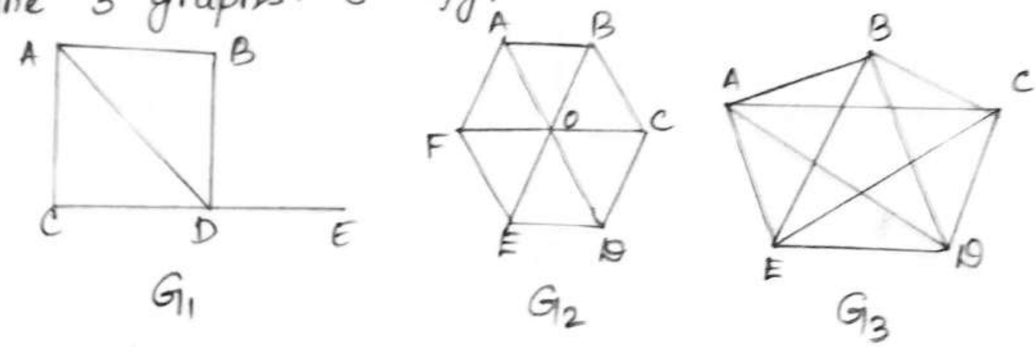
1 — 2 — 3 — 4 — 5 — 6 — 7 — 1

1 — 3 — 6 — 2 — 4 — 7 — 5 — 1

$\therefore$ K_7 is Hamiltonian

K_7 is also Euleion

6. Find an Euler Path or an Euler circuit, if it exists in each of the 3 graphs. Justify?



Soln:

In G_1, there are only 2 vertices A & E are of degree 3,1 and other vertices are of even degree.

∴ there is an Euler Path between A & B

A — C — D — E — B — D — A — B

In G_2, there are 6 vertices of odd degree. Hence G_2 contains neither an Euler Path nor an Euler Circuit.

In G_3, all the vertices of even degree. Hence there exist an Euler circuit, includes each of the 10 edges exactly once

A — B — C — D — E — A — C — E — B — D — A.

Theorem: 2

In a Undirected graph, the number of odd degree vertices are even.

Proof:

Let V_1 and V_2 be the set of all vertices of even degree and set of all vertices of odd degree, respectively in a graph $G = (V, E)$

$$\therefore \sum d(v) = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$$

By handshaking theorem we have $\sum deg(v_i) = 2e$

$$\Rightarrow \quad 2e = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$$

$\because$ each $deg(v_i)$ is even ; $\sum d(v_i)$ is even.

$$\Rightarrow \sum_{v_j \in V_2} d(v_j) \text{ is even}$$

$\because$ each $deg(v_j)$ is odd, the no. of terms contained in

$\sum_{v_j \in V_2} de(v_j)$ must be even.

ie) the number of vertices of odd degree is even.

Theorem: 3

The Maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Proof:

We Prove this theorem by Mathematical Induction

For $n=1$, a graph with one vertex has no edges.

$\therefore$ The result is true for $n=1$.

For $n=2$, a graph with 2 vertices may have atmost one edge.

$\therefore \frac{2(2-1)}{2} = 1$. $\therefore$ The result is true for $n=2$.

Assume that the result is true for $n=K$.

ie) a graph with K vertices has atmost $\frac{K(K-1)}{2}$ edges.

When $n=K+1$, let G be a graph having 'n' vertices and G' be the graph obtained from G by deleting one vertex ie) $v \in V(G)$

$\because$ G' has K vertices, then by the hypothesis G' has atmost $\frac{K(K-1)}{2}$ edges. Now add the vertex 'v' to 'G''

$\ni$: 'v' may be adjacent to all the K vertices of G'

$\therefore$ The total number of edges in G are

$$\frac{K(K-1)}{2}+K = \frac{K^2-K+2K}{2} = \frac{K^2+K}{2} = \frac{K(K+1)}{2}$$

$$= \frac{(K+1)(K+1-1)}{2}$$

$\therefore$ the result is true for $n=K+1$.

Hence the maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

1. If a graph has 'n' vertices and a vertex 'v' is connected to a vertex 'w', then there exists a path from 'v' to 'w' of length not more than $(n-1)$

Soln: let $v, u_1, u_2 \ldots\ldots u_{m-1}, w$ be a path in G from v to w

By the definition of path, the vertices $v, u_1, u_2, \dots u_{m-1}$ and w are all distinct.

As G, contains only 'n' vertices, it follow that

$m+1 \leq n \Rightarrow m \leq n-1$

Theorem: 4

Prove that a simple graph with 'n' vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof:

Let G be a simple graph with 'n' vertices and more than $\frac{(n-1)(n-2)}{2}$ edges.

Suppose if G is not connected, then G must have atleast two components. Let it be G_1 and G_2.

Let V_1 be the vertex set of G_1 with $|V| = m$.

V_2 be the vertex set of G_2 with $|V_2| = n-m$

then

i) $1 \leq m \leq n-1$

ii) There is no edge joining a vertex in V_1 and V_2.

iii) $|V_2| = n-m \geq 1$

Now $|E(G)| = |E(G_1 \cup G_2)|$

$= |E(G_1)| + |E(G_2)|$

$$\leq \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$= \frac{1}{2}\left[m^2 - \cancel{m} + n^2 - nm - n - nm + m^2 + \cancel{m}\right]$$

$$= \frac{1}{2}\left[n(n-1) - nm - m(n-m-1) + m^2 - m\right]$$

$$= \frac{1}{2}\left[n(n-1) - nm - m(n-m) + m^2 + 2(n-1) - 2(n-1)\right]$$

Add & Subtract $2(n-1)$.

$$= \frac{1}{2}\left[(n-1)(n-2) - \cancel{nm} + \cancel{nm} + m^2 + m^2 + 2n - 2\right]$$

$$= \frac{1}{2}\left[n(n-1) + 2m^2 - 2nm + 2(n-1) - 2(n-1)\right]$$

$$= \frac{1}{2}\left[(n-1)(n-2) + 2m^2 - 2nm + 2n - 2\right]$$

$$= \frac{1}{2}\left[(n-1)(n-2) + 2(m^2-1) - 2n(m-1)\right]$$

$$= \frac{1}{2}\left[(n-1)(n-2) + 2\left[(m-1)(m+1)\right] - 2n(m-1)\right]$$

$$= \frac{1}{2}\left[(n-1)(n-2) - 2(m-1)\left[n-m-1\right]\right]$$

$|E(G)| \leq \frac{(n-1)(n-2)}{2}$, $\because (m-1)(n-m-1) \geq 0$ for $1 \leq m \leq n-1$

which is a contradiction as G has more than $\frac{(n-1)(n-2)}{2}$ edges. $\therefore$ Hence G is a Connected Graph.

Theorem: 5

Let G be a simple graph with 'n' vertices, show that if $\delta(G) \geq \frac{n}{2}$ then G is connected where $\delta(G)$ is minimum degree of the graph G.

Proof:

Let u & v be any two distinct vertices in the graph G.

We claim that there is a u–v path.

If uv is an edge in G, then it is a u-v path.

Suppose uv is not an edge of G. Then, X be the set of all vertices which are adjacent to u and Y be the set of vertices which are adjacent to v.

Then $u, v \notin X \cup Y$ [$\because$ G is a simple graph]

and hence $|X \cup Y| \leq n-2$

$$|X| = \deg(u) \geq \delta(G) \geq \left[\frac{n}{2}\right]$$

$$|Y| = \deg(v) \geq \delta(G) \geq \left[\frac{n}{2}\right]$$

$$\therefore |X \cup Y| : |X| + |Y| \geq \left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] \geq n-1$$

We know that $|X \cup Y| = |X| + |Y| - |X \cap Y|$

$$\therefore |X \cap Y| \geq 1 \Rightarrow X \cap Y = \phi$$

$\therefore$ Take a vertex $w \in X \cap Y$. Then uvw is a u-v path in G

$\therefore$ For every pair of distinct vertices of G there is a path between them. $\therefore$ G is connected.

Theorem: 6

A simple graph with 'n' vertices and 'K' Components can have atmost $\frac{(n-K)(n-K+1)}{2}$ edges.

Proof:

Let G be a simple graph with 'n' vertices and 'K' components $G_1, G_2 \ldots\ldots G_K$.

Let the number of vertices of these components be $n_1, n_2 \ldots\ldots n_K$ respectively, so that $n_1+n_2+\ldots n_K = n$ → ①

The Component G_i is a simple Connected graph with n_i vertices.

So the maximum number of edges is $n_iC_2 = \frac{n_i(n_i-1)}{2}$

$$\Rightarrow |E(G_i)| \leq \frac{n_i(n_i-1)}{2}$$

$$\text{But } |E(G)| \leq \sum_{i=1}^{K} |E(G_i)|$$

$$\leq \sum_{i=1}^{K} \frac{n_i(n_i-1)}{2}$$

Consider G_i. Even if all the remaining $(K-1)$ components are isolated vertices, the number of vertices in G_i Cannot exceed $n-(K-1) = n-K+1$

$$\therefore n_i \leq n-K+1$$

$$\therefore |E(G_i)] \leq \sum_{i=1}^{K} \frac{(n-K+1)}{2}(n_i-1)$$

$$\leq \frac{(n-k+1)}{2} \sum_{i=1}^{k} (n_i - 1)$$

$$\leq \frac{(n-k+1)}{2} \sum_{i=1}^{k} (n_i - k)$$

$$\leq \frac{(n-k+1)}{2} [n_1 + n_2 + n_3 \cdots\cdots + n_k - k]$$

$$\leq \frac{n-k+1}{2} (n-k) \quad \text{using } ①$$

$$\Rightarrow \quad \leq \frac{(n-k)(n-k+1)}{2} \text{ edges.}$$

Theorem : 7:

A connected graph G is Eulerian if and only if every vertex of G is of even degree.

Proof:

Let G be an Eulerian graph. We have to prove all vertices are of even degree.

$\because$ G is Eulerian, G contains an Euler Circuit

$$v_0 \; e_1 \; v_1 \; e_2 \cdots\cdots v_{n-1} \; e_n \; v_0$$

Both the edges e_1 and e_n contribute one to the degree of v_0. So $\deg(v_0)$ is atleast two.

In tracing this circuit we find an edge enters a vertex and another edge leaves the vertex contributing 2 to the degree of the vertex.

This is true for all vertices and so each vertex is of degree 2, an even integer.

Conversely, let the graph G be such that all its vertices are of even degrees.

We have to prove G is an Euler graph.

We shall construct an Euler circuit and prove. Let V be an arbitrary vertex in G.

Beginning with V form a circuit $C: V, V_1, V_2 \ldots V_{n-1}, V$

This is possible because every vertex of even degree. We can leave a vertex along an edge not used to enter it. This tracing clearly stops only at the vertex V, because V is also of even degree and it is started from V. Thus we get a cycle or circuit C.

If C includes all the edges of G, then C is an Euler circuit and so G is Eulerian.

If C does not contain all the edges of G, consider the subgraph H of G obtained by deleting all the edges of C from G and vertices not incident with the remaining edges. Note that all the vertices of H have even degree. Since G is connected, H & C must have a common vertex u. Beginning with u construct a circuit C_1 for H.

Now combine C and C_1 to form a larger Circuit C_2.
If it is Eulerian ie) if it contains all the edges of G, then G is Eulerian.

Else continue this process until we get an Eulerian Circuit.

Since G is finite this procedure must come to an end with an Eulerian circuit.

Hence G is Eulerian.

Theorem: 8

If all the vertices of an undirected graph are each of degree k, show that the no. of edges of the graph is a multiple of k.

Proof:

Let 2n be the no. of vertices of the given graph.

Let n_e be the no. of edges of the given graph.

By Handshaking thm, we have $\sum_{i=1}^{n} deg(v_i) = 2n_e$

$$\Rightarrow 2nk = 2n_e$$

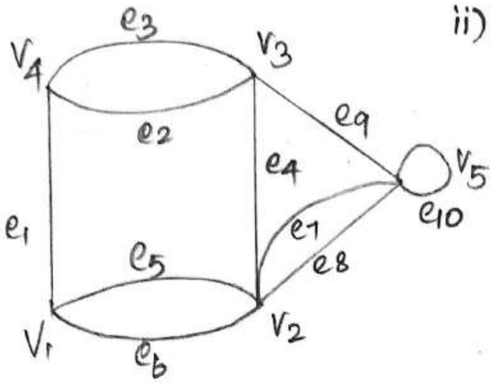
$$\Rightarrow n_e = nk$$

$\Rightarrow$ No. of edges = multiple of k.

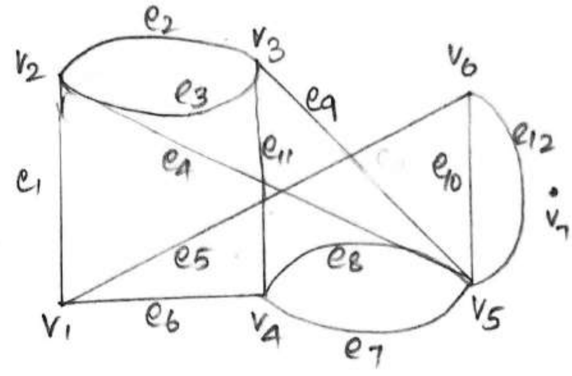
$\therefore$ The no. of edges of the given graph is a multiple of k.

1. Find the no. of vertices, no. of edges and the degree of each vertex. Verify handshaking theorem.

i)



ii)



i) $|V| = 5$; $|E| = 10$

$deg(v_1) = 3$; $deg(v_2) = 5$; $deg(v_3) = 4$

$deg(v_4) = 3$; $deg(v_5) = 5$

$\therefore \sum deg(v) = 3+5+4+3+5 = 20 = 2e$

ii) $|V| = 7$; $|E| = 12$

$deg(v_1) = 3$; $deg(v_2) = 4$; $deg(v_3) = 4$

$deg(v_4) = 4$; $deg(v_5) = 6$; $deg(v_6) = 3$; $deg(v_7) = 0$

$\therefore \sum deg(v) = 3+4+4+4+6+3 = 24 = 2e$

2. If the simple graph G has 4 vertices and 5 edges, How many edges does G^c have?

Soln:

$$G^c = \frac{v(v-1)}{2} - e$$

$$= \frac{4(4-1)}{2} - 5 = 6-5 = 1 \text{ edge.}$$

Unit - IV

ALGEBRAIC STRUCTURES

Algebraic System:

A non-empty set G together with one or more binary operations is called an algebraic system or algebraic structure or Algebra.

denoted by $[G, *]$

Note: $+, -, \cdot, \times, *, \cup, \cap$, etc. are some of binary operations

Properties of Binary Operations:

Let the binary operation be $* : G * G \to G$.

1. Closure Property:

$$a*b = x \in G, \ \forall a, b \in G.$$

2. Commutative Property:

$$a*b = b*a, \quad \forall a, b \in G.$$

3. Associative Property:

$$(a*b)*c = a*(b*c) \ \forall a, b \in G.$$

4. Identity Element:

$$a*e = e*a = a, \ \forall a \in G.$$

'e' is called the identity element.

5. Inverse Element:

If $a*b = b*a = e$ (identity) then 'b' is called the inverse of 'a' and it is denoted by $b = a^{-1}$.

6. Distributive Properties:

$a * (b \cdot c) = (a * b) \cdot (a * c)$ [left distributive law]

$(b \cdot c) * a = (b * a) \cdot (c * a)$ [Right distributive law]

$\forall a, b, c \in G$

7. Cancellation Properties:

$a * b = a * c \Rightarrow b = c$ [left cancellation law]

$b * a = c * a \Rightarrow b = c$ [Right cancellation law]

$\forall a, b, c \in G$.

Note:

If the binary operations defined on G is + and ×, then

	For all $a, b, c \in G$	$(G, +)$	$(G, *)$
1.	Commutativity	$a + b = b + a$	$a * b = b * a$
2.	Associativity	$(a+b)+c = a+(b+c)$	$(a*b)*c = a*(b*c)$
3.	Identity element	$a + 0 = 0 + a = a$ ($0 \rightarrow$ identity)	$a * 1 = 1 * a = a$ ($1 \rightarrow$ identity)
4.	Inverse element	$a + (-a) = 0$ ($-a \rightarrow$ Additive Inverse	$a * \frac{1}{a} = \frac{1}{a} * a = 1$ $\frac{1}{a} \rightarrow$ Multiplicative Inverse

Notations:

Z – the set of all integers

Q – the set of all rational nos.

R – the set of all real nos.

R^+ – the set of all positive real nos.

Q^+ – the set of all positive rational nos.

C – the set of all complex nos.

Semigroups and Monoids:

Semigroup:

If a non-empty set 'S' together with the binary operation $*$ satisfying the following two properties

a) $a * b = b * a$; $a, b \in S$ [closure Property]

b) $(a * b) * c = a * (b * c)$; $a, b, c \in S$ [Associative property]

Monoid:

A Semigroup $(S, *)$ with an identity element w.r.t '$*$' is called Monoid.

It is denoted by $(M, *)$

a) $a * b = b * a$ (closure Property)

b) $(a * b) * c = a * (b * c)$ (Associative Property)

c) $a * e = e * a = a$ (Identity Property)

1. Show that the set $N = \{0, 1, 2 \dots\}$ is a semigroup under the operation $x * y = \max\{x, y\}$. Is it a monoid?

Soln

1. Closure Property:

$$x * y = \max\{x, y\}$$

$$= \begin{cases} x & \text{if } x > y \\ y & \text{if } y > x \end{cases}$$

$\Rightarrow \forall\, x, y \in N \Rightarrow x * y \in N$

$\therefore$ $*$ is closed.

2. Associative Property:

$$x * (y * z) = \max\{x, (y * z)\}$$

$$= \max\{x, \max\{y, z\}\}$$

$$= \max\{x, y, z\} \rightarrow (A)$$

$$(x * y) * z = \max\{(x * y), z\}$$

$$= \max\{\max(x, y), z\}$$

$$= \max\{x, y, z\} \rightarrow (B)$$

$\therefore$ From (A) & (B) We get

$$(x * y) * z = x * (y * z)$$

$\therefore$ $*$ satisfies Associative property.

$\therefore$ $(N, *)$ is a Semigroup.

3. Identity element:

$\because$ $0 \in N$, satisfies

$$x * 0 = \max\{x, 0\} = x = \max\{0, x\} = 0 * x$$

the identity element is 0.

$\therefore$ N is a monoid.

2. Let I be the set of integers. Let Z_m be the set of equivalence classes generated by the equivalence relation "Congruence Modulo m" for any positive integer m. Then $(Z_m, +_m)$ and $(Z_m, \times_m)$ are monoids.

Soln:

The algebraic Systems $(Z_m, +_m)$ and $(Z_m, \times_m)$ are monoids.

For $[i], [j] \in Z_m$

(a) $+_m$ is defined as

$$[i] +_m [j] = [(i+j)(\text{mod } m)]$$

(b) $\times_m$ is defined as

$$[i] \times_m [j] = [(i \times j)(\text{mod } m)]$$

The Composition table for m=5 is given as

$(Z_5, +_5)$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$(Z_5, \times_5)$

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

i) Associative Property:

$(Z_5, +_5), (Z_5, \times_5)$ satisfies associative property

ii) identity element:

$[0]$ is the identity element w.r.t $+_m$.

$[1]$ is the identity element w.r.t $\times_m$.

$\therefore (Z_m, +_m)(Z_m, \times_m)$ are monoids.

3. Let $A = \{0,1\}$ be the given set. Let S denote the set of all mappings from A to A. We have $2^2 = 4$ mappings available, $S = \{I_1, I_2, I_3, I_4\}$ where

1. $I_1(0) = 0$ and $I_1(1) = 1$
2. $I_2(0) = 0$ and $I_2(1) = 0$
3. $I_3(0) = 1$ and $I_3(1) = 1$
4. $I_4(0) = 1$ and $I_4(1) = 0$

Soln:

The Composition of the fn/. is given

o	I_1	I_2	I_3	I_4
I_1	I_1	I_2	I_3	I_4
I_2	I_2	I_2	I_2	I_3
I_3	I_3	I_3	I_3	I_2
I_4	I_4	I_3	I_2	I_1

1. Associative property:

$$[(I_1 \circ I_4) \circ I_2](0) = (I_4 \circ I_2)(0)$$
$$= I_4[I_2(0)]$$
$$= I_4(0)$$
$$= 1$$

$$[I_1 \circ (I_4 \circ I_2)](0) = (I_1 \circ I_3)(0)$$
$$= I_1(I_3(0))$$
$$= I_1(1)$$
$$= 1$$

$\Rightarrow$ It is proved.

2. Identity Element:

I_1 is the identity element in $(S, \circ)$, Hence

i) $(S, \circ)$ is associative

ii) $(S, \circ)$ has identity element I_1.

$\therefore$ $(S, \circ)$ is a semigroup as well as monoid.

Cyclic Monoid:

A monoid $(M, *)$ is said to be cyclic, if every element of M is of the form a^n, $a \in M$, 'n' is an integer. $x = a^n$, such a cyclic monoid is said to be generated by the element 'a'. Here 'a' is called the generator of the cyclic Monoid.

Theorem: 1

Every cyclic Monoid (semigroup) is Commutative.

Proof:

Let $(M,*)$ be a cyclic monoid whose generator is $a \in M$. Then for $x, y \in M$ we have

$$x = a^n \ ; \ y = a^m$$

$$x * y = a^n * a^m = a^{n+m} = a^{m+n} = a^m * a^n$$
$$= y * x$$

$\therefore$ $(M,*)$ is Commutative.

Groups:

Group:

A non-empty set G together with the binary operation $*$, ie) $(G,*)$ is a group if $*$ satisfies the following

i) Closure : $a*b \in G \quad \forall a, b \in G$

ii) Associative : $(a*b)*c = a*(b*c) \ ; \ \forall a, b, c \in G$.

iii) Identity : $\exists$ an element $e \in G$ called the identity element $\ni$: $a*e = e*a = a \ ; \ \forall a \in G$.

iv) Inverse : $\exists$ an element $a^{-1} \in G$ called the inverse of a $\ni$: $a*a^{-1} = a^{-1}*a = e \quad \forall a \in G$.

Abelian Group:

In a group $(G,*)$ if $a*b = b*a$; $\forall a,b \in G$ then the group $(G,*)$ is called an abelian group.

Order of a Group:

The no. of elements in a group G is called the order of the group and it is denoted by $O(G)$.

Finite and Infinite Group:

If $O(G)$ is finite, then G is said to be a finite grp.

If $O(G)$ is infinite, then G is said to be an infinite grp.

1. Show that the set $G = \{1, -1, i, -i\}$ consisting of the 4^{th} roots of unity is a commutative group under multiplication.

Soln:

Consider the multiplication table:

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

All the elements in this table belongs to G. Hence G is closed. '1' is the identity element.

Inverse of 1 is 1

-1 is -1

i is i

$-i$ is $-i$.

2. Show that $(Q^+, *)$ is an abelian group where $*$ is defined by $a*b = \frac{ab}{2}$, $\forall a,b \in Q^+$.

Soln

Q^+ – Set of all positive rational nos.

1. Closure Property : $a*b = \frac{ab}{2} \in Q^+$

2. Associative Property :

$$(a*b)*c = \frac{ab}{2}*c = \frac{\frac{abc}{2}}{2} = \frac{abc}{4}$$

$$a*(b*c) = a*\frac{bc}{2} = \frac{\frac{abc}{2}}{2} = \frac{abc}{4}$$

$\Rightarrow (a*b)*c = a*(b*c)$

3. Identity :

let 'e' be the identity element

then $a*e = a$

$\frac{ae}{2} = a \Rightarrow e = 2 \quad \therefore \; e = 2 \in Q^+$

4. Inverse: Let a^{-1} be the inverse of a.

then $a * a^{-1} = 2$ (identity)

$$\frac{aa^{-1}}{2} = 2 \Rightarrow aa^{-1} = 4$$

$$\Rightarrow a^{-1} = \frac{4}{a} \in Q^+$$

$\therefore$ inverse of a is $a^{-1} = \frac{4}{a} \in Q^+$.

5. Commutative:

$$a*b = \frac{ab}{2} \quad ; \quad b*a = \frac{ba}{2}$$

$$\Rightarrow a*b = b*a \quad \forall a,b \in Q^+$$

$\therefore (Q^+, *)$ is an abelian group.

3. Show that $(R-\{1\}, *)$ is an abelian group, where $*$ is defined by $a*b = a+b+ab \ \forall a,b \in R$.

Soln:

Here $R-\{1\}$ means the set of real nos. except -1.

1. closure property:

$$a*b = a+b+ab \in (R-\{1\}) \quad [a \neq -1 ; b \neq -1]$$

2. Associative Property:

$$(a*b)*c = (a+b+ab)*c$$
$$= a+b+ab+c+ac+bc+abc \rightarrow (1)$$

$$a*(b*c) = a*(b+c+bc)$$
$$= a+b+c+bc+ab+ac+abc \rightarrow (2)$$

① = ②

$\Rightarrow$ Associative Property holds.

3. Identity:

let 'e' be the identity element

then $a * e = a$

$$a + e + ae = a$$

$$e(1+a) = 0$$

$$e = 0$$

$\Rightarrow$ '0' is the identity element and $0 \in (R - \{1\})$

4. Inverse:

Let the inverse of a be a^{-1}

then $a * a^{-1} = 0$

$$a + a^{-1} + aa^{-1} = 0$$

$$a^{-1}(1+a) = -a$$

$$a^{-1} = \frac{-a}{1+a} \in (R - \{1\})$$

$\therefore$ Inverse element is $\frac{-a}{1+a}$

5. Commutative:

$$a * b = a + b + ab = b + a + ba$$

$$= b * a$$

$$\Rightarrow a * b = b * a \quad \forall\, a, b \in (R - (1))$$

$\therefore (R - \{1\})$ is an abelian group.

4. Prove that the set $A = \{1, \omega, \omega^2\}$ is an abelian group of order 3 usual multiplication, where $1, \omega, \omega^2$ are cube roots of unity and $\omega^3 = 1$

Soln:

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

1. Closure Property:

All the elements in the above table are the elements of A. Hence A is closed under '.'.

2. Associative Property:

Multiplication of complex nos. are associative.

3. Identity: Identity element is 1.

4. Inverse: Inverse of 1 is 1
ω is ω^2
ω^2 is ω

5. Commutative: $\omega \cdot \omega^2 = \omega^2 \cdot \omega = \omega^3$

$\therefore$ Commutative is true.

Hence $(A, \cdot)$ is an Abelian group.

$O(A) = 3$.

5. $*$ on R defined by $x*y = x+y+2xy$; $\forall x,y \in R$

Check 1. $(R,*)$ is a Monoid or not.

2. Is it Commutative.

3. Which elements have inverses & what are they?

Soln:

1. i) Closure property:

$\because x,y \in R \Rightarrow x+y+2xy \in R.$

$\Rightarrow x*y \in R$

$\therefore *$ satisfies Closure Property.

ii) Associative Property:

$(x*y)*z = x*(y*z)$

$\Rightarrow (x+y+2xy)*z$

$\Rightarrow x+y+2xy+z+2z(x+y+2xy)$

$\Rightarrow x+y+2xy+z+2xz+2yz+4xyz \rightarrow (1)$

$\therefore x*(y*z)$

$\Rightarrow x*(y+z+2yz)$

$\Rightarrow x+y+z+2yz+2x(y+z+2yz)$

$\Rightarrow x+y+z+2yz+2xy+2xz+4xyz \rightarrow (2)$

$(1) = (2) \Rightarrow (x*y)*z = x*(y*z)$

$\Rightarrow \therefore *$ is Associative.

iii) Identity Property:

let 'e' be the identity element.

$a*e = e*a = a$

$\Rightarrow a*e = a$

$\Rightarrow a+e+2ae = a$

$e(1+2a) = 0 \Rightarrow e = 0 \in R.$

$\therefore$ Identity Element exists.

$\because$ $*$ Satisfies Closure, Associative & Identity element

$(R, *)$ is a Monoid.

2. Now $x*y = x+y+2xy$

$= y+x+2yx$

$= y*x$

$\Rightarrow x*y = y*x \quad \forall x, y \in R.$

$\therefore (R, *)$ is commutative.

3. Let a^{-1} be the inverse element

then $a * a^{-1} = e$

$\Rightarrow a + a^{-1} + 2aa^{-1} = e$

$a^{-1} = \frac{-a}{1+2a}$

$\therefore a^{-1} = -\frac{a}{1+2a}.$

6. Let $S = Z^+ \times Z^+$, Z^+ being set of positive integer and $*$ be an operation on S given by $(a,b)*(c,d) = (a+c, b+d)$ $\forall\, a,b,c,d \in Z^+$. Show that 'S' is a semigroup. Also show that f is a homomorphism, if $f:(S,*) \to (Z,+)$ defined by $f(a,b) = a-b$.

Soln

let x, y, z be the ordered pairs (a,b), (c,d) and (e,f) in $Z^+ \times Z^+$

$$(xy)z = (x*y)*z$$
$$= [(a,b)*(c,d)]*(e,f)$$
$$= [a+c, b+d]*(e,f)$$
$$= [(a+c)+e, (b+d)+f]$$
$$(xy)z = [a+c+e, b+d+f] \longrightarrow ①$$

$$x(yz) = x*(y*z)$$
$$= (a,b)*[(c,d)*(e,f)]$$
$$= (a*b)*[c+e, d+f]$$
$$= [a+(c+e), b+(d+f)]$$
$$= [a+e+c, b+d+f] \longrightarrow ②$$

$① = ② \Rightarrow (xy)z = x(yz)$

$\therefore$ $*$ is Associative.

$\Rightarrow *$ is obviously closure property.

$\therefore$ S is a semigroup.

Claim:

$f : (S, *) \to (Z, +)$ by $f(a,b) = a-b$ is a homomorphism

$\forall$ $x, y \in X$.

$$f(x*y) = f[(a,b)*(c,d)]$$
$$= f[a+c, b+d] = (a+c)-(b+d)$$
$$= (a-b)+(c-d)$$
$$= f(a,b) + f(c,d)$$
$$= f(x) + f(y)$$

$$\therefore f(x*y) = f(x) + f(y)$$

$\therefore$ f is a homomorphism.

7. Let $S = Q \times Q$, be the set of all ordered pairs of rational nos. and given by $(a,b)*(x,y) = (ax, ay+b)$

i) Check $(S,*)$ is a semigroup. Is it commutative?

ii) Also find the identity element of S.

Soln:

i) (1) Closure Property:

Obviously $*$ satisfies closure property.

(2) Associative Property:

$$[(a,b)*(x,y)]*(c,d) = [(ax, ay+b)*(c,d)]$$
$$= [axc, axd + (ay+b)]$$
$$= [acx, adx + ay + b]$$

$(a,b)*[(x,y)*(c,d)] = (a,b)*[cx, dx+y]$

$= [acx, adx+ay+b]$

$\Rightarrow [(a,b)*(x,y)]*(c,d) = (a*b)*[(x,y)*(c,d)]$

$\Rightarrow *$ is associative.

$\therefore (S,*)$ is Semigroup.

(3) Commutative Property:

$(a,b)*(x,y) = [ax, ay+b]$; $(x,y)*(a,b) = xa, xb+y$

$= \cancel{xa}, \cancel{ya+b}$

$[(a,b)*(x,y)] \neq [(x,y)*(a,b)]$

$\therefore (S,*)$ is not Commutative

ii) Identity Property:

let (e_1, e_2) be the identity element of $(S,*)$

Then for any $(a,b) \in S$.

$(a,b)*(e_1,e_2) = (a,b)$

$(ae_1, ae_2+b) = (a,b)$

$\Rightarrow ae_1 = a$ and $ae_2+b = b$

$e_1 = 1$ $\quad$ $e_2 = 0$

$\therefore$ The identity element $= (e_1, e_2) = (1,0)$

8. If M_2 is the set of 2x2 non singular matrices over R. ie) $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a,b,c,d \in R \text{ and } ad-bc \neq 0 \right\}$. Prove that $(M_2, *)$ is a group, where $*$ is usual multiplication. Is it abelian?

i) Closure Property:

let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$; $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$AB = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}$$

$$|AB| = |A| \cdot |B|$$

$$A, B \in M_2 \Rightarrow AB \in M_2$$

$\therefore$ Matrix Multiplication is closed.

ii) Associative Property:

We know that Matrix multiplication is associative.

iii) Identity:

If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $IA = AI = A$

Hence $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element of M_2

iv) Inverse:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in M_2$

$\therefore$ Inverse of A is $A^{-1} \in M_2$

Hence $(M_2, \times)$ is a group

$\because AB \neq BA$ $\therefore (M_2, \times)$ is not abelian.

9. Show that $\{1, 3, 7, 9\}$ is an abelian group under multiplication modulo 10.

Soln

let $G = \{1, 3, 7, 9\}$

$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

From the table it is obvious that closure & associative property holds.

Identity element is 1 and $1 \in G$.

Inverse of 1 is 1

3 is 3

7 is 7

9 is 9.

$\therefore (G, \times_{10})$ is an abelian group.

Properties of Group:

Property 1:

The identity element of a group is unique.

Proof:

let $(G, *)$ be a group

let e_1 and e_2 be two identity elements in G

Suppose e_1 is the identity, then

$$e_1 * e_2 = e_2 * e_1 = e_2$$

Suppose e_2 is the identity, then

$$e_1 * e_2 = e_2 * e_1 = e_1$$

$\therefore \quad e_1 = e_2$

$\therefore$ The identity element is unique.

Property 2:

In a group $(G,*)$ the left and right cancellation laws are true.

i.e) $a*b = a*c \Rightarrow b=c$ [left Cancellation law]

$b*a = c*a \Rightarrow b=c$ [Right Cancellation law]

Proof:

let $(G,*)$ be a group.

let $a \in G$ and hence $a^{-1} \in G$. Then $a*a^{-1} = a^{-1}*a = e \in G$

1) Left Cancellation law:

let $a*b = a*c$.

Pre multiply by a^{-1} on both sides

$$a^{-1}*(a*b) = a^{-1}*(a*c)$$

$$(a^{-1}*a)*b = (a^{-1}*a)*c$$

$$e*b = e*c \Rightarrow b=c$$

2) Right Cancellation law:

let $b*a = c*a$

Post multiply by a^{-1} on both sides

$$(b*a)*a^{-1} = (c*a)*a^{-1}$$

$$b*(a*a^{-1}) = c*(a*a^{-1})$$

$$b*e = c*e \Rightarrow b=c$$

Property 3:

The inverse element of a group is unique.

Proof:

Let $(G, *)$ be a group

let $a \in G$ and e be the identity of G. let a_1^{-1} and a_2^{-1} be the two different inverse of the same element.

$$a_1^{-1} * a = a * a_1^{-1} = e$$

$$a_2^{-1} * a = a * a_2^{-1} = e$$

$$(a_1^{-1} * a) * a_2^{-1} = e * a_2^{-1} = a_2^{-1} \rightarrow (1)$$

$$a_1^{-1} * (a * a_2^{-1}) = a_1^{-1} * e = a_1^{-1} \rightarrow (2)$$

From (1) & (2) $\Rightarrow$ $a_1^{-1} = a_2^{-1}$.

Property 4:

A group cannot have any element which is idempotent except the identity element.

(or)

Prove that in a group the only idempotent element is identity element.

Proof:

let $(G, *)$ be a group.

Assume that $a \in G$ is an idempotent element. Then we have

$$a * a = a \rightarrow (1)$$

let $a = a * e = a * (a * a^{-1}) = (a * a) * a^{-1}$

$$= a * a^{-1}$$

$\Rightarrow$ $a = e$; ie) Idempotent element 'a' is = to identity.

Property 5:

In a group $(a^{-1})^{-1} = a$; $a \in G$

(or)

The inverse of a^{-1} is a.

Proof:

Let $(G,*)$ be a group

let 'e' be the identity element

We know that

$$a^{-1} * a = e = a * a^{-1} \; ; \; a \in G$$

$$(a^{-1})^{-1} * (a^{-1} * a) = (a^{-1})^{-1} * e = (a^{-1})^{-1}$$

$$((a^{-1})^{-1} * a^{-1}) * a = e * a = a$$

$$\Rightarrow (a^{-1})^{-1} = a$$

Hence proved.

Note: The above Property is "involution law".

Property 6:

If a has inverse b and b has inverse c, then a = c.

Proof:

Given 'a' has inverse 'b'.

$$a * b = e = b * a \rightarrow (1)$$

'b' has inverse 'a'

$$b * c = e = c * b \rightarrow (2)$$

Now $a = a * e$

$= a * (b * c)$ [from ②]

$= (a * b) * c$ [Associative]

$= e * c$ [From ①]

$a = c$

Property 7:

Let G be a group. If $a, b \in G$, then $(a*b)^{-1} = b^{-1} * a^{-1}$.

(or)

The inverse of the product of two elements is equal to the product of their inverses in reverse order.

Proof:

let $a, b \in G$ and a^{-1}, b^{-1} be their inverses

$$a * a^{-1} = e = a^{-1} * a$$

and $b * b^{-1} = e = b^{-1} * b$

$$\Rightarrow (a*b) * (b^{-1} * a^{-1}) = a * [b * (b^{-1} * a^{-1})]$$

$$= a * [(b * b^{-1}) * a^{-1}]$$

$$= a * [e * a^{-1}]$$

$$= a * a^{-1}$$

$$\therefore (a*b) * (b^{-1} * a^{-1}) = e$$

Similarly $(b^{-1} * a^{-1}) * (a * b) = e$

$$\therefore (a*b)^{-1} = b^{-1} * a^{-1}$$

ie) the inverse of $a * b$ is $b^{-1} * a^{-1}$.

Property 8:

For any group G, if $a^2 = e$ with $a \neq e$, then G is abelian

(or)

If every element of a group G has its own inverse, then G is abelian. Is the converse true.

Proof:

Let $(G, *)$ be a group

For $a, b \in G$ we've $a * b \in G$.

Given $a = a^{-1}$ and $b = b^{-1}$.

$$(a*b) = (a*b)^{-1}$$
$$= b^{-1} * a^{-1}$$
$$= b * a$$
$$\Rightarrow a*b = b*a$$

$\therefore$ G is abelian.

The converse need not be true since $(Z, +)$ is an abelian group. Except 0, there is no element in G, which has its own inverse.

Property 9:

Prove that $(G, *)$ is a abelian group if and only iff $(a*b)^2 = a^2 * b^2$; $\forall a, b \in G$.

Proof:

Assume that G is abelian.

$\therefore \quad a*b = b*a$

$$a^2 * b^2 = (a*a)*(b*b)$$
$$= a*[(a*b)*b]$$
$$= a*[(b*a)*b]$$
$$= (a*b)*(a*b)$$
$$= (a*b)^2$$

$\Rightarrow \quad a^2 * b^2 = (a*b)^2$.

Conversely assume that $(a*b)^2 = a^2 * b^2$

$$(a*b)*(a*b) = (a*a)*(b*b)$$

$a*[b*(a*b)] = a*[a*(b*b)]$ [left cancellation law]

$$b*(a*b) = (a*b)*b$$

$(b*a)*b = (a*b)*b$ [Right cancellation law]

$\Rightarrow \quad b*a = a*b$.

$\therefore$ G is Abelian

1. Prove that in an abelian group $(ab)^2 = a^2b^2$

Soln

$$(ab)^2 = (ab)(ab)$$
$$= a(ba)b$$
$$= a(ab)b$$
$$= (aa)(bb)$$
$$= a^2b^2$$

$\Rightarrow (ab)^2 = a^2b^2$

Subgroups:

Subgroup:

Let $(G,*)$ be a group. Then $(H,*)$ is said to be a subgroup of $(G,*)$ if $H \subseteq G$ and $(H,*)$ itself is a group under the operation $*$.

ie) $(H,*)$ is said to be a subgroup of $(G,*)$ if

i) $e \in H$, where 'e' is the identity in G.

ii) For any $a \in H$; $a^{-1} \in H$

iii) For $a,b \in H$, $a*b \in H$.

egs:

1. $(Q,+)$ is a subgroup of $(R,+)$
2. $(R,+)$ is a subgroup of $(C,+)$

Proper and improper subgroups.

For any group $(G,*)$

i) The subgroups $(G,*)$ and $(\{e\},*)$ are called improper (or) trivial subgroups.

ii) All the other groups are called the proper (or) non-trivial subgroups.

Theorem-1:

The necessary and sufficient condition that a non-empty subset H of a group G to be a

Subgroup is $a, b \in H \Rightarrow a * b^{-1} \in H \quad \forall a, b \in H$.

Proof: (Necessary condition)

Assume that H is a subgroup of G. $\because$ H itself is a group. we've for $a, b \in H \Rightarrow a * b \in H$ [closure]

$\because b \in H \Rightarrow b^{-1} \in H$

$\therefore$ For $a, b \in H \Rightarrow a, b^{-1} \in H$

$\Rightarrow a * b^{-1} \in H$.

(Sufficient Condition)

let $a * b^{-1} \in H \quad \forall a, b \in H$.

To Prove that H is a Subgroup of G.

i) Identity : let $a \in H$

$\Rightarrow a^{-1} \in H \Rightarrow a * a^{-1} \in H \Rightarrow e \in H$

$\therefore$ the identity element $e \in H$.

ii) Inverse :

let $a, e \in H$

$\Rightarrow e * a^{-1} \in H$

$\Rightarrow a^{-1} \in H$

$\therefore$ Every element 'a' of H has its inverse a^{-1} in H.

iii) Closure : let $b \in H \Rightarrow b^{-1} \in H$

For $a, b \in H \Rightarrow a, b^{-1} \in H$

$\Rightarrow a * (b^{-1})^{-1} \in H \Rightarrow a * b \in H$

$\therefore$ H is closed.

$\therefore$ H is a Subgroup of G.

Theorem - 2:

The intersection of two subgroups of a group is also a subgroup of the group.

(or)

Let G be a group, H_1 and H_2 are subgroups of G. then $H_1 \cap H_2$ is also a subgroup of G.

Proof:

$\because$ H_1 and H_2 are subgroups of G. $\Rightarrow H_1 \cap H_2 \neq \phi$

Let $a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ and $a, b \in H_2$

$\Rightarrow a * b^{-1} \in H_1$ and $a * b^{-1} \in H_2$

$\Rightarrow a * b^{-1} \in H_1 \cap H_2$

$\therefore$ For $a, b^{-1} \in H_1 \cap H_2$ we've $a * b^{-1} \in H_1 \cap H_2$.

$\therefore$ $H_1 \cap H_2$ is a subgroup.

Theorem - 3:

The union of two subgroups of a group need not be subgroup.

Proof:

Let's prove by example.

We know that $(z, +)$ is a group of integers under addition

Define $H_1 = \{x \mid x = 2n ; n \in Z\}$

$= \{0, \pm 2, \pm 4, \pm \cdots\}$

and $H_2 = \{x \mid x = 3n, n \in \mathbb{Z}\}$

$= \{0, \pm 3, \pm 6, \dots\}$

Clearly H_1 and H_2 are subgroups of G.

$H_1 \cup H_2 = \{x \mid x \in H_1 \text{ or } x \in H_2\}$

$= \{0, \pm 2, \pm 3, \pm 4, \dots\}$

Here $2 \in H_1$ and $3 \in H_2 \Rightarrow 2+3 = 5 \notin H_1 \cup H_2$.

$\therefore$ $H_1 \cup H_2$ is not closed under addition.

$\therefore$ $H_1 \cup H_2$ is not a group.

Hence $H_1 \cup H_2$ is not a subgroup of G.

Theorem : 4 :

The identity element of a subgroup is same as that of the group.

Proof :

Let G be a group

Let H be a subgroup of G.

let e and e' be the identity elements in G and H

If $a \in H$, then $a \in G$ and $ae = a$ ($\because$ 'e' is the identity element in G)

Again if $a \in H$, then $ae' = a$ ($\because$ e' is the identity element in H)

$\therefore$ $ae = ae'$

$\Rightarrow e = e'$

Theorem – 5:

The union of two subgroups of a group G is a subgroup iff one is contained in the other.

(or)

Let H and K be two subgroups of a group G. then $H \cup K$ is a subgroup iff either $H \subseteq K$ or $K \subseteq H$.

Proof:

Assume that H and K are two subgroups of G and $H \subseteq K$ or $K \subseteq H$

$\therefore H \cup K = K$ or $H \cup K = H$

Hence $H \cup K$ is a subgroup.

Conversely, suppose $H \cup K$ is a subgroup of G.

We claim that $H \subseteq K$ or $K \subseteq H$

Suppose that H is not contained in K and K is not contained in H.

Then $\exists$ an elements $a, b \ni$:

$a \in H$ and $a \notin K \rightarrow (1)$

$b \in K$ and $b \notin H \rightarrow (2)$

Clearly $a, b \in H \cup K$. $\because H \cup K$ is a subgroup of G, $ab \in H \cup K$

Hence $ab \in H$ (or) $ab \in K$.

Case: 1 :- let $ab \in H$. $\because a \in H$ and $a^{-1} \in H \Rightarrow a^{-1}(ab) = b \in H$ which is a $\Rightarrow\!\Leftarrow$ to (2)

Case: 2 :- let $ab \in K$ $\because b \in K$, $b^{-1} \in K \Rightarrow b^{-1}(ab) = a \in K$ which is $\Rightarrow\!\Leftarrow$ to (1)

$\therefore$ Assumption is wrong. $\therefore H \subseteq K$ (or) $K \subseteq H$

1. Check whether $H_1 = \{0,5,10\}$ and $H_2 = \{0,4,8,12\}$ are subgroups of Z_{15} with respect to $+_{15}$.

Soln:

H_1

$+_{15}$	0	5	10
0	0	5	10
5	5	10	0
10	10	0	5

All the entries in H_1 are the elements of H_1.

$\therefore$ H_1 is a subgroup of Z_{15}

H_2

$+_{15}$	0	4	8	12
0	0	4	8	12
4	4	8	12	1
8	8	12	1	5
12	12	1	5	9

All the entries in H_2 are not equal to the elements of H_2.

$\therefore$ H_2 is not closed

$\therefore$ H_2 is not a subgroup of Z_{15}.

Homomorphism of Groups.

Let $(G,*)$ and (H,Δ) be any two groups.

A mapping $f: G \to H$ is said to be a homomorphism if $f(a*b) = f(a)\Delta f(b)$ $\forall\, a,b \in G$.

Theorem: 1:-

Homomorphism preserves identities.

Proof:

let $a \in G$

let f be a homomorphism from $(G,*)$ into $(G',*)$

Clearly $f(a) \in G'$ then $f(a)*e' = f(a)$

[$\because e'$ – identity]

$$= f(a*e)$$
$$= f(a)*f(e) \qquad [f-\text{homomorphism}]$$
$$\Rightarrow \quad e' = f(e) \qquad [\text{left Cancellation law}]$$

$\therefore$ f preserves identities.

Theorem: 2 :-

Homomorphism preserves inverses.

Proof:

let $a \in G$

$\because$ G is a group, $a^{-1} \in G$.

$$\Rightarrow \quad a*a^{-1} = a^{-1}*a = e$$
$$e' = f(e)$$
$$= f(a*a^{-1})$$
$$= f(a)*f(a^{-1})$$
$$\Rightarrow \quad f(a)*f(a^{-1}) = e'$$

$f(a^{-1})$ is the inverse of $f(a) \in G'$

$$\therefore [f(a)]^{-1} = f(a^{-1}).$$

Theorem: 3 CAYLEY'S THEOREM.

Every Finite group of order 'n' is isomorphic to Permutation group of degree 'n'.

Proof:

We prove this theorem in 3-steps.

Step-1: Find a set G' of Permutation.

Step-2: Prove G' is a group

Step-3: Exhibit an isomorphism $\phi: G \to G'$

Step-1:

Let G be a finite group of order 'n'.

let $a \in G$.

Define $f_a : G \to G$ by $f_a(x) = ax$

$\because f_a(x) = f_a(y) \Rightarrow ax = ay \Rightarrow x = y$.

f_a is 1-1

$\because$ if $y \in G$ then $f_a(a^{-1}y) = aa^{-1}y = y$

f_a is Onto

$\therefore$ f_a is a bijection [1-1 and Onto]

$\because$ G has 'n' elements, f_a is just Permutation on 'n' symbols. let $G' = \{f_a / a \in G\}$.

Step-2:

let G' be a group.

let $f_a f_b \in G'$

$f_a \circ f_b(x) = f_a[f_b(x)] = f_a(bx) = abx = f_{ab}(x)$

Hence $f_a \circ f_b = f_{ab}$

Hence G' is closed.

$f_e = G'$ is the identity element.

The inverse of f_a in G' is f_a^{-1}

$\therefore$ G' is a group.

Step-3:

To Prove G and G' are isomorphic.

Define $\phi : G \to G'$ by $\phi(a) = f_a$

$$\phi(a) = \phi(b) \Rightarrow f_a = f_b \Rightarrow f_a(x) = f_b(x) \Rightarrow ax = bx \Rightarrow a = b$$

Hence ϕ is 1-1.

$\because$ f_a is onto, ϕ is onto

Also $\phi(ab) = f_{ab} = f_a \circ f_b = \phi(a) \circ \phi(b)$

$\therefore$ $\phi : G \to G'$ is an isomorphism.

$\therefore$ $G \simeq G'$

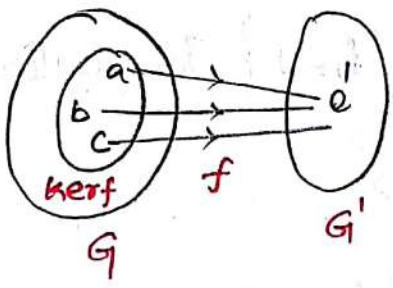
Hence Proved.

Kernel of a Homomorphism

Defn/:

let $f : G \to G'$ be a group homomorphism. The set of elements of G which are mapped into e' is called the kernel of f and it is denoted by $\ker(f)$.

$$\ker(f) = \{x \in G \mid f(x) = e'\}$$



then $\ker(f) = \{a, b, c\}$.

Isomorphism:

A mapping 'f' from a group $(G, *)$ to a group (G', Δ) is said to be an isomorphism if

i) f is a homomorphism.

$$f(a*b) = f(a) \Delta f(b) \quad \forall a, b \in G.$$

ii) f is one - one (injective)

iii) f is onto (surjective)

Left coset of H in G:

Let $(H, *)$ be a subgroup of $(G, *)$. For any $a \in G$, the set aH defined by

$aH = \{a*h / h \in H\}$ is called the left coset of H in G determined by the element $a \in G$.

Right coset of H in G:

Let $(H, *)$ be a subgroup of $(G, *)$. For any set Ha defined by

$Ha = \{h*a / h \in H\}$ is called the right coset of H in G.

Normal Subgroup:

A subgroup $(H, *)$ of $(G, *)$ is called a normal subgroup if for any $a \in G$; $aH = Ha$.

Theorem - 4: Lagrange's Theorem:

The order of a subgroup of a finite group divides the order of the group.

(Or)

If G is a finite group, then $O(H)/O(G)$ for all subgroups H of G.

Proof:

let $O(G) = n$

let $G = \{a_1 = e ; a_2 ; a_3 ; \ldots a_n\}$ and

let H be a subgroup of G, whose order is m.

ie) $O(H) = m$.

Consider the left cosets as follows:

$e * H = \{e * h / h \in H\}$

$a_2 * H = \{a_2 * h_2 / h \in H\}$

$a_n * H = \{a_n * h_n / h \in H\}$

We know that any two left cosets are either identical or disjoint.

Also $O[e * H] = O(H)$

$\therefore O[a_i * H] = O(H) \quad \forall a_i \in G$.

if $a * h_i = a * h_j$ for $i \neq j$; by cancellation laws

We have $h_i = h_j$; which is a contradiction.

Let there be K-disjoint cosets of H in K. Clearly their union equals G

i.e.) $G = (a_1 * H) \cup (a_2 * H) \cup \ldots \cup (a_K * H)$

$\therefore O(G) = O(H) + O(H) + \ldots\ldots + O(H)$

$O(G) = K \cdot O(H)$

$\Rightarrow$ $O(H)$ is a divisor of $O(G)$.

Theorem-5:

Let $(G, *)$ and (H, Δ) be groups and $g: G \to H$ be a Homomorphism. Then the kernel of g is normal subgroups.

Proof:

Let K be the kernel of homomorphism g.

i.e.) $K = \{x \in G / g(x) = e'\}$ where $e' \in H$ is the identity element of H.

To Prove that K is a Subgroup:

let $x, y \in K$ then $g(x) = e'$ and $g(y) = e'$

Claim: $x * y^{-1} \in K$

By definition of homomorphism

$g(x * y^{-1}) = g(x) \Delta g(y^{-1}) = g(x) \Delta [g(y)]^{-1}$

$= e' \Delta (e')^{-1}$

$= e' \Delta e'$

$= e'$

Hence $x * y^{-1} \in K$ and this proves K is a subgroup of G.

To Prove that K is normal:

let $x \in K$, $f \in G$ then $g(x) = e'$

Claim: $f * x * f^{-1} \in K$.

$$g[f * x * f^{-1}] = g(f) * g(x) * g(f^{-1})$$
$$= g(f) * e' * g(f^{-1})$$
$$= g(f)[g(f)]^{-1}$$
$$= e'$$

$\therefore$ ~~$f * x^{-1} *$~~ $f * x * f^{-1} \in K$

$\therefore$ K is a normal Subgroup of G.

Theorem: 6

Fundamental Theorem on homomorphism of Groups:

If f is a homomorphism of G onto G' with Kernel K then $G/K \cong G'$

Proof:

let $f : G \to G'$ be a homomorphism of the group $(G, *)$

to (G', Δ)

Then $K = \text{Ker}(f) = \{x \in G \mid f(x) = e'\}$ is a normal Subgroup of $(G, *)$. Also the Quotient set $(G/K, \circledast)$ is a group.

Define $\phi : G/K \to G'$ is a mapping from the group $(G/K, \circledast)$ to the group $(G', *)$ given by

$$\phi(K * a) = f(a) \quad \text{for any } a \in G.$$

i) ϕ is Well defined:

if $\quad Ka = Kb$

$$\Rightarrow a * b^{-1} \in K$$

$$\Rightarrow f(a * b^{-1}) = e'$$

$$\Rightarrow f(a) * f(b^{-1}) = e'$$

$$\Rightarrow f(a) * [f(b)]^{-1} = e'$$

$$\Rightarrow f(a) * [f(b)]^{-1} * f(b) = e' * f(b)$$

$$f(a) * e' = e' * f(b)$$

$$f(a) = f(b)$$

$$\Rightarrow \phi(Ka) = \phi(Kb)$$

$\therefore \phi$ is well defined.

ii) ϕ is 1-1 :

To prove $\phi(K+a) = \phi(K+b) \Rightarrow K+a = K+b$

We know that $\phi(K * a) = \phi(K * b)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow f(a) * f(b^{-1}) = f(b) * f(b') = e'$

$\Rightarrow f(a) * f(b^{-1}) = e'$

$\Rightarrow f(a * b^{-1}) = e'$

$\Rightarrow a * b^{-1} \in K$

$\Rightarrow K * a = K * b$

$\therefore \phi$ is 1-1

iii) ϕ is onto:

Claim: ϕ is onto ; let $y \in G'$

$\because f$ is onto $\exists\, a \in G \ni: f(a) = y$

$\Rightarrow \phi(K * a) = f(a) = y.$

$\therefore \phi$ is onto.

iv) ϕ is a homomorphism:

$\phi[K * a * K * b] = \phi[K * a * b] = f(a * b) = f(a) * f(b)$

$= \phi(K * a) * \phi(K * b)$

$\therefore \phi$ is a homomorphism.

$\therefore \phi$ is 1-1, onto and homomorphism

ϕ is an isomorphism between $G/K \cong G'$

$\therefore G/K \cong G'.$

Problem:

If $(Z,+)$ and $(E,+)$ where Z is the set of all integers and E is the set of all even integers, show that the two semigroups $(Z,+)$ and $(E,+)$ are isomorphic.

Soln:

let $f:(Z,+)\rightarrow(E,+)$ defined by $f(x)=2x$

Claim: f is 1-1

Assume $f(x)=f(y) \Rightarrow 2x=2y \Rightarrow x=y$.

$\Rightarrow f(x)=f(y)$

$\Rightarrow x=y$.

$\therefore f$ is 1-1.

ii) f is onto

To prove $\forall y\in E$ there exist $(\exists!)$ $x\in X$, $\ni: f(x)=y$.

$f(x)=y \Rightarrow 2x=y$

$\Rightarrow x=y/2$

$\therefore \forall y\in E$, the corresponding preimage is $\frac{y}{2}\in Z$

$\therefore f$ is onto

$\because f$ is 1-1 and onto

f is bijective.

iii) f is homomorphism.

$$f(x+y) = 2(x+y)$$
$$= 2x+2y$$
$$= f(x)+f(y)$$

$\therefore f(x+y) = f(x)+f(y)$.

$\because f:(Z,+) \to (E,+)$ is bijective and homomorphism.

$\therefore f$ is homomorphism

$\therefore (Z,+)$ and $(E,+)$ are isomorphic to each other.

$$\therefore (Z,+) \simeq (E,+)$$

Cyclic Groups:

Let G be a group. let $a \in G$. Then $H = \{a^n / n \in Z\}$ is a subgroup of G. H is called the cyclic subgroup of G generated by a and it is denoted by $\langle a \rangle$

Theorem: 1

Every cyclic group is an abelian group.

Proof:

Let $(G,*)$ be a cyclic group with generator $a \in G$.

$\therefore$ For $x, y \in Z$

$x = a^k$; $y = a^t$ for integers; k, t.

$\therefore x*y = a^k * a^t = a^{k+t} = a^{t+k} = a^t * a^k = y*x$.

$\therefore x*y = y*x \Rightarrow (G,*)$ is an abelian group.

Theorem: 2

Every Subgroup of a cyclic group is cyclic.

(or)

If $(G,*)$ is a cyclic group, then every subgroup of $(G,*)$ is also a cyclic group.

Proof:

Let $G(a)$ is a cyclic group generated by 'a' and H be its subgroup.

If $H = G$ (or) $H = \{e\}$, then H is cyclic.

Let H be a proper subgroup of G.

$\therefore$ the elements of H are integral powers of a.

If $a^s \in H$ then its inverse $a^{-s} \in H$

Let m be the least Positive integer such that $a^m \in H$.
Then we prove that $H = a^m$ is a cyclic group generated by a^m.

$\because$ a^t be any arbitrary element of H. By division algorithm there exists q and r.

$\Rightarrow$: $t = mq + r$

$\therefore$ $a^t = a^{mq+r} = a^{mq} * a^r \Rightarrow a^r = \cancel{a^t} a^t * a^{-mq} = a^{(t-mq)}$

$\because$ $a^m \in H \Rightarrow a^{mq} \in H \Rightarrow a^{-mq} \in H$.

$\therefore$ $a^t, a^{-mq} \in H \Rightarrow a^{(t-mq)} \in H$

$\Rightarrow a^r \in H$.

Unit - 5

Lattices and Boolean Algebra

partial Order Relation:

Let 'X' be any set, 'R' be a relation defined on 'X'. The 'R' is said to be partial order Relation. If it satisfies reflective, antisymmetric, Transistive relations.

i) $xRx \Rightarrow x$

ii) xRy & $yRx \Rightarrow x = y$.

iii) xRy & $yRz \Rightarrow x = z$.

partial Ordered Set (poset):

A set together with a partial order relation define on it is called partially ordered set or poset. It is denoted by $\leq$.

Eg:

1. Let R be the set of real numbers. The Relation $\leq$ is partial order of R. R is poset $(R, \leq)$

2. Let P(A) be the powerset of A. The relation $\subseteq$ (or Inclusio on P(A) is a partial order

$\therefore (P(A), \subseteq)$ is a poset.

Hasse diagram:

pictorial representation of a poset is called Hasse diagram.

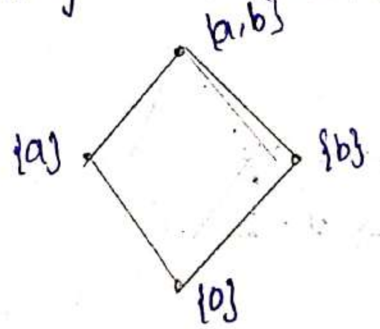
1. Draw the Hasse diagram for $(P(A), \subseteq)$

i) $A = \{a,b\}$ ii) $A = \{a,b,c\}$.

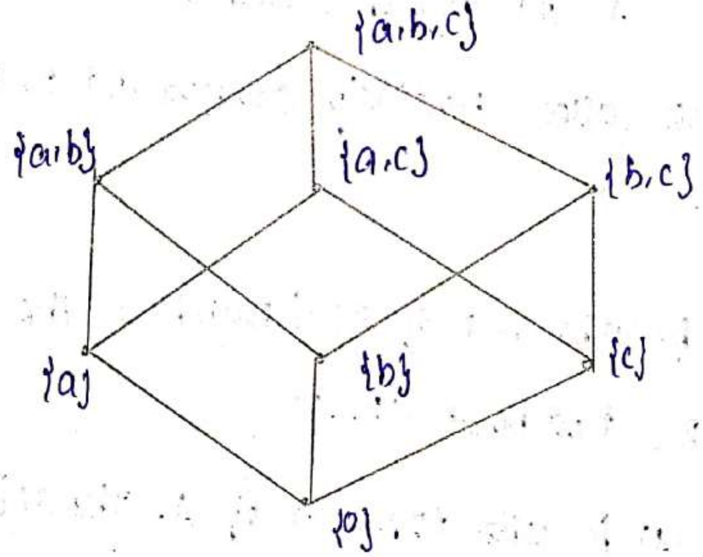
Solution:

i) $P(A) = \{\{a,b\}, \{a\}, \{b\}, \{0\}\}$

The diagram can be represented as $(P(A), \subseteq)$ where $A = \{a,b\}$.



ii) $P(A) = \{\{a,b,c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}, \{0\}\}$.

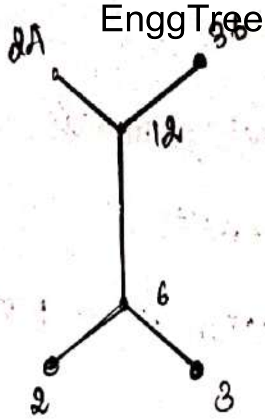


2. If $X = \{2,3,6,12,24,36\}$ and the relation R defined on X by R.

$R = \{\langle a,b\rangle / a|b\}$.

Solution:

$R = \{\langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle, \langle 3,36\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle, \langle 12,24\rangle, \langle 12,36\rangle\}$.



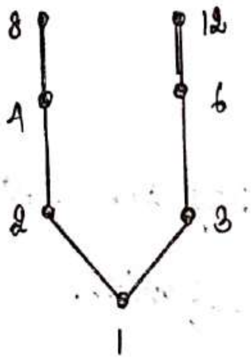
8. Draw the Hasse diagram for $\{(a,b) / a \text{ divides } b\}$.

i) $\{1, 2, 3, 4, 6, 8, 12\}$

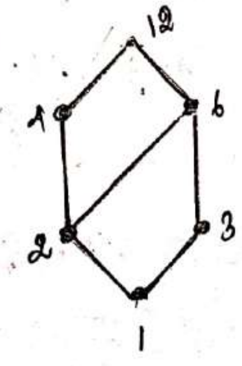
ii) $\{1, 2, 3, 4, 6, 12\}$.

__Solution:__

i) $R = \{(1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (2,4), (2,6), (2,8), (2,12), (3,6), (3,12), (4,8), (4,12), (6,12)\}$.



ii) $R = \{(1,2), (1,3), (1,4), (1,6), (1,12), (2,4), (2,6), (2,12), (3,6), (3,12), (4,12), (6,12)\}$.



Note:

i) $a/a \rightarrow$ Reflexive.

ii) a/b & $b/a \rightarrow a=b$ Antisymmetric

iii) a/b & $b/c \rightarrow a=c$ Transmittive

The divide relation is a partial order relation.

Theorem-1:

Show that $(N, \leq)$ is a partially order set, where N is the set of all +ve integers and $\leq$ defined by $m \leq n$ if and only if, n and m is a non-negative integer.

Solution:

Given that N is the set of all positive integers.

The relation $m \leq n$ if and only if $n-m$ is a non-negative integers

Now, $\forall x \in N$.

$x-x=0$ is a non-negative integers.

xRx, $\forall x \in N$. R is reflexive.

Consider,

$x \in y$ and $y \in x$

Since $xRy \Rightarrow x-y$ is a non-negative integer ... ①.

$yRx \Rightarrow y-x = -(x-y)$ which is also a non-negative integer ... ②.

From eqn ① and ②, we get

$x=y$.

$\therefore$ R is antisymmetric

Assume,

xRy and yRz

$xRy \Rightarrow x-y$ is a non-negative Integer ... ③

$yRz \Rightarrow y-z$ is also a non-negative Integer ... ④

Adding eqn ③ and ④.

$\Rightarrow x-y+y-z$ is a non-negative Integer

$\Rightarrow x-z$ is a non-negative Integer

$\Rightarrow xRz$.

xRy & $yRz \Rightarrow xRz$.

$\therefore$ R is transitive.

$\therefore (N, \leq)$ is a partial order Relation

Least upper Bound (LUB) / Supremum:

Let $(P, \leq)$ be a poset and $A \subseteq P$, an element $a \in P$ is said to be LUB if 'a' is a,

1) upper Bound of a.

2) $a \leq c$, where c is anyother upper bound of a.

Greatest Lower Bound (GLB) / Infimum:

Let $(P, \leq)$ be a poset and $A \subseteq P$, an element $b \in P$ is said to be GLB of 'a'. is

1) If a is lower Bound. A.

2) $b \leq d$ where d is anyother greatest lower bound of A.

Eg: Consider,

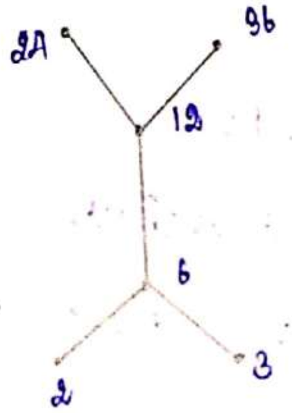
$X = \{2, 3, 6, 12, 24, 36\}$

$R = \{\langle a,b \rangle / a \mid b\}$.

1. Find LUB and GLB of (2, 3) and (24, 36)

Solution:

$R = \{\langle 2,6\rangle, \langle 2,12\rangle, \langle 2,24\rangle, \langle 2,36\rangle, \langle 3,6\rangle, \langle 3,12\rangle, \langle 3,24\rangle,$
$\langle 3,36\rangle, \langle 6,12\rangle, \langle 6,24\rangle, \langle 6,36\rangle, \langle 12,24\rangle, \langle 12,36\rangle\}$



i) LUB:

UB $\{2,3\} \Rightarrow \{6, 12, 24, 36\}$.

LUB $\{2,3\} \Rightarrow \{6\}$.

UB $\{24,36\} \Rightarrow$ does not exist.

LUB $\{24,36\}$ = does not exist.

ii) GLB:

LB $\{2,3\} \Rightarrow$ does not exist.

GLB $\{2,3\} \Rightarrow$ does not exist.

LB $\{24,36\} \Rightarrow \{12, 6, 3, 2\}$.

GLB $\{24, 36\} \Rightarrow \{12\}$.

2. $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and let the relation be a partial ordering D_{24}.

i) Draw the Hasse diagram for D_{24} division.

ii) Find all LB of 8 and 12.

iii) Find all GLB of 8 and 12.

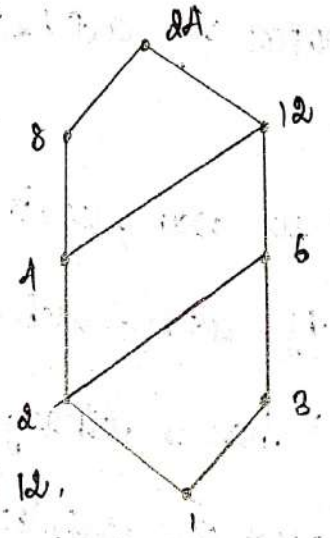
iv) Find all UB of 8 and 12.

v) Find LUB of 8 and 12.

vi) State the greatest and least element of the poset if it exists.

Solution:

i) Hasse diagram:



ii) The LB of 8 and 12.

LB $\{8, 12\} = \{4, 2, 1\}$.

iii) The GLB of 8 and 12.

GLB $\{8, 12\} = \{4\}$.

iv) The UB of 8 and 12.

UB $\{8, 12\} = \{24\}$.

v) The LUB of 8 and 12.

LUB $\{8, 12\} = \{24\}$.

vi) Greatest element of poset $\Rightarrow 24$.
Lowest element of poset $\Rightarrow 1$.

Lattice:

A lattice is a partially ordered set $(L, \leq)$ in which for every pair of elements $a, b \in L$. Both the greatest and lowest bound (GLB) and (LUB).

Note:

1. GLB $\{a, b\}$ is denoted by $a * b$. which is pronunced by 'a' meet 'b' (or) 'a' product 'b'.

Instead of $*$ we can use meet and dot ($\wedge$ or $\cdot$).

$\therefore$ GLB $\{a, b\} = a * b$ (or) $a \wedge b$ (or) $a \cdot b$.

2. LUB $\{a, b\}$ is denoted by $a \oplus b$ which is pronunced by 'a' joint 'b' (or) 'a' sum 'b'.

Instead of $\oplus$ we can use ($\vee$ and $+$)

$\therefore$ LUB $\{a, b\} = a \oplus b = a \vee b = a + b$.

3. Since lattice $(L, \leq)$ has a binary operation $*$ ($\wedge$) and $+$ ($\vee$). a lattice can be denoted by triplet

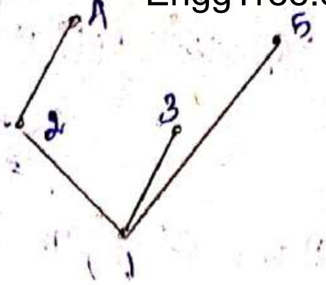
$(L, *, \oplus)$, $(L, \wedge, \vee)$, $(L, \cdot, +)$.

1. Determine whether the poset.

i) $\{\{1, 2, 3, 4, 5\}, /\}$ ii) $\{\{1, 2, 4, 8, 16\}, /\}$ are lattices.

Solution:

i) $R = \{(1,2), (1,3), (1,4), (1,5), (2,4)\}$.



LUB {2,3} does not exits.

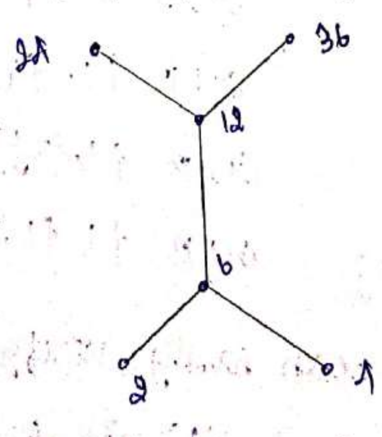
∴ The poset is not a lattice because it has no GLB and LUB.

ii) R = {(1,2), (1,4), (1,8), (1,16), (2,4), (2,8), (2,16), (4,8), (4,16), (8,16)}.

LUB {2,4} = 1.

Hence every pair of elements both GLB and LUB exits. ∴ the poset is lattice.

2. Determine the poset given by the Hasse diagram are lattice or not.



Solution:

Since LUB {21, 36} does not exists and GLB {2,3} does not exists.

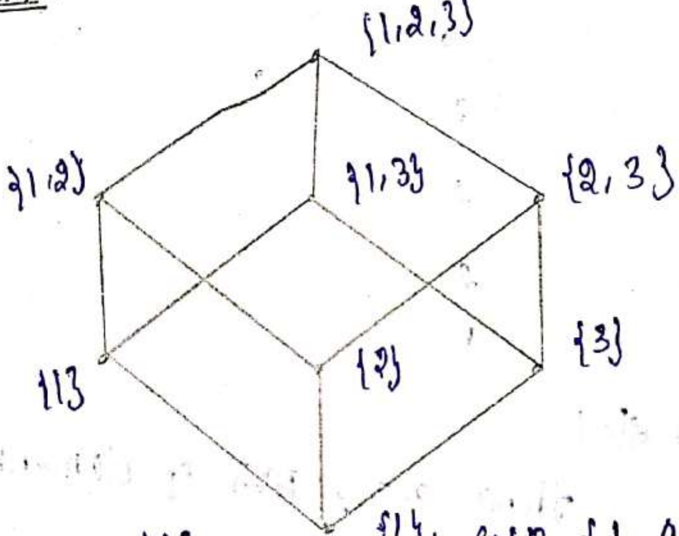
∴ The given Hasse diagram does not exists.

3. Determine whether (P(A), ⊆) is lattice A = {1,2,3}.

Solution:

P(A) = {{1,2,3}, {1,2}, {1,3}, {2,3}, {1}, {2}, {3}, {0}}.

Hasse diagram:



LUB {1, 0} = {1}
LUB {1, {1,2}} = {1,2}
LUB {1, {1,3}} = {1,3}
LUB {1, {2,3}} = {1,2,3}
LUB {1, {2}} = {1,2}
LUB {1, {3}} = {1,3}
LUB {1, {1,2,3}} = {1,2,3}

GLB {1, 0} = {0}
GLB {1, {1,2}} = {1}
GLB {1, {1,3}} = {1}
GLB {1, {2,3}} = {0}
GLB {1, {2}} = {0}
GLB {1, {3}} = {0}
GLB {1, {1,2,3}} = {1}.

Similarly, we can easily verify both GLB and LUB exists for each pair of P(A). It is noticed that, for any two subsets a and b of $P(A)$.

$LUB\{A,B\} = A \vee B$. and

$GLB\{A,B\} = A \wedge B$.

which is exists.

$\therefore (P(A), \subseteq)$ is a lattice.

Properties of lattices:

Let $(L, \wedge, \vee)$ be a given lattice $\wedge, \vee$ satisfies the condition. $\forall a, b, c \in L$.

1. Idempotent law:

$a \vee a = a$.

$a \wedge a = a$.

2. Commutative law:

$a \vee b = b \vee a$.

$a \wedge b = b \wedge a$.

3. Associative law:

$(a \vee b) \vee c = a \vee (b \vee c)$

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

4. Absorption law:

$a \vee (a \wedge b) = a$

$a \wedge (a \vee b) = a$

Property 1:

Idempotent law:

Let $(L, \wedge, \vee)$ be a given lattice. then $a, b, c \in L$,

$a \vee a = a$ and $a \wedge a = a$.

Proof:

$a \vee a \Rightarrow LUB\{a, a\} = LUB\{a\} \Rightarrow a.$

$a \wedge a \Rightarrow GLB\{a, a\} = GLB\{a\} \Rightarrow a.$

Commutative law:

Let $(L, \wedge, \vee)$ be a given lattice and $a, b, c \in L$, then prove $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

proof:

$a \vee b \Rightarrow LUB\{a, b\} \Rightarrow LUB\{b, a\} \Rightarrow b \vee a.$

similarly,

$a \wedge b \Rightarrow GLB\{a, b\} \Rightarrow GLB\{b, a\} \Rightarrow b \wedge a.$

Absorption law:

Let $(L, \wedge, \vee)$ be a given lattice and $\forall a, b, c \in L$ then prove that, $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

proof:

Since $a \wedge b = GLB\{a, b\}$

$\Rightarrow a \wedge b \leq a \cdots (1)$

obviously, $a \leq a \cdots (2)$

By the eqn (1) and (2)

$a \vee (a \wedge b) \leq a \cdots (3)$

By the definition of LUB, we have

$a \leq a \vee (a \wedge b) \cdots (4)$

From ③ and ④

$a = a \vee (a \wedge b)$

$\therefore a \vee (a \wedge b) = a$

Similarly, $a \wedge (a \vee b) = a$.

Theorem-2:

Let $(L, \wedge, \vee)$ be a lattice, in which $\wedge$ and $\vee$ denotes the operation of $\wedge$ and $\vee$ respectively. For any $a, b \in L$, $a \leq b$ iff and only if $a \vee b = b$, if and only if $a \wedge b = a$ (or) $a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$.

Theorem-3:

State and prove distributive inequality of lattice.

Statement:

Let $(L, \wedge, \vee)$ be a given lattice. For any $a, b, c \in L$. The following inequality holds

i) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

proof:

1) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

From the defenition of LUB, it is obvious that,

$a \leq a \vee b \ldots ①$.

and $b \wedge c \leq b \leq a \vee b$

$\Rightarrow b \wedge c \leq a \vee b \ldots ②$

From ① and ②,

$a \vee b$ is a upper bound $\{a, b \wedge c\}$.

Hence, $a \vee b \geq a \vee (b \wedge c) \cdots$ Ⓐ

From the defenition it is obvious that,

$a \leq a \vee c \cdots$ ③

and $b \wedge c \leq c \leq a \vee c$

$\Rightarrow b \wedge c \leq a \vee c \cdots$ ④

From ③ and ④

$a \vee c$ is a upper bound $\{a, b \wedge c\}$.

Hence, $a \vee c \geq a \vee (b \wedge c) \cdots$ Ⓑ

From Ⓐ and Ⓑ

$a \vee (b \wedge c)$ is a lower bound of $(a \vee b)$, $(a \vee c)$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Hence proved.

2) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

From the defenition of GLB, it is obvious that,

$a \geq a \wedge b \cdots$ ①

and $b \vee c \geq b \geq a \wedge b$

$b \vee c \geq a \wedge b \cdots$ ②

From ① and ②,

$a \wedge b$ is a lower bound $\{a, b \vee c\}$.

$a \wedge b \leq a \wedge (b \vee c) \ldots$ Ⓐ

From the defenition, it is obvious that,

$a \geq a \wedge c \ldots$ ③

and $b \vee c \geq c \geq a \wedge c$.

$\Rightarrow b \vee c \geq a \wedge c \ldots$ ④

From ③ and ④

$a \wedge c$ is a lower bound $\{a, b \vee c\}$.

Hence

$a \wedge c \leq a \wedge (b \vee c) \ldots$ Ⓑ.

From Ⓐ and Ⓑ

$a \wedge (b \vee c)$ is a upper bound of $\{a \wedge b, a \wedge c\}$.

$\therefore a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

Hence proved.

Distributive lattice:

A lattice $(L, \wedge, \vee)$ is said to distributive if $\wedge$ and $\vee$ satisifies the following conditions.

$\forall$, $a, b, c \in L$

$D_1 \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$D_2 \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Theorem: 4

prove that any chain is a distributive lattice.

proof: Let $(L, \wedge, \vee)$ be a given chain and $\forall a, b, \in L$.

Since, any two elements of chain are compared in either $a \leq b$ or $b \leq a$.

case i): $a \leq b$

$LUB\{a, b\} = b$

$GLB\{a, b\} = a.$

case ii) $b \leq a.$

$LUB\{b, a\} = a$

$GLB\{a, b\} = b$

In both cases, any two elements of a chain has both GLB and LUB.

$\therefore$ any chain is a lattice.

Next we prove,

$(L, \wedge, \vee)$ satisfies distributive property.

Let $a, b, c, \in L.$

Since, any chain satisfies is a comparable property, we have the following six cases.

case i): $a \leq b \leq c.$

case ii): $a \leq c \leq b.$

Case iii): $b \leq a \leq c$

case iv): $b \leq c \leq a$

case v): $c \leq b \leq a$

case vi): $c \leq a \leq b$

Case i): $a \leq b \leq c$

Prove:

$D_1 \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

LHS:

$a \vee (b \wedge c)$.

$\Rightarrow a \vee (b \wedge c)$

$\Rightarrow a \vee b$ $\quad [\because b \leq c, \; b \wedge c = b]$

$\Rightarrow b$ $\quad [\because a \leq b, \; a \vee b = b]$

RHS: $(a \vee b) \wedge (a \vee c)$

$\Rightarrow b \wedge c$ $\quad [\because a \leq b, \; a \leq c]$.

$\Rightarrow b$

$\therefore$ LHS = RHS.

$\therefore D_1$ condition is true for case 1.

Similarly,

we can easily prove the D_1 property for the remaining five cases.

$\therefore (L, \wedge, \vee)$ is a distributive lattice.

$\therefore$ Any chain is a distributive lattice.

Theorem-5 [Modular inequality]:

If $(L, \wedge, \vee)$ is a lattice, then any a, b, c

$$a \leq c \Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

proof:

assume, $a \leq c$

By the definition of GLB & LUB we get

$$\Rightarrow a \wedge c = a \quad \cdots (1)$$

$$a \vee c = c \quad \cdots (2)$$

By distribution inequality we have,

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \quad \cdots (2)$$

using (1)

$$a \vee (b \wedge c) \leq (a \vee b) \wedge c \quad \cdots (A)$$

Conversely,

Assume

$$a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Now by the definition of LUB and GLB, we have

$$a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$$

$$\Rightarrow a \leq c \quad \cdots (B)$$

From (A) and (B)

$$a \leq c \Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Hence Proved.

Modular lattice:

A lattice $(L, \wedge, \vee)$ is said to be modular lattice, if it satisfies the following condition,

If $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$..

Theorem-6:

Every distributive lattice is modular but not conversely.

proof:

Let $(L, \wedge, \vee)$ be the given distributive lattice

$D_1 \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ holds good, $\forall\, a, b, c \in L$..

Now if, $a \leq c$ then $a \vee c \Rightarrow c$ ... ①

① $\Rightarrow$ $a \vee (b \wedge c) \leq (a \vee b) \wedge c$ ... ②

There fore if $a \leq c \Leftrightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$.

$\therefore$ every distributive lattice is Modular, but converse is not true.

Modular, but

That is every modular lattice need not be distributive.

1. If any distributive lattice $(L, \wedge, \vee)$, $\forall$ $a, b, c \in L$ prove that $a \vee b = a \vee c$, $a \wedge b = a \wedge c \Rightarrow b = c$

<u>Solution:</u>

$b = b \vee (b \wedge a)$ (absorption law)

$b \Rightarrow b \vee (a \wedge b)$ (commutative law)

$b \Rightarrow b \vee (a \wedge c)$ (since by given cond)

$\Rightarrow (b \vee a) \wedge (b \vee c)$ (Di law)

$b \Rightarrow (a \vee b) \wedge (b \vee c)$ (commutative law)

$b \Rightarrow (a \vee c) \wedge (b \vee c)$ (given cond)

$\Rightarrow (c \vee a) \wedge (c \vee b)$ (commutative law)

$\Rightarrow c \vee (a \wedge b)$ (Di $\rightarrow$ law)

$\Rightarrow c \vee (a \wedge c)$ (given cond)

$b \Rightarrow c \vee (c \wedge a)$ (commutative law)

$b \Rightarrow c$ (absorption law)

<u>Theorem-4:</u>

State and prove Isotonicity property.

<u>Solution:</u>

Let $(L, \wedge, \vee)$ be a given lattice.

For any $a, b, c \in L$.

We have, $b \leq c$ i) $a \wedge b \leq a \wedge c$

ii) $a \vee b \leq a \vee c$.

Given,

$b \leq c$

$\therefore$ GLB of given $\{b, c\} \Rightarrow b \wedge c \Rightarrow b$ ... ①

<u>claim 1):</u> $a \wedge b \leq a \wedge c$

It is enough to prove

GLB $\{a \wedge b, a \wedge c\} \Rightarrow (a \wedge b) \wedge (a \wedge c)$

GLB $\{a \wedge b, a \wedge c\} \Rightarrow a \wedge b$

<u>LHS:</u>

$(a \wedge b) \wedge (a \wedge c)$

$\Rightarrow a \wedge (b \wedge a) \wedge c$ (Associative law)

$\Rightarrow a \wedge (a \wedge b) \wedge c$ (Commutative law)

$\Rightarrow (a \wedge a) \wedge (b \wedge c)$ (Associative law)

$\Rightarrow a \wedge (b \wedge c)$ (Idempotent law)

$\Rightarrow a \wedge b$

$\Rightarrow$ RHS

claim 1 is proved.

<u>claim 2:</u> $a \vee b \leq a \vee c$.

It is enough to prove

GLB $\{a \vee b, a \vee c\} \Rightarrow (a \vee b) \vee (a \vee c) \Rightarrow a \vee c$

<u>LHS:</u>

$(a \vee b) \vee (a \vee c)$

$\Rightarrow a \vee (b \vee a) \vee c$ (Associative law)

$\Rightarrow a \vee (a \vee b) \vee c$ (Commutative law)

$\Rightarrow (a \vee a) \vee (b \vee c)$ (associative law)

$\Rightarrow a \vee (b \vee c)$ (Idempotent law)

$\Rightarrow a \vee c$

$\Rightarrow$ RHS

claim 2 is proved.

Lattice as an algebraic system:

A lattice is an algebraic system $(L, \wedge, \vee)$ with two binary operations $\wedge$ and $\vee$ on L which are both commutative, associative and satisfies absorption laws.

Sublattices:

Let $(L, \wedge, \vee)$ be a lattice and $S \subseteq L$ be a subset of L then $(S, \wedge, \vee)$ is a sublattices of $(L, \wedge, \vee)$ if and only if S is closure under both operations $\wedge$ and $\vee$. $\forall\ a, b \in S$ implies $a \wedge b \in S$ and $a \vee b \in S$.

Lattice Homomorphism:

Let $(L_1, \wedge, \vee)$ and $(L_2, *, \oplus)$ be two given lattice. A mapping $f: L_1 \rightarrow L_2$ is called lattice homomorphism if $\forall$, $a, b \in L$.

i) $f(a \wedge b) \Rightarrow f(a) * f(b)$

ii) $f(a \vee b) \Rightarrow f(a) \oplus f(b)$.

Ordered preserving:

A mapping from $L_1 \to L_2$ is said to be ordered preserving map from lattice $(L_1, \leq)$ to $(L_2, \leq)$ if $a \leq b$, then $f(a) \leq f(b)$.

Theorem-8:

prove that any lattice homomorphism is order preserving.

proof:

Let $f: L_1 \to L_2$ be a lattice homomorphism.

$a \leq b$, then the GLB of a, b is,

$GLB\{a,b\} \Rightarrow a \wedge b = a \ldots (1)$

Then $LUB\{a,b\} \Rightarrow (a \vee b) = b \ldots (2)$

Now, $f(a \wedge b) \Rightarrow f(a)$ using (1)

$f(a) \wedge f(b) \Rightarrow f(a)$ [Since f is homomorphism].

$\Rightarrow GLB\{f(a), f(b)\} = f(a)$

$\Rightarrow f(a) \leq f(b)$.

$\therefore f$ is ordered preserving.

Note:

1. Least element is denoted by symbol '0' and it satisfies the condition, $0 \wedge x = 0$ and $0 \vee x = x$.

2. The greatest element is denoted by '1' and it satisfies the condition

$$1 \wedge x = x \text{ and } 1 \vee x = 1.$$

Complement:

Let $(L, \wedge, \vee, 0, 1)$ be given bounded lattice. Let 'a' be any element of L, we say that 'b' is complement of 'a'. If $a \wedge b = 0$ and $a \vee b = 1$ and 'b' is denoted by a symbol a' is

i.e, $a \wedge a' = 0$ and $a \vee a' \Rightarrow 1$.

Complemented Lattice:

A bounded lattice $(L, \wedge, \vee, 0, 1)$ is said to be complemented lattice, if every element of L has atleast one complement.

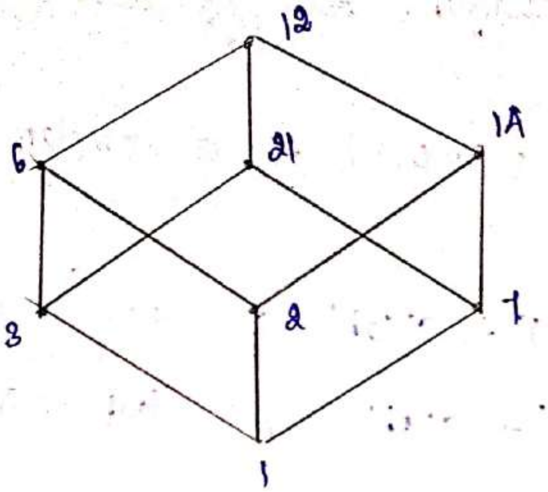
1. If S_{42} is the set of all divisions of 42 and D is relation divisor of on S_{42}. prove that $\{S_{42}, D\}$ is a complemented lattice.

Solution:

$S_{42} = \{$ All divisor of 42 $\}$

$S_{42} = \{1, 2, 3, 7, 14, 21, 42\}$

The Hasse diagram of (S_{42}, D) is



$0 =$ least element $\Rightarrow 1$.

$1 =$ greatest element $\Rightarrow 42$.

$LUB \{1, 42\} = LCM \{1, 42\} = 42$

$GLB \{1, 42\} = GCD \{1, 42\} = 1$

$\therefore$ Complement of 1 is 42.

$\Rightarrow (1)' = 42$.

$LUB \{2, 21\} = LCM \{2, 21\} = 42$

$GLB \{2, 21\} = GCD \{2, 21\} = 1$.

$\therefore$ complement of 2 is 21.

$(2)' \Rightarrow 21$.

$(3)' = 14$.

$(6)' = 7$.

$(7)' = 6$.

$(14)' = 3$.

$(21)' = 2$.

$(42)' = 1$.

Since every element of S_{42} has Complement.

$\therefore (S_{42}, D)$ is complemented lattice

Theorem-9: De Morgan's Law of lattice

Statement:

If $(L, \wedge, \vee, 0, 1)$ is a complemented lattice, then

prove that

i) $(a \wedge b)' \Rightarrow a' \vee b'$ (or) $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

ii) $(a \vee b)' \Rightarrow a' \wedge b'$ (or) $\overline{a \vee b} = \bar{a} \wedge \bar{b}$.

Proof:

Claim i):

$(a \wedge b)' \Rightarrow a' \vee b'$.

It is enough to prove

1) $(a \wedge b) \wedge (a' \vee b') = 0$

2) $(a \wedge b) \vee (a' \vee b') = 1$.

1) $(a \wedge b) \wedge (a' \vee b')$

$\Rightarrow [(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge b']$ → Distributive law

$\Rightarrow [(b \wedge a) \wedge a'] \vee [(a \wedge b) \wedge b']$ → Commutative law

$\Rightarrow [b \wedge (a \wedge a')] \vee [a \wedge (b \wedge b')]$ → Associative law

$\Rightarrow [b \wedge 0] \vee [a \wedge 0]$

$\Rightarrow 0 \vee 0$

$\Rightarrow 0$.

ii) $(a \wedge b) \vee (a' \vee b')$

$\Rightarrow [(a' \vee b') \vee a] \wedge [(a' \vee b') \vee b]$ (Distributive law)

$\Rightarrow [a \vee (a' \vee b')] \wedge [b \vee [b' \vee a']]$ (Commutative law)

$\Rightarrow [(a \vee a') \vee b'] \wedge [(b \vee b') \vee a']$ (associative law)

$\Rightarrow [1 \vee b'] \wedge [1 \vee a']$

$\Rightarrow 1 \wedge 1$

$\Rightarrow 1$

$\therefore$ claim i is proved.

Claim ii):

$(a \vee b)' = a' \wedge b'$.

It is enough to prove

1. $(a \vee b) \wedge (a' \vee b') = 0$
2. $(a \vee b) \vee (a' \vee b') = 1$

i) $(a \vee b) \wedge (a' \wedge b')$:

$\Rightarrow [a \wedge (a' \wedge b')] \vee [b \wedge (a' \wedge b')]$ (Distributive law)

$\Rightarrow [a \wedge (a' \wedge b')] \vee b \wedge [b' \wedge a']$ (commutative law)

$\Rightarrow [(a \wedge a') \wedge b'] \vee [(b \wedge b') \wedge a']$ (associative law).

$\Rightarrow [0 \wedge b'] \vee [0 \wedge a']$

$\Rightarrow 0 \vee 0$

$\Rightarrow 0$.

ii) $(a \vee b) \vee (a' \wedge b')$.

$\Rightarrow [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b']$ (Distributive law)

$\Rightarrow [(b \vee a) \vee a'] \wedge [(a \vee b) \vee b']$ (Commutative law)

$\Rightarrow [b \vee (a \vee a')] \wedge [(a \wedge (b) \vee b')]$ (associative law)

$\Rightarrow [b \vee 1] \wedge [a \wedge 1]$

$\Rightarrow 1 \wedge 1$

$\Rightarrow 1$.

$\therefore$ claim 2 is proved.

De-morgan's Law is proved.

Theorem-10:

prove that in a complemented distributive lattice, complement is unique (or) $(L, \wedge, \vee, 0, 1)$ is a distributive lattice then each element $x \in L$, has atmost one complement

Solution:

Let us assume x and y are two complement.

To prove,

$$x = y$$

Since, x is a complement of 'a'.

$$\left.\begin{array}{l} a \wedge x = 0 \\ a \vee x = 1 \end{array}\right\} \ldots (1)$$

Since, y is a complement of 'a'.

$$\left.\begin{array}{l} a \wedge y = 0 \\ a \vee y = 1 \end{array}\right\} \ldots (2)$$

Now:

$x = x \vee 0$

$\Rightarrow x \vee (a \wedge y)$ Since by (2)

$x \Rightarrow (x \vee a) \wedge (x \vee y)$ (distributive law)

$x \Rightarrow (a \vee x) \wedge (x \vee y)$ (commutative law)

$\Rightarrow 1 \wedge (x \vee y)$

$x \Rightarrow x \vee y \ldots$ (A)

Similarly,

$y = y \vee 0$

$y = y \vee (a \wedge x)$ [by eqn 1]

$y \Rightarrow (y \vee a) \wedge (y \vee x)$

$\Rightarrow (a \vee y) \wedge (y \vee x)$

$y \Rightarrow 1 \wedge (a \vee y)$

$y \Rightarrow a \vee y$ ---- (B)

From equ (A) and (B)

$x = y$

$\therefore$ The complement is unique, in a Complemented distributive lattice.

Theorem-11:

In a complemented distributive lattice, show that following are equivalent.

$a \leq b \Rightarrow a \wedge b' = 0 \Rightarrow a' \vee b \Rightarrow 1 \Rightarrow b' \leq a'$.

(or)

The following are equivalence

i) $a \leq b$ ii) $a \wedge b' = 0$ iii) $a' \vee b = 1$ iv) $b' \leq a'$.

Solution:

Since Given lattice is complemented distributive lattice

$a \wedge a' = 0$

$a \vee a' = 1$.

proof (1) $\Rightarrow$ proof (2):

assume, $a \leq b \Rightarrow a \wedge a = a,$

$a \vee b = b,$

$a \wedge b' = (a \wedge b) \wedge b'$

$\Rightarrow a \wedge (b \wedge b')$

$\Rightarrow a \wedge 0$

$a \wedge b' \Rightarrow 0$.

proof ② ⇒ ③

Let $a \wedge b' \Rightarrow 0$

Taking complement on both sides

$(a \wedge b')' \Rightarrow 0'$

$a' \vee (b')' \Rightarrow 1$

$a' \vee b \Rightarrow 1$

proof ③ ⇒ proof ④

Let $a' \vee b \Rightarrow 1$.

Taking $\wedge b'$ on both sides

$(a' \vee b) \wedge b' = 1 \wedge b'$

$(a' \wedge b') \vee (b \wedge b') \Rightarrow 1 \wedge b'$

$(a' \wedge b') \vee 0 \Rightarrow 1 \wedge b'$

$a' \wedge b' \Rightarrow b'$

$a' \geq b'$

$\Rightarrow b' \leq a'$.

proof ④ ⇒ proof ①

Let $b' \leq a'$

$\Rightarrow a' \wedge b' \Rightarrow b'$

Taking complement on both sides,

$(a' \wedge b') \Rightarrow (b')'$

$(a')' \vee (b')' \Rightarrow b$.

$a \vee b \Rightarrow b$

$a \leq b$.

1. Show that a chain of 3 or more elements is not complemented.

Solution:

Let $(L, \wedge, \vee)$ be the given chain.

We know that, in a chain any 2 elements are comparable.

Let $0, a$ & 1 be any 3 element of $(L, \wedge, \vee)$ with 0 as the least element and 1 as the greatest element.

Now, $0 \leq a \leq 1$

$0 \wedge a \Rightarrow 0$ $\quad$ $a \wedge 1 \Rightarrow a$

$0 \vee a \Rightarrow a$ $\quad$ $a \vee 1 \Rightarrow 1$.

In both cases, a does not have any complement.

Hence, any chain with 3 or more element is not complemented.

Boolean Algebra:

A complemented distributive lattice is called Boolean algebra. A non empty set B with together on two binary operations on $(+, \cdot)$ on B. An unary operation on B and two distinct elements 0 and 1 are called Boolean algebra. If the following axioms satisfies a, b satisfies b.

1. Commutative law:

$$a+b = b+a$$

$$a \cdot b = b \cdot a$$

2. Associative law:

$$a+(b+c) = (a+b)+c.$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

3. Distributive law:

$$a+(b \cdot c) \Rightarrow (a+b) \cdot (a+c).$$

$$a \cdot (b+c) \Rightarrow (a \cdot b) + (a \cdot c)$$

4. Identity law:

There exists $0, 1 \in B$.

$$a+0 = a$$

$$a \cdot 1 = a.$$

5. Complement law:

For any, $a \in B$ there exists an element $a' \in B$,

Such that

$$a \cdot a' \Rightarrow 0$$

$$a + a' \Rightarrow 1.$$

Note: Boolean algebra is usually denoted by $(B, +, \cdot, 0, 1)$.

properties:

1. Idempotent law:

1) $a \cdot a = a$, $\forall a \in B$.

2) $a + a = a$

2. Dominance law (Boundedness laws):

1) $a \cdot 0 = 0$ $\forall a \in B$.

2) $a + 1 = 1$

3. Involution law:

$(a')' \Rightarrow a$ $\forall a \in B$.

4. In a Boolean algebra $0' \Rightarrow 1$ and $1' \Rightarrow 0$.

5. Absorption law:

1. $a \cdot (a+b) = a$ $\forall a, b \in B$.

2. $a + (a \cdot b) = a$

Theorem-12:

In a Boolean algebra, prove that following statement are equivalent.

1) $a+b = b$ 2) $a \cdot b = a$ 3) $a'+b = 1$, 4) $a \cdot b' = 0$.

Solution:

one way of prooving, the equivalence is true.

proof ① ⇒ ②

Let $a+b = b$.

Now $a \cdot b = a(a+b)$

$\Rightarrow a$ (absorption law)

proof ② ⇒ ③

Let $a \cdot b = a$

Now, $a' + b \Rightarrow (a \cdot b)' + b$

$\Rightarrow a' + (b' + b)$ (Demorgan's law)

$a' + b \Rightarrow a' + 1$ (Complement law)

$\Rightarrow (a \cdot 0)'$ (Demorgan's law)

$\Rightarrow 0'$

$a' + b \Rightarrow 1$

proof ③ ⇒ ④.

Let $a' + b \Rightarrow 1$.

Now, $a \cdot b' \Rightarrow 0$.

Taking complement on both the sides,

$(a' + b)' \Rightarrow (1)'$

$(a')' \cdot (b')' \Rightarrow 0$.

$a \cdot b' \Rightarrow 0$

proof ④ ⇒ ①.

Let $a \cdot b' \Rightarrow 0$.

Taking complement on both the sides,

$(a \cdot b')' \Rightarrow 0'$.

$(a') + (b')' \Rightarrow 1$.

$a' + b \Rightarrow 1$.

Now, $a + b \Rightarrow (a + b) \cdot 1$ (Identity law)

$\Rightarrow (a + b)(a' + b)$

$a+b \Rightarrow (b+a)\cdot(b+a')$ (commutative law)

$a+b \Rightarrow b+(a\cdot a')$ (Distributive law)

$a+b \Rightarrow b+0$

$a+b \Rightarrow b$

Hence proved.

1. Prove that D110, the set of all positive divisors of the positive integer 110 is Boolean algebra and find all its subalgebra.

Solution:

$D_{110} = \{1, 2, 5, 10, 11, 22, 55, 110\}$.

Since, D satisfies reflexive, antisymmetric and Transmittive property.

D is a partial order relation on D_{110}.

$\therefore D_{110}$, D is poset

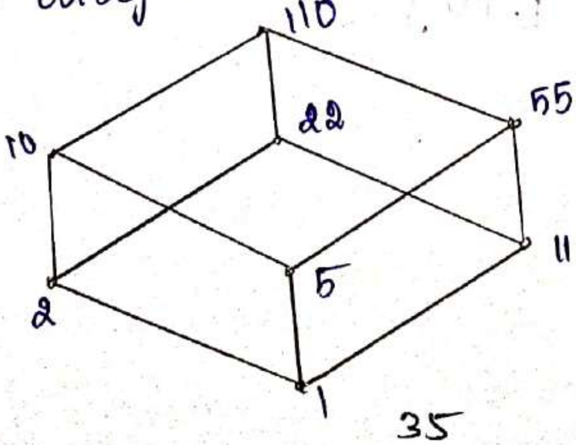
Here,

$a \wedge b = \text{GCD}\{a, b\}$

$a \vee b = \text{LCM}\{a, b\}$ $\forall, a, b \in D_{110}$.

$\therefore (D_{110}, \wedge, \vee)$ is a lattice.

Its Hasse diagram



Here, the least element (0) => 1.

The greatest element (1) => 110.

Each and every element has its complement

Eg:

$gcd\{1, 110\} = 1$

$lcm\{1, 110\} = 110$

$(1)' = 110$ $\quad$ $(22)' = 5$

$(2)' = 55$ $\quad$ $(55)' = 2$

$(5)' = 22$ $\quad$ $(110)' = 1$

$(11)' = 10$ $\quad$ $(10)' = 11.$

$\therefore$ It is complemented lattice.

From the Hasse diagram,

It is obvious that, it is Distributive lattice.

$\therefore (D_{110}, D)$ is a Boolean algebra.

$\therefore$ The sub Boolean algebras are

i) $\{0, 1\}$ is $\{1, 110\}$

ii) $\{1, 2, 5, 10, 11, 22, 55, 110\}$.

iii) $\{a, a', 1, 110\}$ $\forall$ $a \in S$.

9. In a Boolean algebra show that $ab' + a'b = 0$ if and only if $a = b$.

Solution:

Let $a = b$,

Now, $ab' + a'b \Rightarrow aa' + a'a$

$\Rightarrow 0 + 0$

$\therefore ab' + a'b \Rightarrow 0$.

conversely,

assume that, $ab' + a'b \Rightarrow 0$

add a on both sides,

$a + ab' + a'b \Rightarrow a$.

$a + a'b \Rightarrow a$ (absorption law)

$(a + a').(a+b) \Rightarrow a$ (Distributive law)

$1.(a+b) \Rightarrow a$.

$a+b \Rightarrow a$ ... (A)

similarly, $ab' + a'b \Rightarrow 0$

add b on both sides,

$ab' + a'b + b = b$.

$ab' + b = b$ (absorption law)

$(b+a).(b+b') \Rightarrow b$

$(b+a).1 \Rightarrow b$ ... (B)

From eqn (A) and (B)

$a \Rightarrow b$.

Hence Proved.

3. Simplify the Boolean expression $a'b'c + a \cdot b'c + ab'c'$ using Boolean algebraic identities.

Solution:

$a'b'c + ab'c + ab'c'$

$\Rightarrow a'b'c + a \cdot b'(c+c')$.

$\Rightarrow a'b'c + a \cdot b'(1)$

$\Rightarrow (a'b')c + ab'$.

$\Rightarrow (b' \cdot a')c + (b' \cdot a)$ (Commutative law)

$\Rightarrow b'(a'c + a)$

$\Rightarrow b'(a + a'c)$ (Commutative law)

$\Rightarrow b'[(a+a') \cdot (a+c)]$ (Distributive law).

$\Rightarrow b'[1 \cdot (a+c)]$.

$\Rightarrow b'[a+c]$.

$\Rightarrow b'a + b'c$

$\therefore a'b'c + ab'c + ab'c' \Rightarrow ba' + b'c$

4. In any Boolean algebra, show that $(a+b')(b+c')(c+a') \Rightarrow (a'b)(b'+c)(c'+a)$.

Solution:

RHS: $(a+b')(b+c')(c+a')$

$\Rightarrow (a+b'+0)(b+c'+0)(c+a'+0)$

$\Rightarrow \cancel{(a+b'+cc')}\ (a+b'+cc')(b+c'+aa')(c+a'+bb')$

$\Rightarrow (a+b'+c)(a+b'+c')(b+c'+a)(b+c'+a')(c+a'+b)(c+a'+b')$

↳ distributive law

$\Rightarrow [(a'+b+c).(a'+b+c')].[(a+b'+c).(a'+b'+c)].[(a+b'+c').(a+b+c')]$.

$\Rightarrow (a'+b+cc').(b'+c+aa')(c'+a+bb')$ distributive law

$\Rightarrow (a'+b).(b'+c).(c'+a)$

$\Rightarrow$ RHS.

LHS = RHS.

Hence proved.

Theorem-13: Demorgan's law for Boolean algebra.

proof: 1. $(a.b)' = a'+b'$

2. $(a+b)' = a'.b'$.

claim1: $(a.b)' = a'+b'$.

It is enough to prove that,

i) $(a.b).(a'+b') = 0$

ii) $(a.b)+(a'+b') = 1$

i) $(a.b).(a'+b')$.

$\Rightarrow [(a.b).a'] + [(a.b).b']$ distributive law

$\Rightarrow [(b.a).a'] + [(a.b).b']$ commutative law

$\Rightarrow [b.(a.a')] + [a.(b.b')]$ associative law

$\Rightarrow b.0 + a.0$

$\Rightarrow 0$.

ii) $(a.b)+(a'+b')$.

$\Rightarrow [(a'+b')+a].[(a'+b')+b]$ distributive law

$\Rightarrow [(b'+a')+a].[(a'+b')+b]$ commutative law

$\Rightarrow [(a'+a)+b'] \cdot [(b+b')+a']$ associative law

$\Rightarrow (1 \cdot b') \cdot (1 \cdot a')$

$\Rightarrow 1 \cdot 1$

$\Rightarrow 1$ ∴ claim 1 is proved

claim 2): $(a+b)' \Rightarrow a' \cdot b'$.

It is enough to prove that,

i) $(a+b) \cdot (a' \cdot b') \Rightarrow 0$

ii) $(a+b)+(a' \cdot b') \Rightarrow 1$.

i) $(a+b) \cdot (a' \cdot b')$.

$\Rightarrow [(a' \cdot b') \cdot a] + [(a' \cdot b') \cdot b]$ (Distributive law)

$\Rightarrow [(b' \cdot a') \cdot a] + [(a' \cdot b') \cdot b]$ (Commutative law)

$\Rightarrow [b' \cdot (a' \cdot a)] + [a' \cdot (b' \cdot b)]$ (Associative law)

$\Rightarrow (b' \cdot 0) + (a' \cdot 0)$

$\Rightarrow 0+0$

$\Rightarrow 0$

ii) $(a+b)+(a' \cdot b')$.

$\Rightarrow [(a+b)+a'] \cdot [(a+b)+b']$ (Distributive law)

$\Rightarrow [(b+a)+a'] \cdot [(a+b)+b']$ (Commutative law)

$\Rightarrow [b+[a+a']] \cdot [a+(b+b')]$ (Associative law)

$\Rightarrow [b+1] \cdot [(1+a)]$

$\Rightarrow 1 \cdot 1$

$\Rightarrow 1$ ∴ claim 2 is proved.

∴ De morgan's law is verified.