Unit - I

Testing of Hypothesis

Population:

A population in statistics means a set of objects or mainly the set of numbers which are measurements or observations pertaining to the objects.

The population is finite or infinite according to the number of elements of the set is finite or infinite.

Random sampling:

A random sampling is one in which each number of population has an equal chance of being included in it. There are $^{N}C_{n}$ dift samples of size n that can be picked up from a population size N.

Symbols of population and samples:

Characteristic	Population	Sample
	Parameter	Statistics
Symbols	Population size = N Population mean = μ Population standard deviation = σ Population proportion = p	Sample size = n Sample mean = $\bar{x}$ Sample standard deviation = s Sample proportion = $\bar{p}$

Testing a hypothesis:

On the basis of sample information, we make decisions about the population. In taking such decisions, we make certain assumptions. These assumptions are known as statistical hypothesis. These hypothesis are tested. Assuming the hypothesis is correct, we calculate the probability of getting the observed sample. In this Probability is less than a certain assigned value, the hypothesis is to be accepted, otherwise rejected.

Critical region:

A region, corresponding to a statistic t, in the sample space S which amounts to rejection of the null Hypothesis H_0 is called as critical region or region of rejection.

Type I Error

Reject H_0 when it is true

Type II Error

Accept H_0 when it is wrong.

Large Sample Tests: $(n>30)$

1) Test for single mean
2) Test for two means
3) Test for single Proportion
4) Test for two proportions

Formula $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ (or) $\frac{\bar{x} - \mu}{s/\sqrt{n}}$

where $\bar{x}$ - Sample mean
μ - Population mean
σ - Population SD
n - Sample Size.

Assumption of null Hypothesis

$H_0 : \mu = \mu_0$ then the alternate hypothesis is

* $H_1 : \mu \neq \mu_0$ (Two tailed)
* $H_1 : \mu > \mu_0$ (Right tailed)
* $H_1 : \mu < \mu_0$ (Left tailed).

Table for critical values on using normal probability:

Critical values	Level of Significance α		
	1 %	5 %	10 %
Two tailed test	$\|z\| = 2.58$	$\|z\| = 1.96$	$\|z\| = 1.645)$
Right tailed test	$z = 2.33$	$z = 1.645$	$z = 1.28$
Left tailed test	$z = -2.33$	$z = -1.645$	$z = -1.28$

Problems:

① A sample of 900 members has a mean 3.4 cm and Standard deviation 2.61 cm. Is the sample from a large population of mean 3.25 cms & SD 2.61 cms? (Test at 5% LOS. The value of z at 5% level is $|Z_\alpha| < 1.96$).

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Answer:

Given $n = 900$ $\quad \bar{x} = 3.4$

$\mu = 3.25$

$s = 2.61$ $\quad \alpha = 5\%$

Assume $H_0: \bar{x} = \mu$

$H_0: \bar{x} \neq \mu$ (use two tailed test)

Calculation:

$$Z = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{3.4 - 3.25}{\left(\frac{2.61}{\sqrt{900}}\right)}$$

$$= 1.724$$

Table value

$$|Z| = 1.96$$

Conclusion cal Z < Tab Z

$\therefore$ Accept H_0

95% Confidence limits are

$$\bar{x} \pm 1.96 \frac{s}{\sqrt{n}}$$

$$= 3.4 \pm 1.96 \left(\frac{2.61}{\sqrt{900}}\right)$$

$$= 3.4 \pm 0.1705 = 3.57 \ \& \ 3.2295$$

Problem: 2

The mean life time of a sample of 100 light bulbs produced by a company is computed to be 1570 hours with a SD of 120 hours. If μ is the mean life time of all the bulbs produced by the company, test the hypothesis $\mu = 1600$ hours, against the alternative hypothesis $\mu \neq 1600$ hours with $\alpha = 0.05$ & 0.01.

Answer:

Given $n = 100$, $\mu = 1600$, $s = 120$, $\bar{x} = 1570$

$\alpha = 0.05$ & $\alpha = 0.01$

Assume null hypothesis $H_0 : \mu = 1600$

Alternate hypothesis $H_1 : \mu \neq 1600$

Calculation

$$Z = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{1570 - 1600}{120/\sqrt{100}}$$

$$= -2.5$$

Take $|Z| = Z = 2.5$ Cal $Z = 2.5$

Table value

at $\alpha = 5\%$, $|Z| = 1.96$ Tab $Z = 1.96$

at $\alpha = 1\%$, $|Z| = 2.58$ Tab $Z = 2.58$

Conclusion:

5% LoS	1% LoS
Cal Z > Tab Z ∴ Reject Ho	Cal Z < Tab Z ∴ Accept Ho

Problem: 3

The mean breaking strength of the cables supplied by a manufacturer is 1800 with a SD of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. In order to test this claim, a sample of 50 cables is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1 %. LOS?

Answer: Given $n = 50$, $\mu = 1800$, $s = 100$, $\bar{x} = 1850$

$\alpha = 1\%$.

Assume null hypothesis $H_0: \bar{x} = \mu$

Alternate hypothesis $H_1: \bar{x} > \mu$

Calculation:
$$Z = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{1850 - 1800}{(100/\sqrt{50})}$$

$$= 3.54$$

Table value of $Z = 2.33$ at 1 %.

Conclusion:

Cal Z > Tab Z

$\therefore$ we reject H_0

Problem: 4

The guaranteed average life of a certain type of electric light bulbs is 1000 hours with a SD of 125 hours. It is decided to sample the output so as to ensure that 90 percent of the bulbs do not fall short of the guaranteed average by more than 2.5%. what must be the minimum size of the sample?

Answer:

Given $\mu = 1000$ hours

$\sigma = 125$ hours.

Since, we do not want the sample mean to be less than the guaranteed average mean ($\mu = 1000$) by more than 2.5%.

$\therefore \bar{x} > 1000 - 2.5\%$ of 1000

$\Rightarrow \bar{x} > 1000 - 25$

$\Rightarrow \bar{x} > 975$

Formula $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

we want $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{975 - 1000}{125/\sqrt{n}} > -\dfrac{\sqrt{n}}{5}$

According to the given condition

$$P(Z > -\sqrt{n}/5) = 0.90$$

$$P(0 < Z < \sqrt{n}/5) = 0.40$$

$\therefore \sqrt{n}/5 = 1.28$ (from the normal prob. tables).

$\Rightarrow n = 25 \times (1.28)^2$

$= 41$ approximately.

Large Sample test (Normal distribution) for difference of means:

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \quad (\text{or}) \quad \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Problem : 1

Examine whether the difference in the variability in yields is significant at 5% LOS for the following.

	set of 40 plots	set of 60 plots
mean yield per plot	1258	1243
SD per plot	34	28

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Answer: Given $n_1 = 40$, $\bar{x}_1 = 1258$, $s_1 = 34$
$n_2 = 60$, $\bar{x}_2 = 1243$ $s_2 = 28$ $\Big\}$ $\alpha = 5\%$ LOS

Assume null Hypothesis $H_0 : \bar{x}_1 = \bar{x}_2$
There is no difference b/w means of samples.

Alternate Hypothesis $H_1 : \bar{x}_1 \neq \bar{x}_2$

Calculation

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{1258 - 1243}{\sqrt{\frac{1156}{40} + \frac{784}{60}}}$$

$$= 2.32$$

Table value : $Z = 1.96$ at 5% LOS

Conclusion:
Here Cal Z > Tab Z

∴ Reject H_0

Problem : 2

The means of two large samples of 1000 & 2000 members are 67.5 inches and 68.0 inches respectively. Can the samples be regarded as drawn from the Same population of SD 2.5 inches?

Answer:

Given $n_1 = 1000$, $\bar{x}_1 = 67.5$, $s_1 = s_2 = 2.5$

$n_2 = 2000$, $\bar{x}_2 = 68$, $\alpha = 5\%$ LOS

Assume null hypothesis $H_0 : \bar{x}_1 = \bar{x}_2$

Alternate hypothesis $H_1 : \bar{x}_1 \neq \bar{x}_2$

Calculation:

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{67.5 - 68}{\sqrt{\frac{2.5^2}{1000} + \frac{2.5^2}{2000}}}$$

$$= -5.16$$

$$|Z| = 5.16$$

Table value of $Z = 1.96$, 5% LOS

Conclusion

Here Cal $Z >$ Tab Z

$\therefore$ Reject H_0

Problem: 3

A sample heights of 6400 indians has a mean of 67.85 inches & SD 2.56 inches, while a sample heights of 1600 Australians has a mean of 68.55 inches & SD 2.52 inches. Do the data indicate that Australians are on the average taller than indians?

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Answer:

Given $n_1 = 6400$, $\bar{x}_1 = 67.85$ $s_1 = 2.56$

$n_2 = 1600$, $\bar{x}_2 = 68.55$ $s_2 = 2.52$

$\alpha = 5\%$ LOS.

Null Hypothesis H_0:

There is no difference b/w the average heights indians & Australians.

ie. $H_0: \bar{x}_1 = \bar{x}_2$

Alternate hypothesis $H_1: \bar{x}_1 < \bar{x}_2$

Calculation:

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{67.85 - 68.55}{\sqrt{\frac{2.56^2}{6400} + \frac{2.52^2}{1600}}}$$

$$= -9.906$$

$$|Z| = 9.906$$

Table value $Z = 1.645$ (5% LOS)

Conclusion:

Cal Z > Tab Z

∴ Reject H_0

Problem: 3

In a random sample of sige 500, the mean is found to be 20. In another independent sample of sige 400, the mean is 15. Could the samples have been drawn from the sample population with SD 4?

Answer:

$n_1 = 500 \qquad \bar{x}_1 = 20 \qquad \sigma_1 = \sigma_2 = 4$

$n_2 = 400 \qquad \bar{x}_2 = 15$

Null Hypothesis $H_0 : \bar{x}_1 = \bar{x}_2$ (There is no difference b/w the mean values of the samples).

Alternate Hypothesis $H_1 : \bar{x}_1 \neq \bar{x}_2$

Calculation

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{20 - 15}{\sqrt{\frac{4^2}{500} + \frac{4^2}{400}}}$$

$$= 18.6$$

Table value of $Z = 2.58$ (at 1 % LOS)

Conclusion: Cal Z > Tab Z

$\therefore$ Reject H_0

Problem: 4

The sales manager of a large company conducted a sample survey in states A & B taking 400 samples in each case. The results were in the following table. Test whether the average sale is same in the 2 states at 1% Level.

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	State A	State B
Average sales (SD)	Rs. 2500	Rs. 2200
	Rs. 400	Rs. 550

Answer:

Given $n_1 = 400$, $\bar{x}_1 = 2500$, $s_1 = 400$

$n_2 = 400$, $\bar{x}_2 = 2200$, $s_2 = 550$

$\alpha = 1\%$

Null Hypothesis $H_0: \bar{x}_1 = \bar{x}_2$

There is no difference between the average sales of two states.

Alternate Hypothesis $H_1: \bar{x}_1 \neq \bar{x}_2$

Calculation:

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{2500 - 2200}{\sqrt{\frac{400^2}{400} + \frac{550^2}{400}}}$$

$$= 8.82$$

Table value of $Z = 2.58$

Conclusion:

Here Cal Z > Tab Z

$\therefore$ Reject H_0

Problem: 5

Test the significance of the difference b/w the means of the samples, drawn from the two normal populations with the same SD from the following data.

	Size	Mean	SD
Sample I	100	61	4
Sample II	200	63	6

Answer:

$n_1 = 100$, $\bar{x}_1 = 61$, $s_1 = 4$ $\alpha = 5\%$ LOS

$n_2 = 200$, $\bar{x}_2 = 63$, $s_2 = 6$

Null Hypothesis H_0: $\bar{x}_1 = \bar{x}_2$

Alternate Hypothesis H_1: $\bar{x}_1 \neq \bar{x}_2$

Calculation:

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{61-63}{\sqrt{\frac{4^2}{100} + \frac{6^2}{200}}}$$

$$= -3.43$$

$$|Z| = 3.43$$

Table value of $Z = 1.96$

Conclusion: Cal $Z >$ Table Z

$\therefore$ Reject H_0

Large Sample test for single proportion

Formula

$$Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$$

Problem: 1

In a sample of 1000 people in Maharashtra, 540 are rice eaters and the rest are wheat eaters. Can we assume that both rice and wheat are equally popular in this state at 1% LOS.

Answer: Given $n = 1000$, $X = 540$, $p = \frac{X}{n} = \frac{540}{1000} = 0.54$

P = Population proportion of rice eaters in Maharastra = $\frac{1}{2} = 0.5$

$Q = 1 - P = 1 - \frac{1}{2} = 0.5$

Null Hypothesis H_0: Both rice & wheat eaters are equally popular. $(P = 0.5)$

Alternate Hypothesis H_1: $P \neq 0.5$

Calculation $Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{0.54 - 0.50}{\sqrt{\frac{(0.5)(0.5)}{1000}}} = 2.53$

Table value of $Z = 2.58$

Conclusion:

Cal Z < Tab Z

Accept H_0

Problem: 2

Twenty people were attacked by a disease and only 18 survived. Will you reject the Hypothesis that the survival rate, if attacked by this disease, is 85% in favour of the hypothesis that it is more, at 5% Level. (Use Large Sample Test).

Answer: Given $n = 20$

X = Number of Persons who survived after attack by a disease = 18

$$p = \frac{x}{20} = 0.90$$

$$P = 0.85 \quad \Rightarrow Q = 1-P = 0.15$$

Null Hypothesis H_0: $P = 0.85$

Alternate hypothesis H_1: $P > 0.85$

Calculation:
$$Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.90-0.95}{\sqrt{\frac{(0.85)(0.15)}{20}}}$$

$$= 0.626$$

Table value of $Z = 1.645$

Conclusion:

Cal Z < Tab Z

Accept H_0

Problem: 3

A manufacturer claimed that atleast 95% of the equipment which he supplied to a factory conformed to specifications. An examination of a sample of 200 pieces of equipments revealed that 18 were faculty. Test his claim at a significance level of 5% & 1%.

Answer: Given $n = 200$

x = Number of pieces conforming to specifications in the samples $= 200 - 18 = 182$

p = Sample proportion conforming to specifications

$$= \frac{182}{200} = 0.91$$

$P = 0.95$, $Q = 1 - P = 0.05$, $\alpha = 0.05$ & 0.01 LOS.

Null Hypothesis H_0: The proportion of pieces conforming to specification in the population is 95%.

ie. $P = 0.95$

Alternate Hypothesis H_1: $P < 0.95$

Calculation:

$$Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{0.91 - 0.95}{\sqrt{\frac{(0.95)(0.05)}{200}}} = -2.6$$

$$|Z| = 2.6$$

Table value of $Z = 1.645$ (5% LOS)

$Z = 2.33$ (1% LOS)

Conclusion:

① 5% LOS	ⓘⓘ 1% LOS
Cal Z > Tab Z	Cal Z > Tab Z
Reject Ho	Reject Ho

Problem: 4

A coin is tossed 144 times and a Person gets 80 heads. Can we say that the coin is unbiased one?

Answer: Given $n = 144$

P = probability of getting head = $\frac{1}{2}$

$Q = 1 - P = 1 - \frac{1}{2} = \frac{1}{2}$

x = number of successes = number of getting heads

= 80

$$p = \frac{x}{n} = \frac{80}{144} = \frac{5}{9}$$

Null Hypothesis H_0: Coin is unbiased.

Alternate hypothesis H_1: coin is biased.

Calculation:

$$Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{\left(\frac{5}{9}\right)-\left(\frac{1}{2}\right)}{\sqrt{\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{144}}} = 1.33$$

Table value of $Z = 1.96$

Conclusion: Cal Z < Tab Z
Accept Z_0.

Problem - 5

In a big city 325 men out of 600 men were found to be smokers. Does this information Support the conclusion that the majority of men in this city are smokers.

Answer: Given $n = 600$

$$p = \frac{325}{600} = 0.5417$$

P = Population proportion of smokers in the city = 0.5

$\therefore Q = 1 - P = 0.5$

Assume $H_0 : p = 0.5$

$H_1 : p > 0.5$

Calculation:

$$Z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{0.5417 - 0.5}{\sqrt{\frac{(0.5)(0.5)}{600}}} = 2.04$$

Table value of $Z = 2.33$ (1 % LOS)

Conclusion:

Cal $Z <$ Tab Z

∴ Accept H_0.

Large Sample test for difference of Proportions

Formula $Z = \dfrac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$

Problem: 1

During a Country wide investigation the incidence of TB was found to be 1%. In a college of 400 strength 5 were reported to be affected whereas in another college of 1200 strength 10 were reported to be affected. Does this indicate any significant difference.

Answer:

Given $P = 1\% = \frac{1}{100} = 0.01$, $Q = 1 - P = 0.99$

$n_1 = 400$, $n_2 = 1200$, $p_1 = \frac{5}{400} = 0.0125$, $p_2 = \frac{10}{1200} = 0.0083$

Null Hypothesis H_0: There is no difference b/w the Proportions

ie. $P_1 = P_2$

Alternate Hypothesis H_1: $P_1 \neq P_2$

Calculation:

$$Z = \frac{P_1 - P_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{0.0125 - 0.0083}{\sqrt{(0.01)(0.99)\left\{\frac{1}{400}+\frac{1}{1200}\right\}}}$$

$$= 0.7368$$

Table value of $Z = 1.96$ at 5% LOS

Conclusion:

Cal Z < Tab Z

∴ Accept H_0.

Problem: 2

Random Samples of 400 men & 600 women were asked whether they are like to have a residence near flyover. 200 men & 325 women were in favour of the proposals. Test the hypothesis that proportions of men & women infavour of the proposal, are same against that they are not, at 5% LOS.

Answer:

Given

$n_1 = 400$ x_1 = No. of men favouring the proposal = 200

$n_2 = 600$ x_2 = No. of women favouring the proposal = 325.

$p_1 = \frac{x_1}{n_1} = \frac{200}{400} = 0.5$, $p_2 = \frac{x_2}{n_2} = \frac{325}{600} = 0.541$

$$P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{(400)(0.5) + (600)(0.541)}{400+600}$$

$$= 0.525$$

$Q = 1 - P = 1 - 0.525 = 0.475$

Null Hypothesis $H_0 : P_1 = P_2$

There is no difference b/w the Proportions.

Alternate Hypothesis $H_1 : P_1 \neq P_2$

<u>Calculation</u>: $Z = \frac{p_1 - p_2}{\sqrt{PQ\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{0.500 - 0.541}{\sqrt{(0.525)(0.475)\left(\frac{1}{400}+\frac{1}{600}\right)}}$

$$= -1.269$$

$|Z| = 1.269$

Table value of $Z = 1.96$

<u>Conclusion</u>: Cal Z < Tab Z

∴ Accept H_0

Problem : 3

A machine produced 20 defective units in a sample of 400. After overhauling the machine, it produced 10 defective units in a batch of 300. Has the machine improved in production due to overhauling. Test it at 5% LOS.

Given $n_1 = 400$, $n_2 = 300$

$$p_1 = \frac{20}{400} = 0.05$$

$$p_2 = \frac{10}{300} = 0.033$$

$$\therefore \quad P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{(400)(0.05) + (300)(0.033)}{400+300}$$

$$= 0.0427$$

$$\therefore Q = 1 - P = 0.9573$$

Null Hypothesis H_0: $P_1 = P_2$

There is no difference b/w the Proportions

Alternate hypothesis H_1: $P_1 > P_2$

Calculation: $$Z = \frac{P_1 - P_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{0.050 - 0.033}{\sqrt{(0.0427)(0.9573)\left(\frac{1}{400} + \frac{1}{300}\right)}}$$

$$= 1.1$$

Table value of $Z = 1.96$

Conclusion:

cal Z < Tab Z

Accept H_0.

Problem: 4

In two large populations, there are 30 & 25 percent respectively of blue-eyed people. In this difference likely to hidden in samples of 1200 & 900 respectively from the two populations?

Answers:

Given $n_1 = 1200$, $n_2 = 900$

P_1 = Proportion of blue eyed people in the first Population $= 30\% = 0.30$

P_2 = Proportion of blue eyed people in the second Population $= 25\% = 0.25$

$Q_1 = 1 - P_1 = 1 - 0.30 = 0.70$

$Q_2 = 1 - P_2 = 1 - 0.25 = 0.75$

Null Hypothesis H_0: There is no difference b/w the sample proportions $P_1 \neq P_2$

Alternate Hypothesis H_1: $P_1 \neq P_2$

Calculation:

$$Z = \frac{P_1 - P_2}{\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}} = \frac{0.30 - 0.25}{\sqrt{\frac{(0.3)(0.7)}{1200} + \frac{(0.25)(0.75)}{900}}}$$

$= 2.56$

Table value of $Z = 1.96$

Conclusion:

Cal Z > Tab Z

Reject H_0

∴ we conclude that the difference in the population proportions is unlikely to be hidden in sampling.

Hence the problem.

Small Sample Test $(n < 30)$

t- test

Type1 : t-test for single mean

Type 2: t- test for two means.

Formule $t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}}$ where $\bar{x}$ - Sample mean

μ - Population mean.

s - SD

n - number of samples.

Degrees of freedom = n-1

Problem:1

Given a sample mean of 83, a sample SD of 12.5 & a sample size of 22, test the hypothesis that the value of the population mean is 70 against the alternative that it is more than 70. Use the 0.025 significance level.

A/M 2018

Given $n = 22$, $\mu = 70$

$s = 12.5$, $\bar{x} = 83$

$\alpha = 0.025$

$= \frac{25}{1000} \times 100 = 2.5\%$

Null Hypothesis H_0:

There is no difference b/w the sample mean & population mean.

ie. $\bar{x} = \mu$

Alternate hypothesis H_1: $\bar{x} > \mu$

Degrees of freedom $df = 22-1 = 21$

Calculation $t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}}$

$$= \frac{83-70}{(12.5/\sqrt{21})} = 4.77$$

Table value of t = 2.080 (2.5 LOS , Df = 21)

Conclusion:

Cal t > Tab t

Reject Ho

Problem: 2

A machinist is making engine parts with axle diameters of 0.7 inch. A random sample of 10 parts shows a mean diameter of 0.742 inch with a SD of 0.04 inch. Compute the statistic you would use to test, whether the work is meeting the specification.

A/M 2019

Answer: Given $n = 10$

$\bar{x} = 0.742$

$s = 0.04$

$\mu = 0.7$

$\alpha = 5\%$ LOS

$Df = n-1 = 10-1 = 9$

(assume) Null Hypothesis Ho: There is no difference b/w the sample mean & the population mean

$\bar{x} = \mu$

Alternate Hypothesis H₁: $\bar{x} \neq \mu$

Calculation: $t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}}$

$$= \frac{0.742 - 0.7}{(0.04/\sqrt{9})} = 3.15$$

Table value of $Z = 2.262$ (5% LOS , Df = 9)

Conclusion:

Cal t > Tab t

∴ Reject Ho

Problem: 3

A certain injection administered to each of 12 patients resulted in the following increases of blood pressure: 5, 2, 8, −1, 3, 0, 6, −2, 1, 5, 0, 4 can it be concluded that the injection will be, in general, accompanied by an increase in B.P?

Answer: Given $n = 12$

													Total
x	5	2	8	−1	3	0	6	−2	1	5	0	4	$\Sigma x = 31$
x^2	25	4	64	1	9	0	36	4	1	25	0	16	$\Sigma x^2 = 185$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{31}{12} = 2.58$$

Variance $s^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2 = \frac{185}{12} - (2.58)^2$

$$= 8.761$$

$$\therefore \text{SD } s = \sqrt{8.761}$$

$$= 2.96$$

Null Hypothesis Ho: $\bar{x} = \mu$

Alternate hypothesis H_1: $\bar{x} > \mu$

$\alpha = 5\% = 0.05$ LOS

$df = n-1 = 12-1 = 11$

Calculation $t = \frac{\bar{x}-\mu}{(s/\sqrt{n-1})} = \frac{2.58-0}{(2.96/\sqrt{11})}$

$= 2.89$

Table value of $t = 1.796$ (5% LOS, Df = 11)

Conclusion: Cal t > tab t

$\therefore$ Reject H_0

Problem: 4

The mean life time of a sample of 25 bulbs is found as 1550 hours with a SD of 120 hours The company manufacturing the bulbs claims that the average life of their bulbs is 1600 hours. Is the claim acceptable at 5% LOS?

Answer: Given $n = 25$

$\bar{x} = 1550$

$s = 120$ & $\mu = 1600$

Null Hypothesis H_0: $\bar{x} = \mu$

there is no difference b/w the population mean & sample mean.

Alternate Hypothesis H_1: $\bar{x} < \mu$

$\alpha = 5\%$ LOS

$Df = n-1 = 25-1 = 24$

Calculation

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n-1}} = \frac{1550-1600}{[120/\sqrt{25-1}]}$$

$$= -2.04$$

$$|t| = 2.04$$

Table value of $t = 1.711$ (5% LOS & Df = 24)

Conclusion:

Cal t > tab t

Reject Ho

---

* Problem : 5

A test of a breaking strengths of 6 ropes manufactured by a company showed a mean breaking strength of 3515 kg and a SD of 60 kg where as the manufacturer claimed ~~#~~ a mean breaking strength of 3630 kg. Can we support the manufacturer's claim at the level of significance 0.05.

A/M '2019
N/D '2020

Answer: Given $\bar{x} = 3515$

$\mu = 3630$

$s = 60$

$n = 6$

Null Hypothesis Ho: $\bar{x} = \mu$

there is no difference b/w the sample mean & the population mean

Alternate hypothesis H_1: $\bar{x} < \mu$

$\alpha = 5\%$ (5% LOS)

Calculation:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n-1}} = \frac{3515 - 3630}{(60/\sqrt{6-1})} = -4.286$$

$|t| = 4.286$

Cal. $t = 4.286$

Table value of $t = 2.01$

Conclusion:

cal t > tab. t

$\therefore$ Reject Ho

Type: 2

t - test for difference of means.

Formula $t = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ where $s^2 = \left\{\dfrac{\Sigma(x-\bar{x})^2 + \Sigma(y-\bar{y})^2}{n_1+n_2-2}\right\}$

(or)

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}$$

where s_1 & s_2 are SD of given samples.

Problem: 1

Two horses A & B were tested according to the time (in seconds) to run a particular race with the following results:

Horse A	28	30	32	33	33	29	34
Horse B	29	30	30	24	27	29	-

Test whether (A & B were tested according to the time) the horse A is running faster than B at 5%. LOS.

Answer: Given $n_1 = 7, n_2 = 6$

$\Sigma x_1 = 28+30+32+33+33+29+34 = 219$

$\Sigma x_1^2 = 28^2+30^2+32^2+33^2+33^2+29^2+34^2 = 6883$

$\Sigma x_2 = 29+30+30+24+27+29 = 169$

$\Sigma x_2^2 = 29^2+30^2+30^2+24^2+27^2+29^2 = 4787$

$\therefore \bar{x}_1 = \frac{\Sigma x_1}{n_1} = 31.29, \quad \bar{x}_2 = \frac{\Sigma x_2}{n_2} = 28.17$

$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2 = \frac{6883}{7} - (31.29)^2 = 4.23$

$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2 = \frac{4787}{6} - (28.17)^2 = 4.28$

$\therefore S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2} = 5.03$

Null Hypothesis: There is no difference b/w
H0: Population means $(\mu_1 = \mu_2)$

(or)

There is no difference b/w Sample means $(\bar{x}_1 = \bar{x}_2)$

Alternate hypothesis $H_1: \mu_1 \neq \mu_2$

(or)

$\bar{x}_1 \neq \bar{x}_2$

Calculation $t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{S^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{31.29 - 28.17}{\sqrt{5.03\left(\frac{1}{7}+\frac{1}{6}\right)}} = 2.5$

Conclusion :

Cal t > tab t

Reject H_0

Problem : 2

A group of 10 rats fed on diet A & another group of 8 rats fed on diet B, recorded the following increase in weight (gms)

Diet A :	5	6	8	1	12	4	3	9	6	10
Diet B :	2	3	6	8	10	1	2	8		

Does it show superiority of diet A over diet B.

N/D '2017

Answer:

Given $n_1 = 10$

$n_2 = 8$

$\Sigma x_1 = 5+6+8+1+12+4+3+9+6+10 = 64$

$\Sigma x_1^2 = 5^2+6^2+8^2+1^2+12^2+4^2+3^2+9^2+6^2+10^2 = 512$

$\Sigma x_2 = 2+3+6+8+10+1+2+8 = 40$

$\Sigma x_2^2 = 2^2+3^2+6^2+8^2+10^2+1^2+2^2+8^2 = 282$

$\therefore \bar{x}_1 = \frac{\Sigma x_1}{10} = \frac{64}{10} = 6.4$

$\bar{x}_2 = \frac{\Sigma x_2}{8} = \frac{40}{8} = 5$

$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2$

$= 10.24$

$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$

$= 10.25$

$$S^2 = \left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}\right) = \frac{10(10.24)+8(10.25)}{10+8-2}$$

$$= 11.525$$

Null Hypothesis $H_0 : \mu_1 = \mu_2$

Alternate hypothesis $H_1 : \mu_1 > \mu_2$

$\alpha = 5\%$ LOS, $Df = n_1 + n_2 - 2 = 10 + 8 - 2 = 16$

Calculation:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{6.4 - 5}{\sqrt{11.525\left(\frac{1}{10} + \frac{1}{8}\right)}} = 0.869$$

Table value of $t = 1.746$

Conclusion: Cal t < tab t

Accept H_0

Problem: 3

The following random samples, are measurements of the heat producing capacity (in millions of calories per ton) of specimen's of coals from two mines.

Mine 1	8260	8130	8350	8070	8340	-
Mine 2	7950	7890	7900	8140	7920	7840

Use the 0.01 LOS to test whether the difference b/w the means of these two samples is significant

Answer:

$\Sigma x_1 = 8260 + 8130 + 8350 + 8070 + 8340 = 41150$

$\Sigma x_1^2 = 338\,727\,500$

$\Sigma x_2 = 7950 + 7890 + 7900 + 8140 + 7920 + 7840$

$= 47640$

$\Sigma x_2^2 = 378316200$

$\therefore \bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{41150}{5} = 8230$

$\bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{47640}{6} = 7940$

$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2 = 12600$

$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2 = 9100$

$\therefore S^2 = \left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}\right) = \frac{(5)\ (12600) + (6)(9100)}{5+6-2}$

$= 13066.67$, Table value $t = 3.25$

Null Hypothesis $H_0: \mu_1 = \mu_2$

Alternate Hypothesis $H_1: \mu_1 \neq \mu_2$

$\alpha = 1\%$ LOS , $Df = n_1+n_2-2$

$= 5+6-2$

$= 9$

Calculation: $t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{S^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{8230 - 7940}{\sqrt{13066.67\left(\frac{1}{5}+\frac{1}{6}\right)}} = 4.19$

Conclusion: Cal t > Tab t

$\therefore$ Reject H_0

Problem :4 Two independent samples are chosen from two schools A & B, a common test is given in a subject. The scores of the students as follows:

School A	76	68	70	43	94	68	33	-
School B	40	48	92	85	70	76	68	22

Can we conclude that students of school A performed better than students of school B? (Try)

Problem: 5

The independent samples from normal populations with equal variance gave the following

Sample	Size	mean	SD
1	16	23.4	2.5
2	12	24.9	2.8

Is the difference b/w the means significant?

N/D'2019

Answer: Given $\bar{x}_1 = 23.4, \ \bar{x}_2 = 24.9$

$s_1 = 2.5, \ s_2 = 2.8$

$n_1 = 16, \ n_2 = 12$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2} = \frac{(16)(2.5)^2 + (12)(2.8)^2}{16+12-2}$$

$$= 7.4646$$

Null Hypothesis $H_0: \mu_1 = \mu_2$

Alternate hypothesis $H_1: \mu_1 \neq \mu_2$

$\alpha = 0.05, \ df = n_1+n_2-2 = 16+12-2 = 26$

Calculation:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{S^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{23.4 - 24.9}{\sqrt{7.4646\left(\frac{1}{16}+\frac{1}{12}\right)}}$$

$$= -1.4376$$

$$|t| = 1.4376$$

Table value of $t = 2.056$

Conclusion:

cal t < tab t

Accept H_0.

F - Distribution (F - Test)

(Snedecor's F - Distribution)

Test for equality of variances.

Formula: $F = \frac{S_1^2}{S_2^2}$, $S_1^2 > S_2^2$

where $S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}$

$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$

Problem: 1

A group of 10 rats fed on diet A and another group of 8 rats fed on diet B recorded the following increase of in weight. Find if the variances are significantly different.

Diet A	5	6	8	1	12	4	3	9	6	10	-
Diet B	2	3	6	8	10	1	2	8	-	-	-

Answer:

Given $n_1 = 10, n_2 = 8$

											Total
x_1	5	6	8	1	12	4	3	9	6	10	$\Sigma x_1 = 64$
x_1^2	5^2 $=25$	36	64	1	144	16	9	81	36	100	$\Sigma x_1^2 = 512$
x_2	2	3	6	8	10	1	2	8	-	-	$\Sigma x_2 = 40$
x_2^2	4	9	36	64	100	1	4	64	-	-	$\Sigma x_2^2 = 282$

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{64}{10} = \frac{32}{5}$$

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$= \frac{512}{10} - \left(\frac{32}{5}\right)^2$$

$$= 10.24$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{(10)(10.24)}{9}$$

$$= 11.38.$$

$$\bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{40}{8} = 5$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$$

$$= \frac{282}{8} - 5^2$$

$$= 10.25$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{(8)(10.25)}{7}$$

$$= 11.71$$

Null Hypothesis H_0: There is no different b/w the variances.

Calculation

$$F = \frac{S_1^2}{S_2^2} \quad \text{But Here } S_2^2 > S_1^2$$

$$\therefore F = \frac{S_2^2}{S_1^2} = \frac{11.71}{11.38} = 1.03$$

Table value of $F(7, 9) = 3.29$

~~Ho~~

Conclusion: Cal F < Tab F

$\therefore$ Accept H_0

Problem: 2

Two independent samples of sizes 9 & 7 from a normal population has the following values of the variables.

Sample I	18	13	12	15	12	14	16	14	15	-	-
Sample II	16	19	13	16	18	13	15	-	-	-	-

Do the estimates of the population variance differ significantly at 5% LOS.

A/M 2017

Answer:

Given $n_1 = 9$, $n_2 = 7$

										Total
Sample I x_1	18	13	12	15	12	14	16	14	15	$\Sigma x_1 = 129$
x_1^2	18^2	13^2	12^2	15^2	12^2	14^2	16^2	14^2	15^2	$\Sigma x_1^2 = 1879$
Sample II x_2	16	19	13	16	18	13	15	-	-	$\Sigma x_2 = 110$
x_2^2	16^2	19^2	13^2	16^2	18^2	13^2	15^2	-	-	$\Sigma x_2^2 = 1760$

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{129}{9}$$

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \left(\frac{\Sigma x_1}{n_1}\right)^2$$

$$= \left(\frac{1879}{9}\right) - \left(\frac{129}{9}\right)^2 = \frac{10}{3}$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = 3.75$$

$$\bar{x}_2 = \frac{\Sigma x_2}{x_2} = \frac{110}{7}$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \left(\frac{\Sigma x_2}{n_2}\right)^2$$

$$= \frac{1760}{7} - \left(\frac{110}{7}\right)^2 = \frac{220}{49}$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = 5.2381$$

Null Hypothesis $H_0: \mu_1 = \mu_2$

There is no difft b/w Population variances

Calculation

Here $S_2^2 > S_1^2$

$$\therefore F = \frac{S_2^2}{S_1^2} = \frac{5.2381}{3.75} = 1.3968$$

Table value of $F(6,8) = 3.58$

$Df = (n_1-1, n_2-1)$
$= (9-1, 7-1)$
$= (8, 6)$
Here $S_2^2 > S_1^2$
$\therefore Df = (n_2-1, n_1-1)$

Conclusion Cal F < Tab F

Accept H_0

Problem: 3

In one sample of 8 observations, the sum of the squares of deviations of the sample values from the sample mean was 84.4 and in the other sample of 10 observations was 102.6. Test whether this difference is significant at 5% LOS, given that the 5 percent of F for $v_1 = 7$ & $v_2 = 9$ degrees of freedom is 3.29

Ans
Given $n_1 = 8$

$\Sigma(x_1 - \bar{x}_1)^2 = 84.4$

$S_1^2 = \frac{\Sigma(x_1-\bar{x}_1)^2}{n_1-1} = 12.06$

$n_2 = 10$

$\Sigma(x_2 - \bar{x}_2)^2 = 102.6$

$S_2^2 = \frac{\Sigma(x_2-\bar{x}_2)^2}{n_2-1} = 11.42$

Here $S_1^2 > S_2^2$

Null Hypothesis H_0: There is no significant difference b/w the variances of the samples.

$$D.f = (n_1 - 1, n_2 - 1)$$
$$= (8-1, 10-1) = (7, 9)$$

Calculation:
$$F = \frac{S_1^2}{S_2^2} = \frac{12.06}{11.42} = 1.06$$

Table value of $F(7, 9) = 3.29$ at 5%.

Conclusion

Here cal.F < Tab F

Aceept H_0

Problem: 4

In one sample of 10 observations, the sum of the squares of the deviations of the sample values from the sample mean was 120 and in another sample of 12 observations it was 314. Test whether this difference is significant at 5% LOS.

Ans:

Null Hypothesis H_0: There is no significant difft b/w the variances.

Given $n_1 = 10$ | $n_2 = 12$

$\Sigma(x_1 - \bar{x}_1)^2 = 120$ | $\Sigma(x_2 - \bar{x}_2)^2 = 314$

$S_1^2 = \frac{\Sigma(x_1 - \bar{x}_1)^2}{n_1 - 1} = 13.33$ | $S_2^2 = \frac{\Sigma(x_2 - \bar{x}_2)^2}{n_2 - 1} = 28.55$

$Df = (n_1 - 1, n_2 - 1) = (10-1, 12-1) = (9, 11)$

Here $S_2^2 > S_1^2$ $\therefore Df = (n_2 - 1, n_1 - 1) = (11, 9)$

Calculation $F = \frac{S_2^2}{S_1^2} = \frac{28.55}{13.33} = 2.14$

Table value of $F_{(11, 9)}$ = 3.11
5 %

Conclusion: Cal F < Tab F

$\therefore$ Accept Ho.

Problem: 5

Two samples of sizes 9 & 8 gives the sum of squares of deviations from their respective means equal to 160 & 91 respectively. can they be regarded as drawn from the same normal population?

A/M' 2018
N/D' 2019.

Answer:

Null Hypothesis Ho: Samples are drawn from the same normal population.

Given $n_1 = 9$

$\Sigma(x_1-\bar{x}_1)^2 = 160$

$$S_1^2 = \frac{\Sigma(x_1-\bar{x}_1)^2}{n_1-1} = \frac{160}{9-1} = 20$$

$n_2 = 8$

$\Sigma(x_2-\bar{x}_2)^2 = 91$

$$S_2^2 = \frac{\Sigma(x_2-\bar{x}_2)^2}{n_2-1} = \frac{91}{8-1} = 13$$

$Df = (n_1-1, n_2-1) = (9-1, 8-1) = (8, 7)$

Calculation: $F = \frac{S_1^2}{S_2^2}$ Here $S_1^2 > S_2^2$

$$= \frac{20}{13} = 1.54$$

Table value of $F(8,7) = 3.73$

Conclusion:

Cal F < Tab F

Accept Ho.

Hence we conclude that the samples might have come from two populations having the same variance.

TEST BASED ON χ^2 - Distribution

Chi-Square Test for Goodness of fit

Formula $$\chi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right]$$

where O - observed frequency

E - Expected frequency.

Note : In case of

fitting a Binomial Distribution , d.f. = n-1

fitting a Poisson Distribution, df = n-2

fitting a Normal Distribution, df = n-3

Problem:1

A company keeps records of accidents. During a recent Safety review, a random sample of 60 accidents was selected and classified by the day of the week on which they occured.

Day	Mon	Tue	Wed	Thur	Fri
No. of accidents	8	12	9	14	17

Test whether there is any evidence that accidents are more likely on some days than others

N/D 2018

Answer:

Null Hypothesis H_0: Accidents are equally likely to occur on any day of the week.

Alternate Hypothesis H_1: Accidents are not equally likely to occur on the days of the week

$$\text{Expected frequency } E = \frac{60}{5} = 12$$

$$\text{Formula } \psi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right]$$

where O - Observed frequency

E - Expected frequency

						Total
O	8	12	9	14	17	-
E	12	12	12	12	12	-
$\frac{(O-E)^2}{E}$	1.333	0	0.75	0.333	2.083	4.499

Calculation $\chi^2 = \Sigma\left(\frac{(O-E)^2}{E}\right) = 4.499$

Table value of χ^2 = 9.488 (5% LOS)

$Df = n-1$
$= 5-1$
$= 4$

Conclusion Here Cal χ^2 < Tab χ^2

$\therefore$ Accept H_0

Problem: 2

The following data gives the number of aircraft accidents that occured during the various days of the week. Find whether the accidents are uniformly distributed over the week.

Days	Sun	Mon	Tue	Wed	Thurs	Fri	Sat
No. of accidents	14	16	8	12	11	9	14

Answer:

Null Hypothesis H_0: The accidents are uniformly distributed over the week

Alternate Hypothesis H_1: The accidents are not uniformly distributed.

Expected frequency $E = \frac{84}{7} = 12$

								Total
O	14	16	8	12	11	9	14	84
E	12	12	12	12	12	12	12	—
$\frac{(O-E)^2}{E}$	0.333	1.333	1.333	0	0.083	0.75	0.333	4.165

Degrees of freedom $= n-1$

$= 7-1$

$= 6$

Calculation:

$$\chi^2 = \Sigma\left[\frac{(O-E)^2}{E}\right]$$

$$= 4.165$$

Table value of $\chi^2 = 12.592$ (5% LOS)

Df $= 6$

Conclusion:

Cal. χ^2 < Tab χ^2

Accept H_0

Problem: 3

In 120 throws of a single die, the following distribution of faces was observed.

Face	1	2	3	4	5	6
frequency	30	25	18	10	22	15

Can we say that the die is biased.

Answer:

Null Hypothesis H_0: The die is unbiased

Alternate hypothesis H_1: The die is biased.

Expected frequency $E = \frac{120}{6} = 20$

							Total
O	30	25	18	10	22	15	–
E	20	20	20	20	20	20	–
$\frac{(O-E)^2}{E}$	5	1.25	0.2	5	0.2	1.25	12.9

Degrees of freedom $= n-1$

$= 6-1$

$= 5$

Calculation

$$\chi^2 = \Sigma \frac{(O-E)^2}{E}$$

$$= 12.9$$

Table value of $\chi^2 = 11.07$ (5% LOS)

Df = 5

Conclusion:

Cal χ^2 > Tab χ^2

Reject H_0

Problem: 4 A sample analysis of examination results of 1000 students were made and it was found that 260 failed, 110 first class, 420 second class and rest obtained third class. Do these data support the general examination result in the ratio 2:1:4:3

N/D'19

Answer:

Null Hypothesis :

Given data support the general examination result.

ie. The results in the four categories are in the ratio 2:1:4:3

Alternate Hypothesis:

Given data not support the expected result.

To find E

Rough

$\frac{2}{2+1+4+3} \times$ Total stu.

$\frac{2}{10} \times 1000 = 200$

$\frac{1}{10} \times 1000 = 100$

$\frac{4}{10} \times 1000 = 400$, $\frac{3}{10} \times 1000 = 300$

					Total
O	260	110	420	210	-
E	200	100	400	300	-
$\frac{(O-E)^2}{E}$	18	1	1	27	47

Degrees of from = n-1 = 4-1 = 3

Calculation

$$\chi^2 = \Sigma \frac{(O-E)^2}{E} = 47$$

Table value of χ^2 = 7.815

Conclusion: Here cal χ^2 > Tab χ^2

Reject Ho

Problem:5

A survey of 320 families with 5 children each revealed the following distribution.

No. of boys	5	4	3	2	1	0	–
No. of girls	0	1	2	3	4	5	–
No. of families	14	56	110	88	40	12	–

Is this result consistent with the hypothesis that male and female births are equally probable?

Answer:

Null Hypothesis H_0: Male and female births are equally probable.

Alternate Hypothesis: Male and female births are not equally probable.

On the assumption H_0, the expected frequencies are given by the terms of $N(q+p)^n$

$$= 320\left[\frac{1}{2}+\frac{1}{2}\right]^5$$

$$= \frac{320}{32}\left\{5c_0 + 5c_1 + 5c_2 + 5c_3 + 5c_4 + 5c_5\right\}$$

$$= 10\left\{1+5+10+10+5+1\right\}$$

∴ The expected frequencies are

10, 50, 100, 100, 50, 10

O	14	56	110	88	40	12	–
E	10	50	100	100	50	10	–
$\frac{(O-E)^2}{E}$	1.6	0.72	1	1.44	2	0.4	7.16

Degress of freedom $= n-1 = 6-1$
$= 5$

Calculation $\chi^2 = \sum \frac{(O-E)^2}{E}$

$= 7.16$

Table value of $\chi^2 = 11.07$ (5% LOS)
$Df = 5$

Conclusion: Cal χ^2 < Tab χ^2

Accept H_0

χ^2 Test (Independence of attributes)

Formula $\chi^2 = \sum \frac{(O-E)^2}{E}$

(or)

$$\chi^2 = \frac{\cancel{N}\{ad-bc\}^2 (a+b+c+d)}{(a+b)(c+d)(a+c)(b+d)}$$

$Df = (r-1)(c-1) = (2-1)(2-1) = 1$

Problem : 1

Find if there is any association b/w extravagance in fathers and extravagance in sons from the following data.

	Extravagant Father	Miserly Father
Extravagant son	327	741
Miserly son	545	234

M/J' 2013

Determine the co-efficient of association also.

Answer:

H_0: There is no significant difference b/w the extravagance in sons & fathers.

H_1: " significant "

Here $a = 327$
$b = 741$
$c = 545$
$d = 234$

Given $\alpha = 0.05$

$df = (r-1)(c-1) = 1$

$$\chi^2 = \frac{(ad-bc)^2 (a+b+c+d)}{(a+b)(c+d)(a+c)(b+d)}$$

$$= \frac{\{(327)(234)-(741)(545)\}^2 \times \{327 \times 741 + 545 \times 234\}}{(872)(975)(1068)(779)}$$

$$= 279.77$$

Table value for $\chi^2 = 3.841$

Conclusion:

1) cal χ^2 > Tab χ^2

Reject H_0

2) Coefficient of attributes

$$= \frac{ad-bc}{ad+bc}$$

$$= -0.6814$$

Problem: 2

From the following information state whether the condition of the child is associated with the condition of the house.

Condition of House

Condition of child	clean	dirty	Total
Clean	69	51	120
Fairly clean	81	20	101
Dirty	35	44	79
Total.	185	115	300

Answer:

Null Hypothesis H_0:
The given attributes are independent

Alternate Hypothesis H_1:
The given attributes are not independent.

$$\text{Degrees of freedom} = (r-1)(s-1)$$
$$= (3-1)(2-1)$$
$$= 2$$

Observed values:

			Total
	$a = 69$	$b = 51$	120
	$c = 81$	$d = 20$	101
	$e = 35$	$f = 44$	79
Total	185	115	300

To find the expected values.

			Total
	$E(a)=\frac{\text{Row total}\times\text{column total}}{\text{Grand Total}}$ $=74$	$\frac{185\times120}{300}=46$	120
	$\frac{185\times101}{300}=62$	$\frac{115\times101}{300}=39$	101
	$\frac{185\times79}{300}=49$	$\frac{115\times79}{300}=30$	79
Total	185	115	300 Grand total

To find χ^2 value

O	E	$\frac{(O-E)^2}{E}$
69	74	0.34
51	46	0.54
81	62	5.82
20	39	9.26
35	49	4
44	30	6.53
		26.49

Cal $\chi^2 = 26.49$

Table value of $\chi^2 = 5.991$

Conclusion:

Cal χ^2 > Tab χ^2

∴ Reject Ho.

Problem : 3

1000 students at college level were graded according to their IQ and their economic conditions what conclusions can you draw from the following data.

IQ level

Economic conditions	High	Low
Rich	460	140
Poor	240	160

Answer:

Null Hypothesis H_0:

The given attributes are independent

Alternate Hypothesis H_1:

The given attributes are not independent

$$\text{Degrees of freedom} = (r-1)(c-1)$$
$$= (2-1)(2-1)$$
$$= 1$$

Formula $\psi^2 = \dfrac{(ad-bc)^2 (a+b+c+d)}{(a+b)(c+d)(a+c)(b+d)}$

$$= \frac{(460\times160 - 140\times240)^2 \{460+140+240+160\}}{(460+140)(240+160)(460+240)(140+160)}$$

$= 31.75$

Table Value of $\psi^2 = 3.841$

Conclusion

Here cal ψ^2 > Tab ψ^2

Reject Ho.

4) what are the applications /uses of ψ^2 distribution

Ans :

(i) To test the "goodness of fit"

(ii) To test the "independence of attributes"

(iii) To test if the hypothetical value of the population variance is σ^2

(iv) To test the homogeneity of independent estimates of the population variance.

(v) To test the homogeneity of independent estimates of the population correlation co-efficient

— x — x —

Unit - II

Design of Experiments

- One way Classification - CRD
- Two way Classification - RBD
- Latin Square design.
- 2^2 - factorial design.

Basic Principles in the Design of Experiment:-

- Randomization
- Replication
- Local Control (error control)

Randomization:-

* A set of objects is said to be randomized, when they are arranged random order.

* The most frequently used assumption is the one which relates observations are independent.

* This randomization makes the test valid by making it appropriate to analyse the data as though the assumption of independent errors are true.

* Randomization is like insurance. It is always a good idea and even sometimes better than we expect.

Replication:-

* The independent execution of an experiment more than once is called replication.

* Replication is necessary to increase the accuracy of estimates of the treatment effects.

* It also provides an estimate of the error variance which is a function of the differences among observations from experimental units under identical treatments.

* Senstivity of statistical methods for drawing inference also depends on the number of replications.

*Local control :-

- To Provide adequate control of extraneous variables, another essential principle used in the experimental design is the local control.
- This includes techniques such as grouping, blocking and balancing of the experimental units used in the experimental design.
- The Plots in the Same block may be assumed to be relatively homogeneous. we use as many Manures as the number of plots, in a block in a random manner.

Complete Block designs :-

- completely Randomized Design (CRD)
- Randomized Block design (RBD)
- Latin square Design. (LSD)

Analysis of Variance [ANOVA]

*Defn:-

Analysis of Variance is the separation of variance ascribable to one group of causes from the variance ascribable to other gps.

It is nothing but an arithmatical Procedure, used to express the total variation of data, as the sum of its non-negative components.

* Assumptions :-

For the validity of the F-test in ANOVA,

- The observations are independent.
- Parent population from which observations are taken is normal and
- Various treatment and enviromental effects are additive in nature.

Notation :-

SSC – Sum of Squares between Columns

TSS – Total Sum of Squares

MSC – Mean Sum of Squares (between Columns)

MSR – Mean Sum of Squares (between Rows)

SSR – Sum of Squares between Rows

SSE – Error Sum of Squares (or) Within Sum of squares

MSE – Mean Sum of Squares (within Columns).

N – Number of observation.

N_1 – Number of elements in each column.

N_2 – Number of elements in each Row.

ONE WAY CLASSIFICATION

Completely Randomized Design (CRD)

The completely randomized design is the simplest of all the the designs, based on principles of randomization and replication. In this design treatments are allocated at random to the experimental units over the entire experimental material.

* CRD results in the maximum use of the experimental units, since all the experimental material can be used.

* The design is very flexible. Any number of treatments can be used and different treatments can be used unequal number of times without unduly complicating the statistical analysis in most of the cases.

Applications:-

- Completely randomised design is more useful in laboratory technique and methodological studies (Ex) in physics chemistry or cookery, in chemical and biological experiments, in some green house studies, etc.
- CRD is also recommended in situations where an appreciable fraction of units is likely to be destroyed or fail to respond.

Working Procedure:-

1. H_0 : There is no significant difference between the treatments.
2. H_1 : There is significant difference between the treatments.

Step 1:- Find N

Step 2:- Find T

Step 3: Find $\frac{T^2}{N}$

Step 4: Calculate $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \dots + \Sigma x_n^2 - \frac{T^2}{N}$

Step 5: calculate $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \dots + \frac{(\Sigma x_n)^2}{N_1} - \frac{T^2}{N}$

Here N_1 is the number of elements in each column.

Step 6:- Calculate $SSE = TSS - SSC$

Step 7:- ANOVA TABLE

Source of variation.	Sum of Squares	Degrees of freedom	Mean sum of squares	Variance ratio
B/W Columns	SSC	C − 1	$MSC = \frac{SSC}{C-1}$	$F_c = \frac{MSC}{MSE} > 1$
Within Columns	SSE	N − C	$MSE = \frac{SSE}{N-C}$	$F = \frac{MSE}{MSC} > 1$ if MSE > MSC
Total	TSS	N − 1		

Step - 8:- Conclusion

(i) If cal F_c < Table F_c, So we accept H_0

(ii) If cal F_c > Table F_c, so we reject H_0.

Problem : 1

The following are the numbers of mistakes made in 5 successive days of 4 technicians working for a Photographic Laboratory :-

Technician I	Technician II	Technician III	Technician IV
x_1	x_2	x_3	x_4
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

Test at the level of significance $\alpha = 0.01$, whether the differences among the 4 sample means, can be attributed to chance.

soln:-

1. H_0 : There are no significant differences in technicians
2. H_1 : There is significant difference b/w the technicians.

We shift the origin to 10 :-

x_1	x_2	x_3	x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
-4	4	0	-1	-1	16	16	0	1
4	-1	2	2	7	16	1	4	4
0	2	-3	-2	-3	0	4	9	4
-2	0	5	0	3	4	0	25	0
1	4	1	1	7	1	16	1	1
Total:-1	9	5	0	13	37	37	39	10

Step1:-
$N = 20$

Step2: $T = 13$

Step3: $\frac{T^2}{N} = \frac{13^2}{20} = \frac{169}{20} = 8.45$

Step4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 37 + 37 + 39 + 10 - 8.45$

$= 114.55$

Step5: $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(-1)^2}{5} + \frac{(9)^2}{5} + \frac{(5)^2}{5} - 0 - 8.45$

$= \frac{1}{5} + \frac{81}{5} + \frac{25}{5} - 8.45$

$= 12.95$

Step-6 : $SSE = TSS - SSC$
$= 114.55 - 12.95 = 101.6$

Step7 : ANOVA

Source of variation	Sum of squares	degrees of freedom	Mean Square	variance ratio	Table F at 1% level
Between columns	SSC $= 12.95$	$c-1$ $= 4-1$ $= 3$	$MSC = \frac{SSC}{c-1}$ $= \frac{12.95}{3}$ $= 4.32$	$MSE > MSC$ $F_c = \frac{MSE}{MSC}$ $= \frac{6.35}{4.32}$ $= 1.47 > 1$	$F_c(16,3) = 26.9$
Error	SSE $= 101.6$	$N-C$ $= 20-4$ $= 16$	$MSE = \frac{SSE}{N-C}$ $= \frac{101.6}{16}$ $= 6.35$		

Step8: Conclusion:- Cal F_c < Tab F_c i.e, $1.47 < 26.9$
So, we accept H_0
i.e. There are no significant differences in technicians at 1% level

Problem 2 :-

There are three main brands of a certain powder. A set of 120 sample values is examined and found to be allocated among four groups (A, B, C and D) and three brands (I, II, III) as shown here under:-

Brands	Groups			
	A	B	C	D
I	0	4	8	15
II	5	8	13	6
III	8	19	11	13

Is there any significant difference in brands preference? Answer at 5% level.

Soln:-

H_0: There are no significant differences in brands.

H_1: There is significant difference in brands.

Brands	Groups				Total	X_1^2	X_2^2	X_3^2	X_4^2
	A (X_1)	B (X_2)	C (X_3)	D (X_4)					
I (Y_1)	0	4	8	15	27	0	16	64	225
II (Y_2)	5	8	13	6	32	25	64	169	36
III (Y_3)	8	19	11	13	51	64	361	121	169
Total	13	31	32	34	110	89	441	354	430

Step1 :- $N = 12$

Step2 : $T = 110$

Step3 :- $\frac{T^2}{N} = \frac{(110)^2}{12} = 1008.33$

Step4 : $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^4 - \frac{T^2}{N}$

$= 89 + 441 + 354 + 430 - 1008.33$

$= 305.67$

Step:5 :

$SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(27)^2}{4} + \frac{(32)^2}{4} + \frac{(51)^2}{4} - 1008.33$

$= 80.17$

Step6 : $SSE = TSS - SSR$

$= 305.67 - 80.17$

$= 225.50$

Step7 : ANOVA :-

Source of variation	Sum of squares	degree of freedom	Mean Square	Variance ratio	Table F at 5% Level
Between rows	$SSR = 80.17$	$r-1$ $= 3-1$ $= 2$	$MSR = \frac{SSR}{r-1}$ $= \frac{80.17}{2}$ $= 40.09$	$MSR > MSE$ $\therefore F_R = \frac{MSR}{MSE}$ $= \frac{40.09}{25.06}$ $= 1.6$	$MSR > MSE$ $\therefore F_R(2,9)$ $= 4.26$
Error	$SSE = 225.5$	$N-r$ $= 12-3$ $= 9$	$MSE = \frac{SSE}{N-r}$ $= \frac{225.5}{9}$ $= 25.06$		

Step8 : Conclusion Cal F_r < Table F_r i.e $1.6 < 4.26$

So, we accept H_0.

$\therefore$ There are no significant differences in brands.

Problem 3:-

A completely randomised design experiment with 10 plots and 3 treatments gave the following results:

Plot No	1	2	3	4	5	6	7	8	9	10
Treatment	A	B	C	A	C	C	A	B	A	B
Yield	5	4	3	7	5	1	3	4	1	7

Analyse the results for treatment effects.

Soln:-

A	5	7	3	1
B	4	4	7	
C	3	5	1	

Ho: There are no significant differences in treatment.

H₁: There is significant difference in treatments.

A X_1	B X_2	C X_3	Total	X_1^2	X_2^2	X_3^2
5	4	3	12	25	16	9
7	4	5	16	49	16	25
3	7	1	11	9	49	1
1			1	1		
16	15	9	T=40	84	81	35

Step 1: $N = 10$

Step 2: $T = 40$

Step 3: $\frac{T^2}{N} = \frac{(40)^2}{10} = 160$

Step 4:- $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 - \frac{T^2}{N}$

$= 84 + 81 + 35 - 160$

$= 40$

Step5 $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(16)^2}{4} + \frac{(15)^2}{3} + \frac{(9)^2}{3} - 160$

$= 6$

Step6: $SSE = TSS - SSC$

$= 40 - 6$

$= 34$

Step7: ANOVA

Source of Variation	Sum of squares	Degrees of freedom	Mean square	Variance ratio	Table F at 5% Level
Between column	$SSC = 6$	$c-1$ $=3-1$ $=2$	$MSC = \frac{SSC}{c-1}$ $= \frac{6}{2}$ $= 3$	$MSE > MSC$ $\therefore F_c = \frac{MSE}{MSC}$ $= \frac{4.86}{3}$ $= 1.62 > 1$	$MSE > MST$ $\therefore F_c\ (7, 2)$ $= 19.35$
Error	$SSE = 34$	$N-C$ $=10-3$ $=7$	$MSE = \frac{SSE}{N-C}$ $= \frac{34}{7}$ $= 4.86$		

Step. 8 Conclusion:-

Cal F_c < Table F_c

ie $1.62 < 19.35$, So, we accept H_0

∴ There are no significant differences in treatments.

Problem 4:-

The following table shows the lives in hours of four brands of electric lamps:-

Brand A:	1610	1610	1650	1680	1700	1720	1800	
B:	1580	1640	1640	1700	1750			
C:	1460	1550	1600	1620	1640	1660	1740	1820
D:	1510	1520	1530	1570	1600	1680		

Perform an analysis of variance test the homogenity of the mean lives of the four brands of Lamps.

soln:-

H_0: There are no significant differences in brands.

H_1: There is a significant difference b/w the four brands.

Subtract 1600 and then divided by 10, we get

x_1 A	x_2 B	x_3 C	x_4 D	Total	x_1^2	x_2^2	x_3^2	x_4^2
1	-2	-14	-9	-24	1	4	196	81
1	4	-5	-8	-8	1	16	25	64
5	4	0	-7	2	25	16	0	49
8	10	2	-3	17	64	100	4	9
10	15	4	0	29	100	225	16	0
12	–	6	8	26	144	–	36	64
20	–	14	–	34	400	–	196	–
–	–	22	–	22	–	–	484	–
57	31	29	-19	98	735	361	957	267

Step 1: $N = 26$

Step 2: $T = 98$

Step 3: $\frac{T^2}{N} = \frac{(98)^2}{26} = 369.39$

Step 4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 735 + 361 + 957 + 267 - 369.39 = 1950.61$

Step 5: $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(57)^2}{7} + \frac{(31)^2}{5} + \frac{(29)^2}{8} + \frac{(-19)^2}{6} - 369.39$

$= 452.25$

Step 6: $SSE = TSS - SSC = 1950.61 - 452.25 = 1498.36$.

Step 7: ANOVA

Source of Variation	Sum of Squares	Degrees of freedom	Mean Square	Variance ratio	Table F at 5% Level
Between columns	$SSC = 452.25$	$c-1 = 4-1$ $= 3$	$MSC = \frac{SSC}{c-1}$ $= \frac{452.25}{3}$ $= 150.75$	$MSC > MSE$ $\therefore F_c = \frac{MSC}{MSE}$ $= \frac{150.75}{68.11}$ $= 2.21 > 1$	$MSC > MSE$ $\therefore F_c(3,22)$ $= 3.05$
Error	$SSE = 1498.36$	$N - C$ $= 26 - 4$ $= 22$	$MSE = \frac{SSE}{N-C}$ $= \frac{1498.36}{22}$ $= 68.11$		

Step 8: Conclusion

cal $F_c <$ Table F_c

$2.21 < 3.05$

So we accept H_0

The lives of the four brands of lamps do not differ significantly.

Problem 5:-

The accompanying data resulted from an experiment comparing the degree of soiling for fabric co-polymerized with the three different mixtures of metha crcylic acid. Analysis is the given classification.

Mixture 1	·56	1·12	·90	1·07	·94
Mixture 2	·72	·69	·87	·78	·91
Mixture 3	·62	1·08	1·07	·99	·93

Soln:

H_0: The true average degree of soiling is identical for three mixture.

H_1: The true average degree of soiling is not identical for three mixture.

X_1	X_2	X_3	Total	X_1^2	X_2^2	X_3^2
0·56	0·72	0·62	1·9	0·3136	0·5184	0·3844
1·12	0·69	1·08	2·89	1·2544	0·4761	1·1664
0·90	0·87	1·07	2·84	0·8100	0·7569	1·1449
1·07	0·78	0·99	2·84	1·1449	0·6084	0·9801
0·94	0·91	0·93	2·78	0·8836	0·8281	0·8649
4·59	3·97	4·69	T=13·25	4·4065	3·1879	4·5407

Step-1: $N = 15$

Step 2: $T = 13·25$

Step3: $\frac{T^2}{N} = \frac{(13·25)^2}{15} = 11·7042$

Step 4 : $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 - \frac{T^2}{N}$

$= 4.4065 + 3.1879 + 4.5407 - 11.7042$.

$= 0.4309$

Step 5 : $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(4.59)^2}{5} + \frac{(3.97)^2}{5} + \frac{(4.69)^2}{5} - 11.7042$

$= 0.0608$

Step 6 : $SSE = TSS - SSC$

$= 0.4309 - 0.0608$

$= 0.3701$

Step 7 : ANOVA

Source of Variation	Sum of Squares	Degrees of freedom	Mean Square	Variance ratio	Table F at 5% Level
Between Column	$SSC = 0.0608$	$C-1$ $= 3-1$ $= 2$	$MSC = \frac{SSC}{C-1}$ $= \frac{0.0608}{2}$ $= 0.0304$	$MSE > MSC$ $\therefore F_c = \frac{MSE}{MSC}$ $= \frac{0.0308}{0.0304}$ $= 1.0132$	$MSE > MSC$ $\therefore F_c(12,2)$ $= 19.41$
Error	$SSE = 0.3701$	$N-C$ $= 15-3$ $= 12$	$MSE = \frac{SSE}{N-C}$ $= \frac{0.3701}{12}$ $= 0.0308$		

Step 8 :- Conclusion:

Cal F_c < Table F_c

$1.0132 < 19.41$

So, we accept H_0,

The true average degree of soiling is identical for three mixtures.

Two way classification:-

Randomized Block Design [RBD]

Let us consider an agricultural experiment using which we wish to test the effect of k-fertilising treatments on the yield of crops. We assume that we know some information about the soil fertility of the plots. Then we divide the plots into 'h' blocks, according to the soil fertility each bloack containing k blocks. Thus the plots in each block will be of homogeneous fertility as far as possible within each block, the 'k' treatments are given to the 'k' plots in a perfectly random manner, such that each treatment occurs only once in any block. But the same k-treatments are repeated from block to block. This design is called Randomized Block design.

Merits:

* It has a simple layout
* The design controls the variability in the experimental units and gives the treatments equivalence to show their effects.
* The analysis of the design is simple and straight forward as in the case of two-way classification of analysis of variance
* The analysis is possible, even in the case of missing observations.

Demerits:-

* The design is not suitable for large number of treatments, since in this case the block size is large and hence homogeneity of units may not be possible
* Unequal number of replications for equal treatment is not possible.

Working Procedure:-

H_0: There is no significant difference.

H_1: There is significant difference.

Step 1 : Find N

Step 2 : Find T

Step 3 : Find $\frac{T^2}{N}$

Step 4 : $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \dots + \Sigma x_n^2 - \frac{T^2}{N}$

Step 5 : Find $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \dots + \frac{(\Sigma x_n)^2}{N_1} - \frac{T^2}{N}$

Step 6 : Find $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \dots + \frac{(\Sigma y_n)^2}{N_1} - \frac{T^2}{N}$

Step 7 : $SSE = TSS - SSC - SSR$

Step 8 : ANOVA TABLE

Step 9 : Conclusion.

Problem -1

An experiment was designed to study the performance of 4 different detergents for cleaning fuel injectors. The following "cleanliness" readings were obtained with specially designed equipment for 12 tanks of gas distributed over 3 different models of engines:

	Engine 1	Engine 2	Engine 3	Total
Detergent A	45	43	51	139
Detergent B	47	46	52	145
Detergent C	48	50	55	153
Detergent D	42	37	49	128
Total	182	176	207	565

Perform the ANOVA and test at 0.01 level of significance, whether there are differences in the detergents or in the engines.

Soln:-

Subtracting 50 from each

Detergent	Engine (X_1)	(X_2)	(X_3)	Total	X_1^2	X_2^2	X_3^2
A (Y_1)	-5	-7	1	-11	25	49	1
B (Y_2)	-3	-4	2	-5	9	16	4
C (Y_3)	-2	0	5	3	4	0	25
D (Y_4)	-8	-13	-1	-22	64	169	1
Total	-18	-24	7	T=-35	102	234	31

H_0 : There is no significant diff b/w column means as well as row means

H_1 : There is significant diff. b/w column means or the row means

Step-1 : $N = 12$

Step-2 : $T = -35$

Step-3 $\frac{T^2}{N} = \frac{(-35)^2}{12} = 102.08$

Step-4 : $TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 - \frac{T^2}{N}$

$= 102 + 234 + 31 - 102.08$

$= 264.92$

Step5: $SSC = \frac{(\Sigma X_1)^2}{N_1} + \frac{(\Sigma X_2)^2}{N_1} + \frac{(\Sigma X_3)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(-18)^2}{4} + \frac{(-24)^2}{4} + \frac{(7)^2}{4} - 102.08$

$= 135.17$

Step6 : $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(-11)^2}{3} + \frac{(-5)^2}{3} + \frac{(3)^2}{3} + \frac{(-22)^2}{3} - 102.08$

$= 110.92$

Step7: $SSE = TSS - SSC - SSR$

$= 264.92 - 135.17 - 110.92$

$= 18.83$

Step8: ANOVA TABLE

Source of Variation	Sum of Squares	Degrees of freedom	Mean square	Variance ratio	Table F. at 1% Level
Between Columns	SSC $=135.17$	$c-1$ $=3-1$ $=2$	$MSC = \frac{SSC}{C-1} = \frac{135.17}{2}$ $= 67.59$	$MSC > MSE$ $\therefore F_c = \frac{MSC}{MSE}$ $= \frac{67.59}{3.14}$ $= 21.52 > 1$	$MSC > MSE$ $\therefore F_c(2,6)$ $= 10.92$
Between Rows	$SSR = 110.92$	$r-1$ $=4-1=3$	$MSR = \frac{SSR}{r-1}$ $= \frac{110.92}{3}$ $= 36.97$	$MSR > MSE$ $\therefore F_R = \frac{MSR}{MSE}$ $= \frac{36.97}{3.14}$ $= 11.77 > 1$	$MSR > MSE$ $\therefore F_R(3,6)$ $= 9.78$
Error	SSE $=18.83$	$(c-1)(r-1)$ $=(2)(3)$ $=6$	$MSE = \frac{SSE}{(c-1)(r-1)}$ $= \frac{18.83}{6}$ $= 3.14$		

Step9: Conclusion:

(i) cal F_c < Table F_c i.e., $21.52 > 10.92$

So, we reject H_0,
i.e., There are significant differences in the engines.

(ii) Cal F_R < Table F_R i.e., $11.77 > 9.78$

So, we reject H_0
i.e., There are significan differences in the engines.

Problem 2:-

A set of data involving "four tropical feed stuffs A, B, C, D" tried on 20 chicks is given below. All the twenty chicks are treated alike in all respects except the feeding treatments and each feeding treatment is given to 5 chicks. Analyze the data. Weight gain of baby chicks fed on different materials composed of tropical feed stuffs.

						Total T_i
A	55	49	42	21	52	219
B	61	112	30	89	63	355
C	42	97	81	95	92	407
D	169	137	169	85	154	714
				Grand total		1695

Soln:-

Subtract 50 from each value.

	X_1	X_2	X_3	X_4	X_5	Total	X_1^2	X_2^2	X_3^2	X_4^2	X_5^2
$Y_1 = A$	5	−1	−8	−29	2	−31	25	1	64	841	4
$Y_2 = B$	11	62	−20	39	13	105	121	3844	400	1521	169
$Y_3 = C$	−8	47	31	45	42	157	64	2209	961	2025	1764
$Y_4 = D$	119	87	119	35	104	464	14161	7569	14161	1225	10816
Total	127	195	122	90	161	695	14371	13623	15586	5612	12753

Step 1: $N = 20$

Step 2: $T = 695$

Step 3: $\frac{T^2}{N} = \frac{(695)^2}{20} = 24151.25$

Step 4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 + \Sigma x_5^2 - \frac{T^2}{N}$

$= 14371 + 13623 + 15586 + 5612 + 12753 - 24151.25$

$= 37793.75$

Step-5

$$SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} + \frac{(\Sigma x_5)^2}{N_1} - \frac{T^2}{N}$$

$$= \frac{(127)^2}{4} + \frac{(195)^2}{4} + \frac{(122)^2}{4} + \frac{(90)^2}{4} + \frac{(161)^2}{4} - 24151.25$$

$$= 1613.50$$

Step-6:

$$SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} - \frac{T^2}{N}$$

$$= \frac{(-31)^2}{5} + \frac{(105)^2}{5} + \frac{(157)^2}{5} + \frac{(464)^2}{5} - 24151.25$$

$$= 26234.95$$

Step7: $SSE = TSS - SSC - SSR$

$$= 37793.75 - 1613.50 - 26234.95 = 9945.3$$

Source of variation	Sum of Squares	Degrees of freedom	Mean Square	Variance ratio	Table F at 5% Level
B/W Columns	SSC =1613.50	$c-1$ $=5-1$ $=4$	$MSC = \frac{SSC}{c-1}$ $= \frac{1613.5}{4}$ $= 403.38$	$MSE > MSC$ $\therefore F_c = \frac{MSE}{MSC}$ $= 2.06$	$MSE > MSC$ $\therefore F_c(12,4)$ $= 5.91$
B/W Rows	SSR =26234.95	$r-1$ $=4-1$ $=3$	$MSR = \frac{SSR}{r-1}$ $= \frac{26234.95}{3}$ $= 8744.98$	$MSR > MSE$ $F_R = \frac{MSR}{MSE}$ $= \frac{8744.98}{828.78}$ $= 10.55$	$MSR > MSE$ $\therefore F_R(3,12)$ $= 3.49$
Error	SSE =9945.3	$(c-1)(r-1)$ $=(4)(3)$ $=12$	$MSE = \frac{SSE}{12}$ $= \frac{9945.3}{12}$ $= 828.78$		

Step-9: Conclusion.

(i) cal F_c < Table F_c ie., $2.06 < 5.91$, so we accept H_0 B/W columns ie., There is no significant difference B/W columns

(ii) cac F_R > Table F_R ie., $10.55 > 3.49$, so we reject H_0, ie., There are siginificant difference b/w rows.

Problem 3 :-

Three varieties A, B and C of a crop are tested in a randomised block design with four replications. The plot yield in pounds are as follows :-

A	6	C	5	A	8	B	9
C	8	A	4	B	6	C	9
B	7	B	6	C	10	A	6

Analyse the experimental yield and State your conclusion
A/M 2019 – R-13

Soln

A	6	4	8	6
B	7	6	6	9
C	8	5	10	9

Variety		BLOCK			Total of Varieties				
	X1	X2	X3	X4		X_1^2	X_2^2	X_3^2	X_4^2
A	6	4	8	6	24	36	16	64	36
B	7	6	6	9	28	49	36	36	81
C	8	5	10	9	32	64	25	100	81
Total	21	15	24	24	84	149	75	200	198

H_0: There is no significant difference b/w Column means as well as rows

H_1: There is significant difference b/w column means on the rows means.

Step-1 : $N = 12$

Step 2 : $T = 84$

Step 3 : $\frac{T^2}{N} = \frac{(84)^2}{12} = 588$

Step 4 : $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 149 + 77 + 200 + 198 - 588$

$= 36$

Step-5 : $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(21)^2}{3} + \frac{(15)^2}{3} + \frac{(24)^2}{3} + \frac{(24)^2}{3} - 588$

$= 18$

Step 6 : $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(24)^2}{3} + \frac{(28)^2}{3} + \frac{(32)^2}{3} - 588$

$= 8$

Step-7 : $SSE = TSS - SSC - SSR$

$= 36 - 18 - 8$

$= 10$

Step 8 : ANOVA

Source of variation	Sum of squares	Degrees of freedom	Mean Square	Variance ratio.	Table F at 5% Level
Between Blocks	$SSC = 18$	$c-1$ $= 4-1$ $= 3$	$MSC = \frac{SSC}{c-1}$ $= 18/3$ $= 6$	$MSC > MSE$ $\therefore F_c = \frac{MSC}{MSE}$ $= \frac{6}{1.67}$ $= 3.59$	$MSC > MSE$ $\therefore F_c(3,6)$ $= 4.76$
Between Varieties	$SSR = 8$	$r-1$ $= 3-1$ $= 2$	$MSR = \frac{SSR}{r-1}$ $= \frac{8}{2}$ $= 4$	$MSR > MSE$ $\therefore F_R = \frac{MSR}{MSE}$ $= \frac{4}{1.67}$ $= 2.4$	$MSR > MSE$ $\therefore F_R(2,6)$ $= 5.14$
Error.	$SSE = 10$	$(c-1)(r-1)$ $= (3)(2)$ $= 6$	$MSE = \frac{SSE}{(c-1)(r-1)}$ $= \frac{10}{6}$ $= 1.67$		

Step 9: Conclusion

(i) Cal F_C < Table F_C i.e., $3.59 < 4.76$, So, we accept H_0, i.e., There is no significant difference b/w blocks.

(ii) Cal F_R < Table F_R i.e., $2.4 < 5.14$. So, we accept H_0, i.e, There is no significant difference b/w varieties.

Problem 4:-

The table shows the yield of paddy in arbitrary units obtained from four different varieties planted in five blocks where each block is divided into four plots. Test at 5% level whether the yields vary significantly with (i) soil differences (ii) differences in the type of paddy.

Blocks	Types of Paddy			
	I	II	III	IV
A	12	15	10	14
B	15	19	12	11
C	14	18	15	12
D	11	16	12	16
E	16	17	11	14

soln:-

H_0: There is no significant difference b/w columns and rows

H_1: There is significant difference b/w colums and rows.

	X_1	X_2	X_3	X_4	Total	X_1^2	X_2^2	X_3^2	X_4^2
Y_1	-2	1	-4	0	-5	4	1	16	0
Y_2	1	5	-2	-3	1	1	25	4	9
Y_3	0	4	1	-2	3	0	16	1	4
Y_4	-3	2	-2	2	-1	9	4	4	4
Y_5	2	3	-3	0	2	4	9	9	0
Total	-2	15	-10	-3	0	18	55	34	17

Step-1: $N = 20$

Step-2: $T = 0$

Step-3: $\frac{T^2}{N} = 0$

Step-4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 18 + 55 + 34 + 17 - 0$

$= 124$

Step5: $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(-2)^2}{5} + \frac{(15)^2}{5} + \frac{(-10)^2}{5} + \frac{(-3)^2}{5} - 0$

$= 67.6$

Step6: $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma x_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} + \frac{(\Sigma y_5)}{N_2} - \frac{T^2}{N}$

$= \frac{(-5)^2}{4} + \frac{(1)^2}{4} + \frac{(3)^2}{4} + \frac{(-1)^2}{4} + \frac{(2)^2}{4} - 0$

$= 10$

Step7: $SSE = TSS - SSC - SSR.$

$= 124 - 67.6 - 10$

$= 46.4$

Step8: ANOVA:

Source of variation	Sum of Squares	Degrees of freedom	Mean square	Variance of ratio	Table(F) at 5% Level
B/w columns	$SSC = 67.6$	$c-1 = 4-1$ $= 3$	$MSC = \frac{SSC}{c-1}$ $= \frac{67.6}{3}$ $= 22.53$	$MSC > MSE$ $\therefore F_c = MSC/MSE$ $= \frac{22.53}{3.87}$ $= 5.82$	$\therefore F_c\ (3, 12)$ $= 3.49$
B/w Rows	$SSR = 10$	$r-1$ $= 5-1$ $= 4$	$MSR = \frac{SSR}{r-1}$ $= \frac{10}{4}$ $= 2.5$	$MSE > MSR.$ $\therefore F_R = \frac{MSE}{MSR}$ $= 3.87/2.5$ $= 1.55$	$\therefore F_R\ (12, 4)$ $= 5.91$
Error.	$SSE = 46.4$	$(c-1)(r-1)$ $= (3)(4)$ $= 12$	$MSE = \frac{SSE}{(c-1)(r-1)}$ $= \frac{46.4}{12} = 3.87$		

Step9:- Conclusion.

(i) Cal F_c > Table F_c ie., $5.82 > 3.49$. So, we reject at Ho.
ie., There are significant difference b/w column & rows

(ii) Cal F_R < Table F_R ie., $1.55 < 5.91$, So we accept Ho.
ie., There is no significant difference b/w column and rows

Problem - 5:-

Table below shows the seeds of 4 different types of corns planted in 3 blocks. Test at 0.05 level of significance whether the yields in kilograms per unit area vary significantly with different types of corns.

		Types of corns			
Blocks	~~A~~	I	II	III	IV
	A	4.5	6.4	7.2	6.7
	B	8.8	7.8	9.6	7.0
	C	5.9	6.8	5.7	5.2

Soln:-

H_0: There is no significant difference b/w blocks and types of corns

H_1: There is significant difference b/w blocks and types of corns.

Subtract 6.8 from each elements.

	X_1	X_2	X_3	X_4	Total	X_1^2	X_2^2	X_3^2	X_4^2
Y_1	-2.3	-0.4	0.4	-0.1	-2.4	5.29	0.16	0.16	0.01
Y_2	2	1	2.8	0.2	6	4	1	7.84	0.04
Y_3	-0.9	0	-1.1	-1.6	-3.6	0.81	0	1.21	2.56
Total	-1.2	0.6	2.1	-1.5	0	10.1	1.16	9.21	2.61

Step1 : $N = 16$

Step2 : $T = 0$

Step3 : $\frac{T^2}{N} = 0$

Step4: $TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - \frac{T^2}{N}$

$= 10.1 + 1.16 + 9.21 + 2.61$

$= 23.08$

Step5: $SSC = \frac{(\Sigma X_1)^2}{N_1} + \frac{(\Sigma X_2)^2}{N_1} + \frac{(\Sigma X_3)^2}{N_1} + \frac{(\Sigma X_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(-1.2)^2}{3} + \frac{(0.6)^2}{3} + \frac{(2.1)^2}{3} + \frac{(-1.5)^2}{3} - 0 = 2.82$

Step 6: $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(-2.4)^2}{4} + \frac{(6)^2}{4} + \frac{(-3.6)^2}{4} - 0$

$= 13.68$

Step 7: $SSE = TSS - SSC - SSR$

$= 23.08 - 2.82 - 13.68$

$= 6.58$

Step 8 ANOVA.

Source of variation	Sum of squares	Degrees of freedom	Mean Square	Variance ratio	Table F at 5% Level
B/W Columns	SSC = 2.82	$c-1 = 4-1$ $= 3$	$MSC = \frac{SSC}{c-1}$ $= \frac{2.82}{3}$ $= 0.94$	MSE > MSC $\therefore F_c = \frac{MSE}{MSC}$ $= \frac{1.09}{0.94}$ $= 1.15$	$\therefore F_c(6,3)$ $= 8.94$
B/W Rows	SSR = 13.68	$r-1 = 3-1$ $= 2$	$MSR = \frac{SSR}{r-1}$ $= \frac{13.68}{2}$ $= 6.84$	MSR > MSE $\therefore F_R = \frac{MSR}{MSE}$ $= \frac{6.84}{1.09}$ $= 6.27$	$F_R(2,6)$ $= 5.14$
Error	SSE = 6.58	$(c-1)(r-1)$ $= (3)(2)$ $= 6$	$MSE = \frac{SSE}{(c-1)(r-1)}$ $= \frac{6.58}{6}$ $= 1.09$		

Step 9: Conclusion

(i) cal F_c < Table F_c ie., $1.15 < 8.94$, So, we accept H_0
ie., There is no significant difference b/w types of corns

(ii) cal F_R ≥ Table F_R ie, $6.27 > 5.14$, so we reject H_0.
ie., There are significant different in blocks.

Latin Square Design (LSD)

The term Latin square takes its name from a figure of mathematical puzzle that was studied many years before, use as a plan of experiment.

Latin squares are very extensively used in agricultural trials in order to eliminate fertility trends in two directions simultaneously. The data are classified according to the different criteria (ie,) according to columns, rows and varieties and are arranged in a square known as Latin square.

Merits:-

- * Latin square design controls variability in two directions of the experimental material.
- * The analysis of the design is simple and straight forward and is a three way classification of analysis of variance.

Demerits:-

- * The process of randomization is not as simple as in RBD
- * The number of treatments should be equal to the number of rows and columns.
- * The experimental area should be in the form of a square.
- * It is suitable only in the case of smaller number of treatments.
- * A 2×2 Latin ~~square~~ square is not possible.

Working Procedure.

H_0: There is no significant difference b/w column means row means and treatments.

H_1: There is significant difference b/w column means row means and treatments.

Step 1 Find N

Step 2 Find T

Step 3 Find $\frac{T^2}{N}$

Step 4 $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \dots + \Sigma x_n^2 - \frac{T^2}{N}$

Step 5 $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \dots + \frac{(\Sigma x_n)^2}{N_1} - T^2/N$

Step 6 $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \dots + \frac{(\Sigma y_n)^2}{N_2} - \frac{T^2}{N}$

Step 7 $SSK = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \dots + \frac{(\Sigma y_n)^2}{N_2} - T^2/N$

Step 8 $SSE = TSS - SSC - SSR - SSE$

Step 9 ANOVA Table.

Step 10 Conclusion.

Problem 1:

The following is a Latin square of a design, when 4 varieties of seeds are being tested. Set up the analysis of variance table and State your conclusion. You may carry out suitable change of origin and scale.

A	105	B	95	C	125	D	115
C	115	D	125	A	105	B	105
D	115	C	95	B	105	A	115
B	95	A	135	D	95	C	115.

Soln:-

Subtract 100 and then divided by 5, we get

A	1	B	-1	C	5	D	3
C	3	D	5	A	1	B	1
D	3	C	-1	B	1	A	3
B	-1	A	7	D	-1	C	3

H_0: There is no significant difference b/w column means, row means and treatments.

H_1: There is significant difference b/w column means or row means or treatments.

	X_1	X_2	X_3	X_4	Total	X_1^2	X_2^2	X_3^2	X_4^2
Y_1	1	-1	5	3	8	1	1	25	9
Y_2	3	5	1	1	10	9	25	1	1
Y_3	3	-1	1	3	6	9	1	1	9
X_4	-1	7	-1	3	8	1	49	1	9
Total	6	10	6	10	32	20	76	28	28

Step-1: $N=16$

Step 2: $T=32$

Step 3: $\frac{T^2}{N} = \frac{(32)^2}{16} = 64$

Step 4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 20+76+28+28-64$

$= 88$

Step 5: $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(6)^2}{4} + \frac{(10)^2}{4} + \frac{(6)^2}{4} + \frac{(10)^2}{4} - 64$

$= 4$

Step 6: $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(8)^2}{4} + \frac{(10)^2}{4} + \frac{(6)^2}{4} + \frac{(8)^2}{4} - 64$

$= 2$

Step 7 :- SSK :-

A	1	1	3	7	12
B	-1	1	1	-1	0
C	5	3	-1	3	10
D	3	5	3	-1	10

$SSK = \frac{(\Sigma A)^2}{N_1} + \frac{(\Sigma B)^2}{N_1} + \frac{(\Sigma C)^2}{N_1} + \frac{(\Sigma D)^2}{N_1} - T^2/N$

$= \frac{(12)^2}{4} + \frac{0}{4} + \frac{(10)^2}{4} + \frac{(10)^2}{4} - 64$

$= 22$

Step 8: $SSE = TSS - SSC - SSR - SSK$

$= 84 - 4 - 2 - 22$

$= 60$

Step 9.

Source of variation	Sum of Squares	Degrees of freedom	Mean Square	Variance ratio	Table (F) at 5% Level
Between Column	$SSC = 4$	$K-1$ $= 4-1$ $= 3$	$MSC = \frac{SSC}{K-1}$ $= 4/3$ $= 1.33$	$MSE > MSC$ $\therefore F_C = \frac{MSE}{MSC}$ $= \frac{10}{1.33} = 7.52$	$\therefore F_C(6,3)$ $= 8.94$
Between Rows	$SSR = 2$	$K-1$ $= 4-1$ $= 3$	$MSR = \frac{SSR}{K-1}$ $= 2/3$ $= 0.67$	$F_R = \frac{MSE}{MSR}$ $= \frac{10}{0.67}$ $= 14.9$	$F_R(6,3)$ $= 8.94$
Between treatments	$SSK = 22$	$K-1$ $= 4-1$ $= 3$	$MSK = \frac{SSK}{K-1}$ $= \frac{22}{3}$ $= 7.33$	$F_T = \frac{MSE}{MSK}$ $= \frac{10}{7.33} = 1.36$	$F_T(6,3) = 8.94$
Error	$SSE = 60$	$(K-1)(K-2)$ $= (3)(2)$ $= 6$	$MSE = \frac{SSE}{(K-1)(K-2)}$ $= \frac{60}{6} = 10$		

Step 10: Conclusion

(i) cal Fc < Table Fc, ie., 7.52 < 8.94. So, we accept Ho.
ie., There is no significant difference b/w columns.

(ii) cal Fc > Table Fc i.e., 14.93 > 8.94. So we reject Ho.
ie, There are significant differences in rows

(iii) cal F_T < Table F_T i.e., 1.36 < 8.94. So, we accept Ho.
ie., There is no significant difference b/w treatments.

Problem 2:

A variable trial was conducted on wheat with 4 varieties in a Latin Square Design. The plan of the experiment and the per plot yield are given below.

C	25	B	23	A	20	D	20
A	19	D	19	C	21	B	18
B	19	A	14	D	17	C	20
D	17	C	20	B	21	A	15

Analyse data and interpret the result.

soln:- Subtract 20 from all the items.

	x_1	x_2	x_3	x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
Y_1	5	3	0	0	8	25	9	0	0
Y_2	-1	-1	1	-2	-3	1	1	1	4
Y_3	-1	-6	-3	0	-10	1	36	9	0
Y_4	3	0	1	-5	-7	9	0	1	25
Total	0	-4	-1	-7	-12	36	46	11	29.

H_0: There is no significant difference b/w rows, columns and treatments.

H_1 : There is significant difference b/w rows or columns or treatments.

Step 1 : $N = 16$

Step 2 : $T = -12$

Step 3 : $\frac{T^2}{N} = \frac{(-12)^2}{16} = 9$

Step 4 : $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 36 + 46 + 11 + 29 - 9$

$= 113$

Step 5 : $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(0)^2}{4} + \frac{(-4)^2}{4} + \frac{(-1)^2}{4} + \frac{(-7)^2}{4} - 9$

$= 7.5$

Step 6 : $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma X_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(8)^2}{4} + \frac{(-3)^2}{4} + \frac{(-10)^2}{4} + \frac{(-7)^2}{4} - 9$

$= 46.5$

Step 7 : SSK

					Total
A	0	-1	-6	-5	-12
B	3	-2	4	1	1
C	5	1	0	0	6
D	0	-1	-3	-3	-7

$SSK = \frac{(\Sigma A)^2}{N_2} + \frac{(\Sigma B)^2}{N_2} + \frac{(\Sigma C)^2}{N_2} + \frac{(\Sigma D)^2}{N_2} - \frac{T^2}{N}$

$$= \frac{(-12)^2}{4} + \frac{(1)^2}{4} + \frac{(6)^2}{4} + \frac{(-7)^2}{4} - 9$$

Step-8: $= 48.5$

$SSE = TSS - SSC - SSR - SSK$

$= 113 - 7.5 - 46.5 - 48.5$

$= 10.5$

Step9:- ANOVA TABLE

Source of variation	Sum of squares	Degrees of freedom	Mean square	Variance ratio	Table(F) at 5% Level
Between Columns	$SSC = 7.5$	$K-1$ $=4-1$ $=3$	$MSC = \frac{SSC}{K-1}$ $= \frac{7.5}{3}$ $= 2.5$	$F_C = \frac{MSC}{MSE}$ $= \frac{2.5}{1.75}$ $= 1.43$	$\therefore F_C(3,6)$ $= 4.76$
Between Rows	$SSR = 46.5$	$K-1$ $=4-1$ $=3$	$MSR = \frac{SSR}{K-1}$ $= \frac{46.5}{3}$ $= 15.5$	$F_R = \frac{MSR}{MSE}$ $= \frac{15.5}{3}$ $= 8.86$	$F_R(3,6)$ $= 4.76$
Between Treatment	$SSK = 48.5$	$K-1$ $=4-1$ $=3$	$MSK = \frac{SSK}{K-1}$ $= \frac{48.5}{3}$ $= 16.17$	~~MSS~~ $F_T = \frac{MSK}{MSE}$ $= \frac{16.17}{1.75}$ $= 9.24$	$F_T(3\,6)$ $= 4.76$
Error	$SSE = 10.5$	$(K-1)(K-2)$ $= (3)(2)$ $= 6$	$MSE = \frac{SSE}{(K-1)(K-2)}$ $= \frac{10.5}{6}$ $= 1.75$		

Step 10 Conclusion:-

(i) cal $F_C <$ Tab F_C i.e., $1.43 < 4.76$, so, we accept H_0, i.e., There is no significant difference b/w columns.

(ii) cal $F_R >$ Tab F_R i.e. $8.86 > 4.76$, so we reject H_0. i.e., There are significant difference in rows

(iii) cal $F_T >$ Tab F_T i.e., $9.24 > 4.76$, so we reject H_0. i.e., There are siginificant difference in Treatments.

Problem 3:-

A farmer wishes to test the effects of four different fertilizers A, B, C, D on the yield of wheat. In order to eliminate sources of error due to variability in soil fertility. he uses the fertilizers, in a Latin square arrangement. as indicated in the following table where the numbers indicate yields in bushels per unit area

A 18	C 21	D 25	B 11
D 22	B 12	A 15	C 19
B 15	A 20	C 23	D 24
C 22	D 21	B 10	A 17

Perform an analysis of variance to determine, if there is a significant difference b/w the fertilizers at $\alpha = 0.05$ level of significance.

Soln:-

Subtract 20 we get

A −2	C 1	D 5	B −9
D 2	B −8	A −5	C −1
B −5	A 0	C 3	D 4
C 2	D 1	B −10	A −3

H_0: There is no significant diff. b/w rows, columns and treatments.

H_1: There is significant diff. b/w rows, columns or treatments.

	x_1	x_2	x_3	x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
Y_1	-2	1	5	-9	-5	4	1	25	81
Y_2	2	-8	-5	-1	-12	4	64	25	1
Y_3	-5	0	3	4	2	25	0	9	16
Y_4	2	1	-10	-3	-10	4	1	100	9
Total	-3	-6	-7	-9	-25	37	66	159	107

Step 1: $N = 16$

Step 2: $T = -25$

Step 3: $\frac{T^2}{N} = \frac{(-25)^2}{16} = 39.06$

Step 4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 37 + 66 + 159 + 107 - 39.06$

$= 329.94.$

Step 5: $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(-3)^2}{4} + \frac{(-6)^2}{4} + \frac{(-7)^2}{4} + \frac{(-9)^2}{4} - 39.06$

$= 4.69$

Step 6: $SSR = \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(-5)^2}{4} + \frac{(-12)^2}{4} + \frac{(2)^2}{4} + \frac{(-10)^2}{4} - 39.06$

$= 29.19$

Step 7: SSK.

					Total
A	-2	-5	0	-3	-10
B	-9	-8	-5	-10	-32
C	1	-1	3	2	5
D	5	2	4	1	12

$$SSK = \frac{(\Sigma A)^2}{N_2} + \frac{(\Sigma B)^2}{N_2} + \frac{(\Sigma C)^2}{N_2} + \frac{(\Sigma D)^2}{N_2} - \frac{T^2}{N}$$

$$= \frac{(-10)^2}{4} + \frac{(-32)^2}{4} + \frac{(5)^2}{4} + \frac{(12)^2}{4} - 39.06$$

$$= 284.19$$

Step 8: $SSE = TSS - SSC - SSR - SSK$

$$= 329.94 - 4.69 - 29.19 - 284.19$$

$$= 11.87$$

Step 9 : ANOVA TABLE

Source of variation	Sum of squares	Degrees of freedom	Mean square	Variance ratio	Table F at 5% Level
Between columns	$SSC = 4.69$	$K-1 = 4-1 = 3$	$MSc = \frac{SSC}{K-1} = \frac{4.69}{3} = 1.56$	$\therefore F_C = \frac{MSE}{MSC} = \frac{1.98}{1.56} = 1.27$	$F_C(6,3) = 8.94$
Between Rows	$SSR = 29.19$	$K-1 = 4-1 = 3$	$MSR = \frac{SSR}{K-1} = \frac{29.19}{3} = 9.73$	$F_R = \frac{MSR}{MSE} = \frac{9.73}{1.98} = 4.91$	$F_R(3,6) = 4.76$
Between Treatments	$SSK = 284.19$	$K-1 = 4-1 = 3$	$MS_K = \frac{SSK}{K-1} = \frac{284.19}{3} = 94.73$	$F_T = \frac{MSK}{MSE} = \frac{94.73}{1.98} = 47.84$	$F_T(3,6) = 4.76$
Error	$SSE = 11.87$	$(K-1)(K-2) = (3)(2) = 6$	$MSE = \frac{SSE}{(K-1)(K-2)} = \frac{11.87}{6} = 1.98$		

Step 10: Conclusion:-

(i) cal F_C < Table F_C i.e., $1.27 < 8.94$, so we accept Ho.
i.e, There is no significant diff. b/w the columns.

(ii) cal F_T < Table F_T i.e., $4.91 > 4.76$, so we reject Ho.
i.e., There are significant difference in rows.

(iii) cal F_R > Table F_R ie $47.84 > 4.76$, So we reject Ho.
i.e, There are significant difference in treatments.

Problem 4 :-

Analyze the variance in the Latin square of yields (in kgs) Paddy where P, Q, R, S denote the different methods of cultivation.

S	122	P	121	R	123	Q	122
Q	124	R	123	P	122	S	125
P	120	Q	119	S	120	R	121
R	122	S	123	Q	121	P	122.

Examine whether the different methods of cultivation have given significantly different yields.

Solns:-

Subtract 120 we get

S	2	P	1	R	3	Q	2
Q	4	R	3	P	2	S	5
P	0	Q	-1	S	0	R	1
R	2	S	3	Q.	1	P	2

	x_1	x_2	x_3	x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
Y_1	2	1	3	2	8	4	1	9	4
Y_2	4	3	2	5	14	16	9	4	25
Y_3	0	-1	0	1	0	0	1	0	1
Y_4	2	3	1	2	8	4	9	1	4
Total	8	6	6	10	30	24	20	14	34

H_0: There is no significant difference between rows, columns and treatments.

H_1: There is significant difference b/w rows, or columns or treatments.

Step 1: $N = 16$

Step 2: $T = 30$

Step 3: $\frac{T^2}{N} = \frac{(30)^2}{16} = 56.25$

Step 4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 24 + 20 + 14 + 34 - 56.25$

$= 35.75$

Step 5: $SSC = \frac{(\Sigma x_1)^2}{N_1} + \frac{(\Sigma x_2)^2}{N_1} + \frac{(\Sigma x_3)^2}{N_1} + \frac{(\Sigma x_4)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(8)^2}{4} + \frac{(6)^2}{4} + \frac{(6)^2}{4} + \frac{(10)^2}{4} - 56.25$

$= 2.75$

Step 6: $SSR = \frac{(\Sigma y_1)^2}{N_2} + \frac{(\Sigma y_2)^2}{N_2} + \frac{(\Sigma y_3)^2}{N_2} + \frac{(\Sigma y_4)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(8)^2}{4} + \frac{(14)^2}{4} + \frac{(0)^2}{4} + \frac{(8)^2}{4} - 56.25$

$= 24.75$

Step 7

SSK

					Total
P	0	1	2	2	5
Q	4	-1	1	2	6
R	2	3	3	1	9
S	2	3	0	5	10

$SSK = \frac{(5)^2}{4} + \frac{(6)^2}{4} + \frac{(9)^2}{4} + \frac{(10)^2}{4} - 56.25$

$= 4.25$

Step 8: $SSE = TSS - SSC - SSR - SSK$

$= 35.75 - 2.75 - 24.75 - 4.25$

$= 4$

Step 9: ANOVA TABLE

Source of Variation	Sum of Squares	Degrees of freedom	Mean square	Variance of ratio	Table F at 5% Level
Between columns	SSC = 2.75	K−1 = 4−1 = 3	$MSC = \frac{SSC}{K-1}$ $= \frac{2.75}{3}$ = 0.92	$F_C = \frac{MSC}{MSE}$ $= \frac{0.92}{0.67}$ = 1.37	$F_c(3,6)$ = 4.76
Between rows	SSR = 24.75	K−1 = 4−1 = 3	$MSR = \frac{SSR}{K-1}$ $= \frac{24.75}{3}$ = 8.25	$F_R = \frac{MSR}{MSE}$ $= \frac{8.25}{0.67}$ = 12.31	$F_R(3,6)$ = 4.76
Between treatments	SSK = 4.25	K−1 = 4−1 = 3	$MSK = \frac{SSK}{K-1}$ $= \frac{4.25}{3}$ = 1.42	$F_T = \frac{MSK}{MSE}$ $= \frac{1.42}{0.67}$ = 2.12	$F_T(3,6)$ = 4.76
Error	SSE = 4	(K−1)(K−2) = (3)(2) = 6	$MSE = \frac{SSE}{}$		

Step 10: Conclusion:-

(i) cal F_C < Table F_C, i.e, 1.37 < 4.76, so we accept H_0.
i.e., There is no significant difference in columns.

(ii) cal F_R > Table F_R i.e., 12.31 > 4.76 so we reject H_0.
i.e., There ~~are~~ significant difference in rows

(iii) cal F_T < Table F_T i.e., 2.12 < 4.76 so, we accept H_0.
i.e., There is no significant difference in treatments.

Problem 5 :-

The following data resulted from an experiment to compare 3 burners B1, B2, B3. A Latin square Design was used as the tests were made on 3 engines and were spread over 3 days. Perform an analysis of variance at 5% level of significance on the data.

Engine.

		1	2	3
	1	B_1 - 16	B_2 - 17	B_3 - 20
Day	2	B2 - 16	B3 - 21	B_1 - 15
	3	B3 - 15	B1 - 12	B_2 - 13

Soln:-

H_0 : There is no significant difference between rows, columns and treatments.

H_1 : There is significant difference between rows or columns or treatments.

Subtract 16 from each value.

	X_1	X_2	X_3	Total	X_1^2	X_2^2	X_3^2
Y_1	0	1	4	5	0	1	16
Y_2	0	5	-1	4	0	25	1
Y_3	-1	-4	-3	-8	1	16	9
Total	-1	2	0	1	.	42	26

Step 1: $N = 9$

Step 2: $T = 1$

Step 3: $\frac{T^2}{N} = \frac{(1)^2}{9} = 0.11$

Step 4: $TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 - \frac{T^2}{N}$

$= 1 + 42 + 26 - 0.11 = 68.89$

Step5: SSC $= \frac{(\Sigma X_1)^2}{N_1} + \frac{(\Sigma X_2)^2}{N_1} + \frac{(\Sigma X_3)^2}{N_1} - \frac{T^2}{N}$

$= \frac{(-1)^2}{3} + \frac{(2)^2}{3} + \frac{(0)^2}{3} - 0.11$

$= 1.56$

Step6 SSR $= \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} - \frac{T^2}{N}$

$= \frac{(5)^2}{3} + \frac{(4)^2}{3} + \frac{(-8)^2}{3} - 0.11$

$= 34.89$

Step7: SSK.

				Total
B_1	0	-1	-4	-5
B_2	1	0	-3	-2
B_3	4	5	-1	8

SSK $= \frac{(-5)^2}{3} + \frac{(-2)^2}{3} + \frac{(8)^2}{3} - 0.11$

$= 30.89$

SSE = TSS − SSC − SSR − SSK

$= 68.89 - 1.56 - 34.89 - 30.89$

$= 1.55$

Source of Variation	Sum of squares	Degrees of freedom	Mean Square	Variance of ratio	Table (F) at 5% Level
Between columns	SSC $=1.56$	$K-1$ $=3-1$ $=2$	$MSC = \frac{SSC}{K-1}$ $= \frac{1.56}{2}$ $= 0.78$	$\therefore F_c = \frac{MSC}{MSE}$ $= 0.78$	$\therefore F_c(2,2)$ $=19$
Between Rows	SSR $=34.89$	$K-1$ $=3-1$ $=2$	$MSR = \frac{SSR}{K-1}$ $= \frac{34.89}{2}$ $=17.45$	$\therefore F_R = \frac{MSR}{MSE}$ $= \frac{17.45}{0.775}$ $= 22.52$	$F_R(2,2)=19$
Between treatments	SSK $=30.89$	$K-1$ $=3-1$ $=2$	$MSK = \frac{SSK}{K-1}$ $= \frac{30.89}{2}$ $= 15.45$	$\therefore F_T = \frac{MSK}{MSE}$ $= \frac{15.45}{0.775}$ $=19.94$	$\therefore F_T(2,2)$ $=19$
Error	SSE $=1.55$	$(K-1)(K-2)$ $=(2)(1)$ $=2$	$MSE = \frac{SSE}{(K-1)(K-2)}$ $=\frac{1.55}{2}$ $=0.775.$		

Step 10 :- Conclusion

(i) cal F_c < table F_c i.e., $1.006 < 19$, So we accept H_0.
i.e., There is no significant difference b/w Columns

(ii) Cal F_R > Table F_R i.e., $22.52 > 19$, So we reject H_0.
i.e., There are significant difference in rows

(iii) Cal F_T > Table F_T i.e., $19.94 > 19$. So we reject H_0.
ie., There are significant difference in treatments.

2^2 - Factorial Design:-

A major conceptual advancement in an experimental design is exemplified by factorial design. Factorial designs are frequently used in experiments involving several factors, where it is necessary to study the joint effect of the factors on a response.

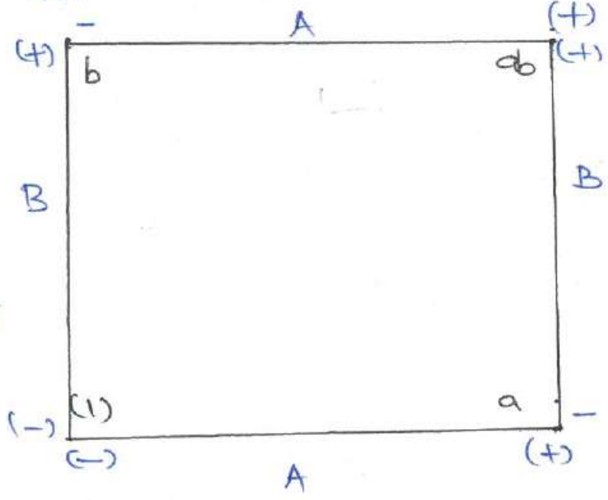
A complete replicate of such a design requires $2\times2\times\ldots\times2 = 2^k$ observation is called a 2^k factorial design.

2^2 Design:-

2^2 is the simplest design of 2^k. Two factors A and B each at two levels that it may be low and high levels of B the factor.

2^2 design can be represented geometrically as a square with the $2^2 = 4$ runs, (or) treatment, combinations, forming the corners of the square.

Here we denote low and high levels of A and B by the signs − and +. This is sometimes called the geometric notation for the design.



Working Procedure:-

Step 1 : Find N

Step 2 : Find T

Step 3 : Find $\frac{T^2}{N}$

Step 4 : Find TSS $= \Sigma x_1^2 + \Sigma x_2^2 + \dots + \Sigma x_n^2 - \frac{T^2}{N}$

Step 5 :
Contrast A $= a + ab - a - (1)$
Contrast B $= b + ab - a - (1)$
Contrast AB $= ab + (1) - a - (b)$

Step 6

$$A = \frac{1}{2n}[a + ab - b - (1)]$$

$$B = \frac{1}{2n}[b + ab - a - (1)]$$

$$AB = \frac{1}{2n}[ab + (1) - a - b]$$

Step 7 :

$$SSA = \frac{[a + ab - b - (1)]^2}{4n}$$

$$SS\,B = \frac{[b + ab - a - (1)]^2}{4n}$$

$$SSAB = \frac{[ab + (1) - a - b]}{4n}$$

Step 8 :

$$SSE = TSS - SS_A - SS_B - SS_{AB}$$

Step 9 : ANOVA

Step 10 : Conclusion :-

Problem 1:

Find out the main effects and interactions in the following 2^2 factorial experiment and write down the analysis of variance table.

	1 00	a 10	b 01	ab 11
BLOCK I	64	25	30	6
II	75	14	50	33
III	76	12	41	17
IV	75	33	25	10

Soln:

Subtracting

Blocks

Treatment	I X_1	II X_2	III X_3	IV X_4	Total	X_1^2	X_2^2	X_3^2	X_4^2
Y_1 (1)	27	38	39	38	142	729	1444	1521	1444
Y_2 a	−12	−23	−25	−4	−64	144	529	625	16
Y_3 b	−7	13	4	−12	−2	49	169	16	144
Y_4 ab	−31	−4	−20	−27	−82	961	16	400	729
Total	−23	24	−2	−5	T=−6	1883	2158	2562	2333

Step1 : $N = 16$

Step2 : $T = -6$

Step3 : $\frac{T^2}{N} = \frac{36}{16} = 2.25$

Step4: $TSS = \Sigma x_1^2 + \Sigma x_2^2 + \Sigma x_3^2 + \Sigma x_4^2 - \frac{T^2}{N}$

$= 1883 + 2158 \quad 2562 + 2333 - 2.25$

$= 8933.75$

Step-5:-

contrast A : $a + ab \cancel{+} b - (1) = -64 - 82 + 2 - 142 = -286$

contrast B : $b + ab - a - (1) = -2 - 82 + 64 - 142 = -162$

contrast AB = $ab + (1) - a - b = -82 + 142 + 64 + 2 = 126$

Step6:

Main effect A : $\frac{1}{2n}[a + ab - b - (1)]$

$= \frac{1}{2(4)}[-286]$

$= -35\ 75$

Main effect B $= \frac{1}{2n}[b + ab - a - (1)]$

$= \frac{1}{(2)(4)}[-162]$

$= -20.25$

Main effect AB $= \frac{1}{2n}[ab + (1) - a - b]$

$= \frac{1}{(2)(4)}[126]$

$= 15.75$

Step7:-

$$SS_A = \frac{[a + ab - b - (1)]^2}{4n} = \frac{(-286)^2}{16} = 5112.25$$

$$SS_B = \frac{[b + ab - a - (1)]^2}{4n} = \frac{(-162)^2}{16} = 1640.25$$

$$SS_{AB} = \frac{[ab + (1) - a - b]^2}{4n} = \frac{(126)^2}{16} = 992.25$$

Step 8:-

$SS_E = T_{SS} - SS_A - SS_B - SS_{AB}$

$= 8933.75 - 5112.25 - 1640.25 - 992.25$

$= 1189.$

Step 9: Analysis of variance table for the 2^2 experiment.

Source of variation	Sum of Squares	Degrees of freedom	Mean square	Variance ratio	Table F	
					5%	1%
A	SSA $=5112.25$	$a-1$ $=2-1$ $=1$	$MSS_A=\frac{SSA}{a-1}$ $=\frac{5112.25}{1}$ $=5112.25$	$M_{SSA} > MSSE$ $\therefore F_A=\frac{MSS_A}{MSS_E}$ $=\frac{5112.25}{99.08}$ $=51.60$	$F_A(1,12)$ $=4.75$	$F_A(1,12)$ $=9.33$
B	SSB $=1640.25$	$b-1$ $=2-1$ $=1$	$MSS_B=\frac{SSB}{b-1}$ $=\frac{1640.25}{1}$ $=1640.25$	$\therefore F_B=\frac{MSSB}{MSSE}$ $=\frac{1640.25}{99.08}$ $=16.56$	$F_B(1,12)$ $=4.75$	$F_B(1,12)$ $=9.33$
AB	$SSAB$ $=992.25$	$(a-1)(b-1)$ $=(1)(1)$ $=1$	$MSS_{AB}=\frac{SSAB}{(a-1)(b-1)}$ $=\frac{992.25}{1}$ $=992.25$	$\therefore F_{AB}=\frac{MSSAB}{MSSE}$ $=\frac{992.25}{99.08}$ $=10.02$	$F_{AB}(1,12)$ $=4.75$	$F_{AB}(1,12)$ $=9.33$
Error	SS_E $=1189$	$ab(n-1)$ $=(2)(2)(3)$ $=12$	$MSS_E=\frac{SSE}{ab(n-1)}$ $=\frac{1189}{12}$ $=99.08$			

Step 10: Conclusion:-

	5%	1%
Cal F_A > Tab F_A	51.60 > 4.75	51.60 > 9.33
Cal F_B > Tab F_B	16.56 > 4.75	16.67-933
Cal F_{AB} > Tab $_{AB}$	10.02 > 4.75	10.02 > 9.33

Problem 2:-

An experiment was planned to study the effect of sulphate of potash and super phosphate on the yield of potates. All the combinations of 2 levels of super phosphate [0 cent (b_0) and 5 cent (b_1) / acre] and two levels of sulphate of potash [0 cent (a_0) and 5 cent (a_1) / acre] were studied in a randomised block design with 4 replications for each. The (1/70) yields [lb per plot = (1/70) acre obtained are given in Table. Analyse the data and give your conclusions.

BLOCK	Yields (lbs per plot)			
I	(1) 23	a 25	b 22	ab 38
II	b 40	(1) 26	a 36	ab 38
III	(1) 29	a 20	ba 30	b 20
IV	ab 34	a 31	b 24	(1) 28

Soln: subtract 29 from each value.

Treatment combination	Blocks I x_1	II x_2	III x_3	IV x_4	Total	x_1^2	x_2^2	x_3^2	x_4^2
(Y_1) (1)	−6	−3	0	−1	−10	36	9	0	1
(Y_2) a	−4	7	−9	2	−4	16	49	81	4
(Y_3) b	−7	11	−9	−5	−10	49	121	81	25
(Y_4) ab	9	9	1	5	24	81	81	1	25
Total	−8	24	−17	1	0	182	260	163	55

Step 1 : $N=16$

Step 2 : $T=0$

Step 3 : $\frac{T^2}{N}=0$

Step 4 : $TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - \frac{T^2}{N}$

$= 182+260+163+55-0$

$= 660$

Step 5:

Contrast A $= a+ab-b-(1) = -4+24+10+10 = 40$

Contrast B $= b+ab-a-(1) = -10+24+4+10 = 28$

Contrast AB $= ab+(1)-a-(b) = 24-10+4+10 = 28$

Step 6:-

Main effect A : $\frac{1}{2n}\left[a+ab-b-(1)\right]$

$= \frac{1}{(2)(4)}[40]$

$= 5$

Main effect B $= \frac{1}{2n}\left[b+ab-a-(1)\right]$

$= \frac{1}{(2)(4)}[28]$

$= 3.5$

Main effect AB $= \frac{1}{2n}\left[ab+(1)-a-b\right]$

$= \frac{1}{(2)(4)}[28]$

$= 3.5$

Step 7:

$$SS_A = \frac{\left[a+ab-b-(1)\right]^2}{4n} = \frac{(40)^2}{4(4)} = 100$$

$$SS_B = \frac{\left[b+ab-a-(1)\right]^2}{4n} = \frac{(28)^2}{4(4)} = 49$$

$$SS_{AB} = \frac{\left[ab+(1)-a-b\right]^2}{4n} = \frac{(28)^2}{16} = 49.$$

Step 8:

$$SSE = TSS - SS_A - SS_B - SS_{AB}$$

$$= 660 - 100 - 49 - 49$$

$$= 462.$$

Step 9: ANOVA

Source of Variation	Sum of Squares	Degrees of freedom	Mean square	Variance ratio	Table F at	
					5%	1%
A	$SS_A = 100$	$a-1$ $=2-1$ $=2$	$MSS_A = \frac{SS_A}{a-1}$ $= \frac{100}{1}$ $= 100$	$MSS_A > MSSE$ $\therefore F_A = \frac{MSSA}{MSSE}$ $= \frac{100}{38.5} = 2.6$	$F_A(1,12)$ $= 4.75$	$F_A(1,12)$ $= 9.33$
B	$SS_B = 49$	$b-1$ $=2-1$ $=$	$MSS_B = \frac{SS_B}{b-1}$ $= \frac{49}{1}$ $= 49$	$F_B = \frac{MSS_B}{MSSE}$ $= \frac{49}{38.5}$ $= 1.27$	$F_B(1,12)$ $= 4.75$	$F_B(1,12)$ $= 9.33$
AB	$SSAB = 49$	$(a-1)(b-1)$ $=(1)(1)$ $=1$	$MSS_{AB} = \frac{SSAB}{(a-1)(b-1)}$ $= \frac{49}{1}$ $= 49$	$MSSAB > MSSE$ $F_{AB} = \frac{MSSAB}{MSSE}$ $= 49/38.5 = 1.27$	$F_{AB}(1,12)$ $= 4.75$	$F_{AB}(1,12)$ $= 9.33$
Error	$SSE = 462$	$ab(n-1)$ $=(2)(2)(3)$ $=12$	$MSS_E = \frac{SSE}{ab(n-1)}$ $= \frac{462}{12}$ $= 38.5$			

Step 10: conclusion:

	5%	1%
Cal F_A > Tab F_A	$2.6 < 4.75$	$2.6 < 9.33$
Cal F_B > Tab F_B	$1.27 < 4.75$	$1.27 < 9.33$
Cal F_{AB} > Tab F_{AB}	$1.27 < 4.75$	$1.27 < 9.33$

CHENNAI INSTITUTE OF TECHNOLOGY
Transforming Lives

DEPARTMENT OF SCIENCE AND HUMANITIES

SUBJECT CODE : MA3251

SUBJECT NAME : STATISTICS AND NUMERICAL METHODS

DEPARTMENT : COMMON TO ALL BRANCES

YEAR/SEM : I/II

ACEDEMIC YEAR : 2021-2022

BATCH XII

HAND WRITTEN MATERIAL

UNIT III SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

- Fixed point iteration method
- Newton Raphson method
- Gauss elimination method
- Gauss Jordan method
- Gauss Jacobi method
- Gauss Seidal method
- Matrix Inversion of Gauss Jordan method
- Power method for Eigen values and Eigen vectors
- Jacobi method for Eigen values and Eigen vectors

Unit - III

Solutions of Equations and Eigenvalue problems

Contents

- Fixed point Iteration method
- Newton - Raphson method
- Direct methods
  1. Gauss Elimination
  2. Gauss Jordan
- Indirect methods [Iterative method]
  1. Gauss Jacobi
  2. Gauss Seidal
- Matrix Inversion
  1. Gauss Jordan method
- Eigenvalues and Eigenvectors
  1. power method
  2. Jacobi method [Symmetric Matrices]

Solution of Algebraic and Transcendental Equations

An expression of the form,

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x^1 + a_n x^0$ is known as a polynomial equation in 'x' with degree 'n'. Here 'n' is a positive Integer and $a_0, a_1, a_2, \dots a_n$ are constants with $a_0 \neq 0$ and some of $a_1, a_2, \dots a_n$ may be zero. In the above equation if $f(x) = 0$, then this equation is known as an Algebraic equation of degree 'n'.

In Algebraic equation of degree 'n', 'x' in $f(x)$ occurs in an Integral form then the function is known as Rational Integral Equation. The rational Integral equation classified into two parts.

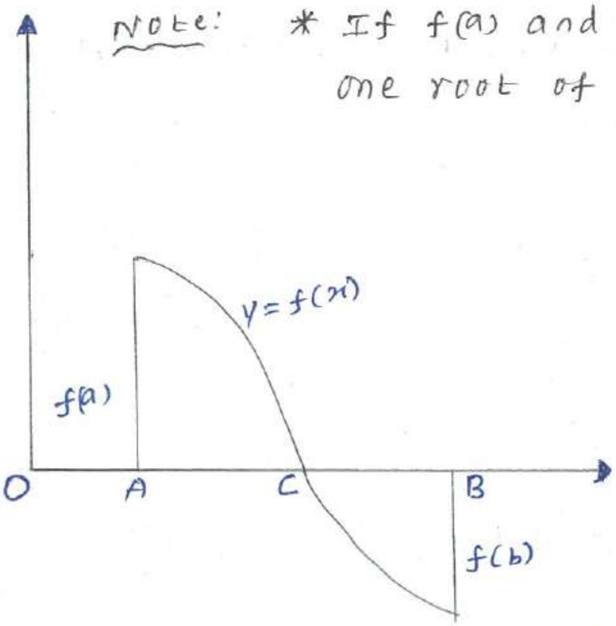
(i) Algebraic Equations.

Ex: $x^3 + 4x^2 + 2x + 1 = 0$

(ii) Transcendental Equations.

Ex: $2x - 3\cos x + 5 = 0$, $x \log_{10}^{x} - 2 + e^x = 0$.

Note: * If f(a) and f(b) have opposite signs then one root of f(x) lies between a & b *.



To solve Algebraic and Transcendental equations, we have two methods.

(i) Fixed point Iteration method
(ii) Newton-Raphson method (or) Newton method (or) method of Tangents.

Problems

1. Solve the equation $x^3+x^2-1=0$ by using iteration method. NOV/DEC 2019.

Soln:

$$\text{Let } f(x) = x^3 + x^2 - 1$$

$$f(0) = -1 \text{ (-ive)}$$

$$f(1) = 1 \text{ (+ive)}$$

∴ According to the "Note", mentioned above, The root of f(x) lies between 0 and 1.

Let's rewrite the given function f(x),

$$x^2(x+1)-1 = 0$$

$$x^2(x+1) = 1$$

$$x^2 = \frac{1}{x+1}$$

$$x = \frac{1}{\sqrt{x+1}}$$

$\therefore$ Let $g(x) = \frac{1}{\sqrt{x+1}} = x$ —— ①

$\therefore g'(x) = \frac{-1}{2(x+1)\sqrt{x+1}}$ $\qquad \because \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-1}{2x\sqrt{x}}$

$\therefore |g'(x)| = \left|\frac{1}{2(x+1)^{3/2}}\right| < 1$ in $(0,1)$

$g'(0) = \frac{1}{2} < 1$

$g'(1) = 0.1767 < 1$

$\therefore$ The Iteration method may be applied to this problem, let the Initial approximation be $x_0 = 0.5$

$\therefore$ ① $\Rightarrow$ $x_1 = g(x_0) = \frac{1}{\sqrt{x_0+1}} = \frac{1}{\sqrt{0.5+1}} = 0.81649$

$x_2 = g(x_1) = \frac{1}{\sqrt{0.81649+1}} = 0.74196$

$x_3 = g(x_2) = \frac{1}{\sqrt{0.74196+1}} = 0.75767$

$x_4 = g(x_3) = \frac{1}{\sqrt{0.75767+1}} = 0.75427$

$x_5 = g(x_4) = \frac{1}{\sqrt{0.75427+1}} = 0.75500$

$x_6 = g(x_5) = \frac{1}{\sqrt{0.75500+1}} = 0.75485$

$x_7 = g(x_6) = \frac{1}{\sqrt{0.75485+1}} = 0.75488$

The difference between x_6 and x_7 is very small.

$\therefore$ The root of the given equation is 0.75488.

2. Find a real root of the equation $\cos x = 3x - 1$ correct to 5 decimal places by fixed point Iteration method.

AU. DEC 2019, April/May 2016, 2017.

Soln:

Let $f(x) = \cos x - 3x + 1 = 0$

$f(0) = 2$ (+ive)

$f(1) = -1.4597$ (-ive)

The root lies between '0' and '1'.

Let $3x = 1 + \cos x$

$$\therefore g(x) = x = \frac{1+\cos x}{3} \quad \text{———} \quad ①$$

$$\therefore g'(x) = \frac{-\sin x}{3}$$

$$|g'(x)| = \frac{\sin x}{3}, \quad |g'(0)| = 0 < 1$$

$$|g'(1)| = 0.2804 < 1$$

Hence, the Iteration method may be applied.

Let $x_0 = 0.5$

① ⇒

$$x_1 = g(x_0) = \frac{1+\cos x_0}{3} = \frac{1+\cos(0.5)}{3} = 0.62586$$

$$x_2 = g(x_1) = \frac{1+\cos(0.62586)}{3} = 0.60349$$

$$x_3 = g(x_2) = \frac{1+\cos(0.60349)}{3} = 0.60779$$

$$x_4 = g(x_3) = \frac{1+\cos(0.60779)}{3} = 0.60697$$

$$x_5 = g(x_4) = \frac{1+\cos(0.60697)}{3} = 0.60713$$

$$x_6 = g(x_5) = \frac{1+\cos(0.60713)}{3} = 0.60710$$

$$x_7 = g(x_6) = \frac{1+\cos(0.60710)}{3} = 0.60710$$

Here $x_6 = x_7 = 0.60710$.

Hence the root is 0.60710.

3. Use the method of fixed point iteration to solve the equation $3x - \log_{10} x = 6$.

Soln:

Let $f(x) = 3x - \log_{10} x - 6$

$f(1) = 3 - 0 - 6 = -3$ (-ive)

$f(2) = -0.3010$ (-ive)

$f(3) = 2.5229$ (+ive)

The root lies between 2 & 3.

$\therefore$ Let $3x = 6 + \log_{10} x$

$$x = \frac{6 + \log_{10} x}{3} = g(x) \quad \text{———} \quad ①$$

$$\therefore \; g'(x) = \frac{1}{3}\left(\frac{\log_{10} e}{x}\right)$$

& $|g'(x)| < 1$ in the interval $(2, 3)$

Let $x_0 = 2$,

$① \Rightarrow$

$$x_1 = \frac{1}{3}\left(6 + \log_{10} x_0\right) = 2.1003$$

$$x_2 = \frac{1}{3}\left(6 + \log_{10}(2.1003)\right) = 2.1074$$

$$x_3 = \frac{1}{3}\left(6 + \log_{10}(2.1074)\right) = 2.1079$$

$$x_4 = \frac{1}{3}\left(6 + \log_{10}(2.1079)\right) = 2.1080$$

$$x_5 = \frac{1}{3}\left(6 + \log_{10}(2.1080)\right) = 2.1080$$

Hence the approximate value of the root is 2.1080.

4. Find a positive root of $3x - \sqrt{1 + \sin x} = 0$ by iteration method.

Soln:

Let $f(x) = 3x - \sqrt{1 + \sin x}$

$f(0) = -1$ (-ive)

$f(1) = 1.6429$ (+ive)

The root is lies between '0' and '1'.

Let $x = \frac{1}{3}\sqrt{1+\sin x} = g(x)$ —— ①

$g'(x) = \frac{1}{3} \frac{1}{2\sqrt{1+\sin x}} . \cos x$

$\therefore g'(0) = \frac{1}{6}(1) = 0.16 < 1$

$g'(1) = 0.066 < 1$

$\therefore |g'(x)| < 1$

Hence, we can use iteration method.

Let $x_0 = 0.4$,

$x_1 = \frac{1}{3}\sqrt{1+\sin(0.4)} = 0.39291$

$x_2 = \frac{1}{3}\sqrt{1+\sin(0.39291)} = 0.39199$

$x_3 = \frac{1}{3}\sqrt{1+\sin(x_2)} = 0.39187$

$x_4 = \frac{1}{3}\sqrt{1+\sin(x_3)} = 0.39185$

$x_5 = \frac{1}{3}\sqrt{1+\sin(x_4)} = 0.39185$

Here $x_4 = x_5 = 0.39185$

$\therefore$ The root is 0.39185.

5. Solve $x^3 - 2x - 5 = 0$ for the positive root by using iteration method.

Soln:

Let $f(x) = x^3 - 2x - 5$

$f(2) = -1$ (-ive)

$f(3) = 16$ (+ive)

$\therefore$ The root lies between 2 and 3

Let $x^3 = 2x+5$, $x = (2x+5)^{1/3} = g(x)$ —— ①

$$g'(x) = \frac{1}{3}(2x+5)^{\frac{1}{3}-1} \cdot 2$$

$$= \frac{2}{3}(2x+5)^{-2/3}$$

$$g'(x) = \frac{2}{3}\left[\frac{1}{(2x+5)^{2/3}}\right]$$

$|g'(x)| < 1$ for all x in $(2,3)$

Let $x_0 = 2$

$\therefore x_1 = (2x_0+5)^{1/3} = 2.0801$

$x_2 = (2x_1+5)^{1/3} = 2.0924$

$x_3 = (2x_2+5)^{1/3} = 2.0924$

$x_4 = (2x_3+5)^{1/3} = 2.0945$

$x_5 = (2x_4+5)^{1/3} = 2.0945$

$\therefore$ The root is 2.0945.

6. Solve by iteration method for the equation $2x - \log_{10}x = 7$.

Soln:

Let $f(x) = 2x - \log_{10}x - 7$

$f(3) = 6 - \log_{10}3 - 7 = -1.4771$ (-ive)

$f(4) = 8 - \log_{10}4 - 7 = 0.3979$ (+ive)

Hence the root lies between 3 & 4.

Let $x = \frac{1}{2}(\log_{10}x + 7) = g(x)$ —— ①

$$g'(x) = \frac{1}{2}\left(\log_{10}e \ \frac{1}{x}\right)$$

$|g'(x)| < 1$, x lies in $(3,4)$.

$\therefore$ Let $x_0 = 3.7$, $|g'(x)| < 1$ $\because \log_{10}e = 0.4343 < 1$

Let the 1st approximation, $x_0 = 3.6$

① ⇒ $x_1 = \frac{1}{2}(\log_{10}3.6 + 7) = 3.77815$

$$x_2 = \frac{1}{2}\left(\log_{10} x_1 + 7\right) = 3.78863$$

$$x_3 = \frac{1}{2}\left(\log_{10} x_2 + 7\right) = 3.78924$$

$$x_4 = \frac{1}{2}\left(\log_{10} x_3 + 7\right) = 3.78927$$

Since the difference between x_3 and x_4 is very small, the root is 3.78927.

Newton - Raphson Method

This method is used to improve the result obtained by the Fixed Iteration Method. In Newton-Raphson method to find an approximate root of an equation $f(x)=0$, use the formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots$$

Note:

* If $f'(x)$ is zero (or) nearly zero, then this method fails.

* This method is also used to obtain complex roots.

* N-R method is used to improve the results obtained by other methods.

Problems

1. Find the real positive root of $3x - \cos x - 1 = 0$ by Newton's method correct to '6' decimal places.

Soln:

Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

& $f(0) = -2$ (-ive)

$f(1) = 1.459698$ (+ive)

Hence the root is lies between '0' and '1'

Also $|f(0)| > |f(1)|$ as $2 > 1.459698$

Hence the root is nearer to 1.

∴ We can take $x_0 = 0.6$ which is nearer to '1' than '0'.

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0,1,2,3,\dots$

Put $n=0$, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0.6 - \frac{f(0.6)}{f'(0.6)}$

$= 0.6 - \frac{[3(0.6) - \cos(0.6) - 1]}{3 + \sin(0.6)}$

$\boxed{x_1 = 0.607108}$

Put $n=1$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.607108 - \frac{[3(0.607108) - \cos(0.607108) - 1]}{3 + \sin(0.607108)}$

$\boxed{x_2 = 0.607102}$

Put $n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 0.607102 - \frac{[3(0.607102) - \cos(0.60702)]}{3 + \sin(0.607102)}$

$\boxed{x_3 = 0.607102}$

Here $x_2 = x_3 = 0.607102$

∴ The root of $f(x)=0$ is 0.607102 correct to '6' Decimal places.

2. Find a root of $x\log_{10} x - 1.2 = 0$ by N-R method correct to three decimal places.

Soln:

Let $f(x) = x\log_{10} x - 1.2$

$f(1) = -1.2$ (−ive)

$f(2) = -0.598$ (−ive)

$f(3) = 0.231$ (+ive)

The root is lies between 2 and 3, also it is nearer to '3'. Let $x_0 = 2.7$.

$$f'(x) = x \frac{1}{x} \log_{10}^{e} + \log_{10}^{x} (1)$$

$$f'(x) = \log_{10}^{e} + \log_{10}^{x} \quad \text{——} \quad ①$$

WKT, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, By N-R method

Put $n=0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.7 - \frac{f(2.7)}{f'(2.7)}$$

$$= 2.7 - \frac{(2.7)\log_{10}(2.7) - 1.2}{\log_{10}^{e} + \log_{10} 2.7}$$

$$= 2.7 - \left(\frac{-0.035}{0.867}\right)$$

$$\boxed{x_1 = 2.740}$$

Put $n=1$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.740 - \frac{(2.740)\log_{10}(2.740) - 1.2}{\log_{10}^{e} + \log_{10} 2.740}$$

$$= 2.740 - \left(\frac{-0.0006}{0.872}\right)$$

$$\boxed{x_2 = 2.741}$$

Put $n=2$,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.741 - \frac{f(2.741)}{f'(2.741)}$$

$$= 2.741 - \left[\frac{(2.741)\log_{10}(2.741) - 1.2}{\log_{10} e + \log_{10} 2.741}\right]$$

$$= 2.741 - \left(\frac{0.003}{0.872}\right)$$

$$\boxed{x_3 = 2.741}$$

Here $x_2 = x_3 = 2.741$, ∴ The root is 2.741.

3. Find the positive root of $f(x) = 2x^3 - 3x - 6$ by N-R method correct to 5 decimal places.

Soln:

Let $f(x) = 2x^3 - 3x - 6$

∴ $f(1) = -7$ (-ive)

$f(2) = 4$ (+ive)

∴ The root is lies between 1 and 2.

∴ Let $x_0 = 2$

Now, $f'(x) = 2(3x^2) - 3 = 6x^2 - 3$

Let $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Put $n=0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{4}{21}$$

$$\boxed{x_1 = 1.809524}$$

Put $n=1$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.809524 - \left[\frac{2(1.809524)^3 - 3(1.809524) - 6}{6(1.809524)^2 - 3}\right]$

$= 1.809524 - \left(\frac{0.421556}{16.646263}\right)$

$\boxed{x_2 = 1.784200}$

Put $n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 1.784200 - \left[\frac{2(1.784200)^3 - 3(1.784200) - 6}{6(1.784200)^2 - 3}\right]$

$= 1.784200 - \left[\frac{0.006936}{16.100218}\right]$

$\boxed{x_3 = 1.783769}$

Put $n=3$, $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$

$= 1.783769 - \left[\frac{2(1.783769)^3 - 3(1.783769) - 6}{6(1.783769)^2 - 3}\right]$

$= 1.783769 - \left[\frac{-0.000001}{16.090991}\right]$

$\boxed{x_4 = 1.783769}$

$\therefore$ $x_3 = x_4 = 1.783769$

$\therefore$ The root is 1.783769.

4. Evaluate $\sqrt{12}$ to three decimal places using Newton-Raphson method.

Soln:

Let $x = \sqrt{12}$

$\therefore \quad x^2 - 12 = 0$

$f(x) = x^2 - 12$

$f(1) = -11$

$f(2) = -8$

$f(3) = -3$ (-ive)

$f(4) = 4$ (+ive)

The root lies between 3 & 4.

Let $x_0 = 4$

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Put $n=0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 4 - \frac{f(4)}{f'(4)}$$

$$= 4 - \frac{4}{2 \cdot 4}$$

$$\boxed{x_1 = 3.5}$$

Put $n=1$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3.5 - \frac{f(3.5)}{f'(3.5)}$$

$$= 3.5 - \frac{0.25}{7}$$

$$\boxed{x_2 = 3.46429}$$

Put $n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$$= 3.46429 - \frac{f(3.46429)}{f'(3.46429)}$$

$$= 3.46429 - \frac{0.0013052}{6.92858}$$

$$\boxed{x_3 = 3.464102}$$

Hence the root is repeating upto three decimal places.

Hence $x_3 = 3.464102$ is the required root.

5. Find the iterative formula for finding the value of $\frac{1}{N}$ where 'N' is a real number, using N-R method. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Soln:

Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x}$

$$(i.e)\ \frac{1}{x} - N = 0$$

$$\therefore\ f(x) = \frac{1}{x} - N$$

$$f'(x) = \frac{-1}{x^2}$$

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \left[\frac{\frac{1}{x_n} - N}{\frac{-1}{x_n^2}}\right]$$

$$= x_n + x_n^2\left(\frac{1}{x_n} - N\right)$$

$$= x_n + x_n - Nx_n^2$$

$$= 2x_n - Nx_n^2$$

$$\boxed{x_{n+1} = x_n(2 - Nx_n)} \text{ ——— } ①$$

This is the iterative formula.

Let's find $\frac{1}{26}$. Take $N = 26$

Let $x_0 = 0.04$ $\qquad \because \frac{1}{25} = 0.04$

① ⇒ $x_{n+1} = x_n(2 - Nx_n)$

Put $n=0$

$x_1 = x_0(2 - Nx_0)$

$= 0.04(2 - 26(0.04))$

$\boxed{x_1 = 0.0384}$

Put $n=1$,

$x_2 = x_1(2 - Nx_1)$

$= 0.0384(2 - 26(0.0384))$

$\boxed{x_2 = 0.0385}$

Hence the value of $\frac{1}{26}$ is 0.0385.

6. Use the Newton's iterative formula to find $\sqrt{N}$ where N is a positive real number. Hence evaluate $\sqrt{142}$.

Soln:

Let $x = \sqrt{N}$

$x^2 - N = 0$

$\therefore f(x) = x^2 - N.$

$f'(x) = 2x$

Formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \left(\frac{x_n^2 - N}{2x_n}\right)$

$= x_n - \frac{x_n^2}{2x_n} + \frac{N}{2x_n} = x_n - \frac{x_n}{2} + \frac{N}{2x_n}$

$x_{n+1} = \frac{x_n}{2} + \frac{N}{2x_n}$

$$\therefore \boxed{x_{n+1} = \frac{1}{2}\; x_n + \frac{N}{x_n}} \text{ ——— } ①$$

To find $\sqrt{142}$:

Let $x_0 = 12$, since 12 is very near to $\sqrt{142}$. Put $n=0$,

$$① \Rightarrow \quad x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right)$$

$$= \frac{1}{2}\left(12 + \frac{142}{12}\right)$$

$$\boxed{x_1 = 11.9167}$$

Put $n=1$,

$$x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right)$$

$$= \frac{1}{2}\left(11.9167 + \frac{142}{11.9167}\right)$$

$$\boxed{x_2 = 11.9164}$$

Put $n=2$,

$$x_3 = \frac{1}{2}\left(x_2 + \frac{N}{x_2}\right)$$

$$= \frac{1}{2}\left(11.9164 + \frac{142}{11.9164}\right)$$

$$\boxed{x_3 = 11.9164}$$

Here $x_2 = x_3 = 11.9164$.

Hence $\sqrt{142} = 11.9164$, correct to '4' decimal places.

7. Derive Newton's Algorithm for finding the p^{th} root of a Number N.

Soln: Let $x = N^{1/p}$ [p^{th} root of N]

$$\therefore x^p = N. \Rightarrow \boxed{f(x) = x^p - N}, \; f'(x) = p x^{p-1}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow x_n - \left(\frac{x_n^p - N}{p x_n^{p-1}}\right) \Rightarrow \frac{p x_n^p - x_n^p + N}{p x_n^{p-1}}$$

$$\therefore x_{n+1} = \frac{(p-1)x_n^p + N}{p x_n^{p-1}}$$

Solution of Linear system by Gauss Elimination and Gauss Jordan Methods.

There are two types to solve simultaneous linear Algebraic equations.

- * Direct method [Exact method]
- * Indirect method [Iterative method]

There are two methods under direct methods, namely

- * Gauss Elimination method.
- * Gauss - Jordan method.

There are two methods under Indirect methods also, namely

- * Gauss Jacobi method
- * Gauss Seidal method

Gauss Elimination method

Let consider the 'n' linear equations with 'n' unknowns.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right] \text{——} ①$$

where a_{ij}'s and b_i's are constants and x_i's are variables. The system ① is equivalent to the matrix equation $AX = B$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \& \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Here the matrix $[A, B]$ is the augumented matrix and we have to reduce it to upper triangular matrix by using row and column operations.

Gauss – Jordan method

In this method, we make the matrix A not only upper triangular matrix, but also lower triangular matrix, thereby the matrix A is brought to a diagonal matrix or unit matrix.

1. Solve the system of equations by (i) Gauss elimination method (ii) Gauss-Jordan method. $x+2y+z=3$; $2x+3y+3z=10$; $3x-y+2z=13$.

Soln:

(i) Gauss Elimination method

The given system of equations is equivalent to AX=B.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 3 \end{bmatrix}$$

The Augmented matrix is given by

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_3 \to R_3 - 7R_2$$

By back substitution method, we get

$$-8z = -24, \quad \boxed{z=3}$$

$$-y+z = 4, \quad -y+3=4, \quad \boxed{y=-1}$$

$$x+2y+z = 3, \quad x-2+3 = 3, \quad \boxed{x=2}$$

∴ $x=2$, $y=-1$ & $z=3$ is the required solution.

Verification: $x+2y+z=3$ (Given) $\Rightarrow 2-2+3=3$ ✓, Verified ✓

Gauss - Jordan method

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right] \quad R_1 \to 2R_2 + R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad R_3 \to \frac{1}{8} R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad \begin{array}{l} R_1 \to 3R_3 + R_1 \\ R_2 \to R_3 + R_2 \end{array}$$

By comparing, we have $-z = -3$, $\boxed{z = 3}$, $-y = 1$, $\boxed{y = -1}$ & $\boxed{x = 2}$. ∴ $x = 2$, $y = -1$ & $z = -3$ is the required solution.

2. Solve the system of the following equations using Gauss Jordan method connect to two decimal places.

$$2x_1 + 2x_2 - x_3 + x_4 = 04$$
$$4x_1 + 3x_2 - x_3 + 2x_4 = 6$$
$$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12$$
$$3x_1 + 3x_2 - 2x_3 + 2x_4 = 6$$

Soln:

The augmented matrix is given by,

$$[A, B] = \left[\begin{array}{cccc|c} 2 & 2 & -1 & 1 & 4 \\ 4 & 3 & -1 & 2 & 6 \\ 8 & 5 & -3 & 4 & 12 \\ 3 & 3 & -2 & 2 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & -1/2 & 1/2 & 2 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & -3 & 1 & 0 & -4 \\ 0 & 0 & -1/2 & 1/2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/2 \\ R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 8R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & -1/2 & 1/2 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \Rightarrow -R_2 \\ R_2 \to R_1 - R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1/2 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 1/2R_3 \\ R_2 \to R_2 + R_3 \\ R_3 \to R_3/-2 \\ R_4 \to R_4 + 1/2R_3 \end{array} \quad \begin{array}{l} R_3 \to R_3/-2. \\ R_1 \to R_1 - 1/2R_3 \\ R_2 \to R_2 + R_3 \\ R_4 \to R_4 + \frac{1}{2R_3} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

By comparing, we have $x_1 = 1$, $x_2 = 1$, $x_3 = -1$ & $x_4 = -1$

2. Using Gauss-Jordan method, solve the following system of equations

$$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ x + y + 5z &= 7 \end{aligned}$$

Soln:

The given system of equations can be written in the matrix form as $AX = B$.

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right] \to \text{[The augmented form]}.$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right] \quad R_3 \leftrightarrow R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 10R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{array}\right] \quad R_2 \rightarrow \frac{R_2}{8}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 1 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{array}\right] \quad \begin{array}{l} R_1 \rightarrow R_1 - R_2 \\ R_3 \rightarrow R_3 + 9R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3 \rightarrow \frac{-R_3}{59.125}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \rightarrow R_1 - 6.125R_3 \\ R_1 \rightarrow R_2 + 1.125R_3 \end{array}$$

∴ By comparing we have $x=1, \; y=1, \; z=1$.

4. Solve $x+3y+3z=16$, $x+4y+3z=18$, $x+3y+4z=19$ by Gauss-Jordan method.

Soln:

The Augmented matrix is given by,

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \rightarrow R_1 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_1 \rightarrow R_1 - 3R_3$$

Hence $x=1, \; y=2, \; z=3$.

5. Solve the system of equations by (i) Gauss Elimination method (ii) Gauss - Jordan method.

Soln:

(i) Gauss Elimination method

The augmented matrix is given by,

$$[A \mid B] = \left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{array}\right] \quad \begin{array}{l} R_2 \to 5R_2 - R_1 \\ R_3 \to 10R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{array}\right] \quad R_3 \to 52R_3 + 34R_2$$

This is an upper triangular matrix. So back substitution we get,

$$3780z = 11340 \qquad , \qquad 52y - 28z = -188$$

$$\boxed{z = 3} \qquad\qquad 52y - 28(3) = -188$$

$$\boxed{y = -2}$$

$$10x - 2y + 3z = 23$$

$$10x - 2(-2) + 3(3) = 23$$

$$\boxed{x = 1}$$

$\therefore$ $x = 1$, $y = -2$, $z = 3$.

(ii) Gauss - Jordan method

$$[A \mid B] \sim \left[\begin{array}{ccc|c} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & 14040 \\ 0 & 0 & 3780 & 11340 \end{array}\right] \quad \begin{array}{l} R_1 \to 1260R_1 - R_3 \\ R_2 \to 135R_2 + R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 88458000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{array}\right] \quad R_1 \rightarrow 7020R_1 + 2520R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

By comparing, we have $x=1,\ y=-2,\ z=3.$

6. Solve the following system of equations by Gauss-Jordan method: $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$; $x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$.

Soln:

We can interchange the 1st and 4th equations, so that the co-efficient of x_1 in the 1st equation is 1.

$$[A, B] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right] \quad R_2 \rightarrow \frac{R_2}{6}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right] \quad \begin{array}{l} R_1 \rightarrow R_1 - R_2 \\ R_4 \rightarrow R_4 + 4R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 4 & -21 & 46 \end{array}\right] \quad R_3 \to \frac{R_3}{5}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -23.4 & 46.8 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_4 \to R_4 + 4R_3 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \quad R_4 \to \frac{R_4}{23.4}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 5.1R_4 \\ R_2 \to R_2 + 0.5R_4 \\ R_3 \to R_3 + R_4\left(\frac{-3}{5}\right) \end{array}$$

By comparing, we have, $x_1 = 1$, $x_2 = 2$, $x_3 = -1$ & $x_4 = -2$.

Inverse of a matrix by using Gauss-Jordan method.

* Let's consider the augmented matrix $[A, I]$, where I is the identity matrix of the same order as A. By applying row operations, we can reduce A into a unit matrix and I will be changed into a matrix X.

* This matrix X is the inverse of the given matrix A. It is advisable to change the pivotal element as unity before applying the row operations at each and every step.

1. Find the inverse of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$ by using Gauss-Jordan method.

Soln:

The Augmented Matrix is

$$[A, I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 7 & -3 & 2 & 0 \\ 0 & -2 & -1 & -1 & 0 & 2 \end{array}\right] \begin{array}{l} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 7 & -3 & 2 & 0 \\ 0 & 0 & 5 & -5 & 2 & 4 \end{array}\right] R_3 \to 2R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & 10 & -2 & -4 \\ 0 & 20 & 0 & 20 & -4 & -28 \\ 0 & 0 & 5 & -5 & 2 & 4 \end{array}\right] \begin{array}{l} R_1 \to 5R_1 - R_3 \\ R_2 \to 5R_2 - 7R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/5 & -2/5 \\ 0 & 1 & 0 & 1 & -1/5 & -7/5 \\ 0 & 0 & 1 & -1 & 2/5 & 4/5 \end{array}\right] \begin{array}{l} R_1 \to \frac{R_1}{10} \\ R_2 \to R_2/20 \\ R_3 \to R_3/5 \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1/5 & -2/5 \\ 1 & -1/5 & -7/5 \\ -1 & 2/5 & 4/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & -1 & -2 \\ 5 & -1 & -7 \\ -5 & 2 & 4 \end{bmatrix}$$

Verification:

$$AA^{-1} = I$$

$$AA^{-1} = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 5 & -1 & -2 \\ 5 & -1 & -7 \\ -5 & 2 & 4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Using Gauss - Jordan method, find the inverse of the matrix $\begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$

Soln:

The Augmented matrix [A, I] is given by,

$$[A, I] = \left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \quad R_1 \to \frac{R_1}{2}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 2 & 7/2 & -1/2 & 0 & 1 \end{array}\right) \quad R_2 \to -R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -1/2 & -5/2 & 2 & 1 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right) \quad R_3 \to -2R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -9 & 7 & 4 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 + \frac{R_3}{2} \\ R_2 \to R_2 - 2R_3 \end{array}$$

Hence $A^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{pmatrix}$

Verification: $AA^{-1} = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. Find the Inverse of the matrix $\begin{pmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{pmatrix}$ using Gauss - Jordan method.

Soln:

The augmented matrix is given by,

$$[A, I] = \left(\begin{array}{ccc|ccc} 8 & -4 & 0 & 1 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/8 & 0 & 0 \\ -4 & 8 & -4 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right) \quad R_1 \to \frac{R_1}{8}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/8 & 0 & 0 \\ 0 & 6 & -4 & 1/2 & 1 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right) \quad R_2 \to R_2 + 4R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1/8 & 0 & 0 \\ 0 & 1 & -2/3 & 1/12 & 1/6 & 0 \\ 0 & -4 & 8 & 0 & 0 & 1 \end{array}\right) \quad R_2 \to \frac{R_2}{6}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 1/6 & 1/12 & 0 \\ 0 & 1 & -2/3 & 1/12 & 1/6 & 0 \\ 0 & 0 & 16/3 & 1/3 & 2/3 & 1 \end{array}\right) \quad R_3 \to \cancel{\frac{8}{16}} R_3 + 4R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 1/6 & 1/12 & 0 \\ 0 & 1 & -2/3 & 1/12 & 1/6 & 0 \\ 0 & 0 & 1 & 1/16 & 1/8 & 3/16 \end{array}\right) \quad R_3 \to \frac{3}{16} R_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} + \frac{1}{48} & \frac{1}{12} + \frac{1}{24} & \frac{3}{16}\left(\frac{1}{3}\right) \\ 0 & 1 & 0 & \frac{1}{12} + \frac{2}{48} & \frac{1}{6} + \frac{2}{24} & \frac{3}{6}\left(\frac{2}{3}\right) \\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16} \end{array}\right)$$

$R_1 \to R_1 + \frac{1}{3} R_3$

$R_2 \to R_2 + \frac{2}{3} R_3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/16 & 1/8 & 1/16 \\ 0 & 1 & 0 & 1/8 & 1/4 & 1/8 \\ 0 & 0 & 1 & 1/16 & 1/8 & 3/16 \end{array}\right)$$

Hence, the inverse of the given matrix is,

$$A^{-1} = \begin{pmatrix} 3/16 & 1/8 & 1/16 \\ 1/8 & 1/4 & 1/8 \\ 1/16 & 1/8 & 3/16 \end{pmatrix}$$

Verification: $AA^{-1} = I$

$$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{pmatrix} \begin{pmatrix} 3/16 & 1/8 & 1/16 \\ 1/8 & 1/4 & 1/8 \\ 1/16 & 1/8 & 3/16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Find the Inverse of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ by using Gauss - Jordan method.

Soln:

The augmented matrix is given by

$$[A, I] = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1/2 & 3/2 & -3/2 & 1 & 0 \\ 0 & 7/2 & 17/2 & -1/2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{3}{2} R_1 \\ R_3 \to R_3 - \frac{1}{2} R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1/2 & 3/2 & -3/2 & 1 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \quad R_3 \to R_3 - 7R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 6 & -17/4 & 3/4 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \quad R_2 \to R_2 + \frac{3}{4} R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 6 & -7/2 & 1/2 \\ 0 & 1/2 & 0 & 6 & -17/4 & 3/4 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \quad R_1 \to R_1 + \frac{1}{2} R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -6 & 5 & -1 \\ 0 & 1/2 & 0 & 6 & -17/4 & 3/4 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 5/2 & -1/2 \\ 0 & 1 & 0 & 12 & -17/2 & 3/2 \\ 0 & 0 & 1 & -5 & 7/2 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/2 \\ R_2 \to 2R_2 \\ R_3 \to -\frac{R_2}{2} \end{array}$$

Hence the inverse of the given matrix is,

$$A^{-1} = \begin{bmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{bmatrix}$$

Verification: $AA^{-1} = I$

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Iterative Methods

The types of iterative methods are

(i) Gauss – Jacobi method
(ii) Gauss – seidal method

To apply iterative methods successfully, each equation of the system must possess one large co-efficient and the large co-efficient must be attached to a different unknown in that equation.

Consider the system of equations,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

These equations will be solvable by Iterative method If,

$$|a_1| > |b_1| + |c_1|$$
$$|b_2| > |a_2| + |c_2| \quad \Rightarrow \underline{\text{Diagonally Dominant}}$$
$$|c_3| > |a_3| + |b_3|$$

Then Iterative method can be used for the above system of equations. [Gauss – Jacobi]

i.e., $$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$
$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad \text{———} \ ①$$
$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

If $x^{(0)}, y^{(0)}, z^{(0)}$ are the initial values of x, y, z respectively, then

$$x^{(1)} = \frac{1}{a_1}[d_1 - b_1 y^{(0)} - c_1 z^{(0)}]$$
$$y^{(1)} = \frac{1}{b_2}[d_2 - a_2 x^{(0)} - c_2 z^{(0)}] \quad \text{———} \ ②$$
$$z^{(1)} = \frac{1}{c_3}[d_3 - a_3 x^{(0)} - b_3 y^{(0)}]$$

Again using these values $x^{(1)}, y^{(1)}, z^{(1)}$ in ② we get,

$$x^{(2)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(1)} - c_1 z^{(1)}\right]$$

$$y^{(2)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(1)} - c_2 z^{(1)}\right]$$

$$z^{(2)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right]$$

Continuing in the same way, if the r^{th} iterates are $x^{(r)}, y^{(r)}, z^{(r)}$ the iteration scheme reduces to,

$$x^{(r+1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(r)} - c_1 z^{(r)}\right]$$

$$y^{(r+1)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(r)} - c_2 z^{(r)}\right]$$

$$z^{(r+1)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(r)} - b_3 y^{(r)}\right]$$

This procedure is continued till the convergence is assured [correct to required decimals].

The refinement of Gauss - Jacobi method is called Gauss - seidal method.

* To find the value of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the r^{th} iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}\left[d_1 - b_1 y^{(r)} - c_1 z^{(r)}\right]$$

$$y^{(r+1)} = \frac{1}{b_2}\left[d_2 - a_2 x^{(r+1)} - c_2 z^{(r)}\right]$$

$$z^{(r+1)} = \frac{1}{c_3}\left[d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)}\right]$$

This Process of iteration is continued untill the convergence is confirmed. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss-seidal method is very fast when compared to Gauss-Jacobi method and this method is roughly two times faster than that of Gauss-Jacobi method.

Problems

1. Solve the following system of equations by Gauss-Jacobi method and Gauss-seidal method.

$27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$.

soln:

As the co-efficient matrix is not diagonally dominant, we write the equations,

$$\textcircled{27}x + 6y - z = 85$$
$$6x + \textcircled{15}y + 2z = 72$$
$$x + y + \textcircled{54}z = 110$$

Here $|27| > |6| + |1|$

$|15| > |6| + |2|$

$|54| > |1| + |1|$

∴ Now, the diagonal elements are dominant in the co-efficient matrix, we write x, y, z as follows:

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

(i) Gauss - Jacobi method

Let the initial values be $x=0, y=0, z=0$

First Iteration:

$$x^{(1)} = \frac{1}{27}(85) = 3.148$$

$$y^{(1)} = \frac{1}{15}(72) = 4.8$$

$$z^{(1)} = \frac{1}{54}(110) = 2.037$$

Second Iteration

$$x^{(2)} = \frac{1}{27}\left[85 - 6y^{(1)} + z^{(1)}\right]$$

$$= \frac{1}{27}\left[85 - 6(4.8) + 2.037\right]$$

$$\boxed{x^{(2)} = 2.157}$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6x^{(1)} - 2z^{(1)}\right]$$

$$= \frac{1}{15}\left[72 - 6(3.148) - 2(2.037)\right]$$

$$\boxed{y^{(2)} = 3.269}$$

$$z^{(2)} = \frac{1}{54}\left[110 - x^{(1)} - y^{(1)}\right]$$

$$= \frac{1}{54}\left[110 - 3.148 - 4.8\right]$$

$$\boxed{z^{(2)} = 1.889}$$

Third Iteration

$$x^{(3)} = \frac{1}{27}\left[85 - 6y^{(2)} - z^{(2)}\right]$$

$$= \frac{1}{27}\left[85 - 6(3.269) - 1.889\right]$$

$$\boxed{x^{(3)} = 2.492}$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(2)} + 2z^{(2)}\right]$$

$$= \frac{1}{15}\left[72 - 6(2.157 - 2(1.889))\right]$$

$$\boxed{y^{(3)} = 3.685}$$

$$z^{(3)} = \frac{1}{54}\left[110 - x^{(2)} - y^{(2)}\right]$$

$$= \frac{1}{54}(110 - 2.157 - 3.269)$$

$$\boxed{z^{(3)} = 1.937}$$

Fourth Iteration:

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right]$$

$$= \frac{1}{27}(85 - 6(3.685) + 1.937)$$

$$\boxed{x^{(4)} = 2.401}$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(3)} + 2z^{(3)}\right]$$

$$= \frac{1}{15}\left[72 - 6(2.492) + 2(1.937)\right]$$

$$\boxed{y^{(4)} = 3.545}$$

$$z^{(4)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right]$$

$$= \frac{1}{54}\left[110 - 2.492 - 3.685\right]$$

$$\boxed{z^{(4)} = 1.923}$$

Fifth Iteration

$$x^{(5)} = \frac{1}{27}\left[85 - 6y^{(4)} - z^{(4)}\right]$$

$$= \frac{1}{27}\left[85 - 6(3.545) - 1.923\right]$$

$$\boxed{x^{(5)} = 2.432}$$

$$y^{(5)} = \frac{1}{15}\left[72 - 6\,x^{(4)} - 2z^{(4)}\right]$$

$$= \frac{1}{15}\left[72 - 6\,(2.401) - 2(1.923)\right]$$

$$\boxed{y^{(5)} = 3.583}$$

$$z^{(5)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right]$$

$$= \frac{1}{54}\left(110 - 2.401 - 3.545\right)$$

$$\boxed{z^{(5)} = 1.927}$$

Sixth Iteration

$$x^{(6)} = \frac{1}{27}\left[85 - 6\,y^{(5)} + z^{(5)}\right]$$

$$= \frac{1}{27}\left(85 - 6\,(3.583) + 1.927\right)$$

$$\boxed{x^{(6)} = 2.423}$$

$$y^{(6)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(5)}\right]$$

$$= \frac{1}{15}\left(72 - 6\,(2.432) - 2(1.927)\right)$$

$$\boxed{y^{(6)} = 3.570}$$

$$z^{(6)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right]$$

$$= \frac{1}{54}\left(110 - 2.432 - 3.583\right)$$

$$\boxed{z^{(6)} = 1.926}$$

Seventh Iteration

$$x^{(7)} = \frac{1}{27}\left(85 - 6\,y^{(6)} + z^{(6)}\right)$$

$$= \frac{1}{27}\left(85 - 6\,(3.570) + 1.926\right.$$

$$\boxed{x^{(7)} = 2.426}$$

$$y^{(7)} = \frac{1}{15}\left(72 - 6x^{(6)} - 2z^{(6)}\right)$$

$$= \frac{1}{15}\left(72 - 6(2.423) - 2(1.926)\right)$$

$$\boxed{y^{(7)} = 3.574}$$

$$z^{(7)} = \frac{1}{54}\left(110 - x^{6} - y^{(6)}\right)$$

$$= \frac{1}{54}\left(110 - 2.423 - 3.570\right)$$

$$\boxed{z^{(7)} = 1.926}$$

Eighth Iteration

$$x^{(8)} = \frac{1}{27}\left(85 - 6y^{(7)} + z^{(7)}\right)$$

$$= \frac{1}{27}\left(85 - 6(3.574) + 1.926\right)$$

$$\boxed{x^{(8)} = 2.425}$$

$$y^{(8)} = \frac{1}{15}\left(72 - 6x^{(8)} - 2z^{(7)}\right)$$

$$= \frac{1}{15}\left(72 - 6(2.425) - 2(1.926)\right)$$

$$\boxed{y^{(8)} = 3.573}$$

$$z^{(8)} = \frac{1}{54}\left(110 - x^{7} - y^{7}\right)$$

$$= \frac{1}{54}\left(110 - 2.425 - 3.574\right)$$

$$\boxed{z^{(8)} = 1.926}$$

Ninth Iteration

$$x^{(9)} = \frac{1}{27}\left(85 - 6y^{(8)} - z^{(8)}\right)$$

$$= \frac{1}{27}\left(85 - 6(3.573) - 1.926\right)$$

$$\boxed{x^{(9)} = 2.425}$$

$$y^{(9)} = \frac{1}{15}\left(72 - 6x^{(8)} - 2z^{(8)}\right)$$

$$= \frac{1}{15}\left(72 - 6(2.425) - 2(1.926)\right)$$

$$\boxed{y^{(9)} = 3.573}$$

$$z^{(9)} = \frac{1}{54}\left(110 - x^{(8)} - y^{(8)}\right)$$

$$= \frac{1}{54}\left(110 - 2.425 - 3.573\right)$$

$$\boxed{z^{(9)} = 1.926}$$

Eighth and Ninth iteration coinsides.

Hence, $x = 2.425$, $y = 3.573$, $z = 1.926$. [correct to 3 Decimal places].

Gauss-seidal method

Let the initial values be $y=0, z=0$

First Iteration

$$x^{(1)} = \frac{1}{27}\left[85 - 6y^{(0)} + z^{(0)}\right]$$

$$= \frac{1}{27}(85)$$

$$\boxed{x^{(1)} = 3.148}$$

$$y^{(1)} = \frac{1}{15}\left[72 - 6x^{(1)} - 2z^{(0)}\right]$$

$$= \frac{1}{15}\left[72 - 6(3.148) - 2(0)\right]$$

$$\boxed{y^{(1)} = 3.541}$$

$$z^{(1)} = \frac{1}{54}\left[110 - x^{(1)} - y^{(1)}\right]$$

$$= \frac{1}{54}\left[110 - 3.148 - 3.541\right)$$

$$\boxed{z^{(1)} = 1.913}$$

Second Iteration

$$x^{(2)} = \frac{1}{27}\left[85 - 6y^{(1)} + z^{(1)}\right]$$

$$= \frac{1}{27}\left(85 - 6(3.541) + 1.913\right)$$

$$\boxed{x^{(2)} = 2.432}$$

$$y^{(2)} = \frac{1}{15}\left(72 - 6x^{(2)} - 2z^{(1)}\right)$$

$$= \frac{1}{15}\left(72 - 6(2.432) - 2(1.913)\right)$$

$$\boxed{y^{(2)} = 3.572}$$

$$z^{(2)} = \frac{1}{54}\left(110 - x^{(2)} - y^{(2)}\right)$$

$$= \frac{1}{54}\left(110 - 2.432 - 3.572\right)$$

$$\boxed{z^{(2)} = 1.926}$$

Third Iteration

$$x^{(3)} = \frac{1}{27}\left[85 - 6\,y^{(2)} - z^{(2)}\right]$$

$$= \frac{1}{27}\left(85 - 6(3.572) - 1.926\right)$$

$$\boxed{x^{(3)} = 2.426}$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(2)}\right]$$

$$= \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right]$$

$$\boxed{y^{(3)} = 3.573}$$

$$z^{(3)} = \frac{1}{54}\left(110 - x^{(3)} - y^{(3)}\right)$$

$$= \frac{1}{54}\left(110 - 2.426 - 3.573\right)$$

$$\boxed{z^{(3)} = 1.926}$$

Fourth Iteration

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right]$$

$$= \frac{1}{27}\left(85 - 6(3.573) + 1.926\right)$$

$$\boxed{x^{(4)} = 2.425}$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(3)}\right]$$

$$= \frac{1}{15}\left(72 - 6(2.425) - 2(1.926)\right)$$

$$\boxed{y^{(4)} = 3.573}$$

$$z^{(4)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right]$$

$$= \frac{1}{54}\left(110 - 2.425 - 3.573\right)$$

$$\boxed{z^{(4)} = 1.926}$$

Fifth Iteration:

$$x^{(5)} = \frac{1}{27}\left(85 - 6y^{(4)} + z^{(4)}\right)$$

$$= \frac{1}{27}\left(85 - 6(3.573) + 1.926\right)$$

$$\boxed{x^{(5)} = 2.425}$$

$$y^{(5)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(4)}\right]$$

$$= \frac{1}{15}\left[72 - 6(2.425) - 2(1.926)\right]$$

$$\boxed{y^{(5)} = 3.573}$$

$$z^{(5)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right]$$

$$= \frac{1}{54}\left(110 - 2.425 - 3.573\right)$$

$\boxed{z^{(5)} = 1.926}$, Fourth and Fifth iteration values are same.

This shows that the convergence is rapid in Gauss-seidal method when compared to Gauss-Jacobi method.

2. Solve the following equations by Gauss-seidal method

$①x + 2y + z = 14$; $x + ⑤y - z = 10$; $x + y + ⑧z = 20$.

Soln:

As the co-efficient matrix is diagonally dominant, solving for x, y, z we get,

$$x = \frac{1}{4}(14 - 2y - z)$$

$$y = \frac{1}{5}(10 - x + z)$$

$$z = \frac{1}{8}(20 - x - y)$$

Let the initial values be $y=0$, $z=0$

First Iteration

$$x^{(1)} = \frac{1}{4}\left(14 - 2(0) - (2)(0)\right)$$

$$\boxed{x^{(1)} = 3.5}$$

$$y^{(1)} = \frac{1}{5}\left(10 - x^{(1)} + z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{5}(10 - 3.5 + 0)$$

$$\boxed{y^{(1)} = 1.3}$$

$$z^{(1)} = \frac{1}{8}\left(20 - x^{(1)} - y^{(1)}\right)$$

$$= \frac{1}{8}(20 - 3.5 - 1.3)$$

$$\boxed{z^{(1)} = 1.9}$$

Second Iteration

$$x^{(2)} = \frac{1}{4}\left[14 - 2y^{(1)} - z^{(1)}\right]$$

$$= \frac{1}{4}\left[14 - 2(1.3) - (1.9)\right]$$

$$\boxed{x^{(2)} = 2.38}$$

$$y^{(2)} = \frac{1}{5}\left[10 - x^{(2)} + z^{(1)}\right]$$

$$= \frac{1}{5}(10 - 2.38 + 1.9)$$

$$\boxed{y^{(2)} = 1.90}$$

$$z^{(2)} = \frac{1}{8}\left(20 - x^{(2)} - y^{(2)}\right)$$

$$= \frac{1}{8}(20 - 2.38 - 1.90)$$

$$\boxed{z^{(2)} = 1.97}$$

Third Iteration

$$x^{(3)} = \frac{1}{4}\left(14 - 2y^{(2)} - z^{(2)}\right)$$

$$= \frac{1}{4}(14 - 2(1.90) - 1.97)$$

$$\boxed{x^{(3)} = 2.06}$$

$$y^{(3)} = \frac{1}{5}\left(10 - x^{(3)} + z^{(2)}\right)$$

$$= \frac{1}{5}(10 - 2.06 + 1.97)$$

$$\boxed{y^{(3)} = 1.98}$$

$$z^{(3)} = \frac{1}{8}\left(20 - x^{(3)} - y^{(3)}\right)$$

$$= \frac{1}{8}(20 - 2.06 - 1.98)$$

$$\boxed{z^{(3)} = 2}$$

Fourth Iteration:

$$x^{(4)} = \frac{1}{4}\left(14 - 2y^{(3)} - z^{(3)}\right)$$

$$= \frac{1}{4}(14 - 2(1.98) - (2))$$

$$\boxed{x^{(4)} = 2.01}$$

$$y^{(4)} = \frac{1}{5}\left(10 - x^{(4)} + z^{(3)}\right)$$

$$= \frac{1}{5}(10 - 2.01 + 2)$$

$$\boxed{y^{(4)} = 2}$$

$z^{(4)} = \frac{1}{8}\left(20 - x^{(4)} - y^{(4)}\right)$

$= \frac{1}{8}\left(20 - 2.01 - 2\right)$

$\boxed{z^{(4)} = 2}$

Fifth Iteration

$x^{(5)} = \frac{1}{4}\left(14 - 2y^{(4)} - z^{(4)}\right)$

$= \frac{1}{4}\left(14 - 2(2) - 2\right)$

$\boxed{x^{(5)} = 2}$

$y^{(5)} = \frac{1}{5}\left(10 - x^{(5)} + z^{(4)}\right)$

$= \frac{1}{5}\left(10 - 2 + 2\right)$

$\boxed{y^{(5)} = 2}$

$z^{(5)} = \frac{1}{8}\left(20 - x^{(5)} - y^{(5)}\right)$

$= \frac{1}{8}\left(20 - 2 - 2\right)$

$\boxed{z^{(5)} = 2}$

Sixth Iteration

$x^{(6)} = \frac{1}{4}\left(14 - 2y^{(5)} - z^{(5)}\right)$

$= \frac{1}{4}\left(14 - 2(2) - (2)\right)$

$\boxed{x^{(6)} = 2}$

$y^{(6)} = \frac{1}{5}\left(10 - x^{(6)} + z^{(5)}\right)$

$= \frac{1}{5}\left(10 - 2 + 2\right)$

$\boxed{y^{(6)} = 2}$

$z^{(6)} = \frac{1}{8}\left(20 - x^{(6)} - y^{(6)}\right)$

$= \frac{1}{8}\left(20 - 2 - 2\right)$

$\boxed{z^{(6)} = 2}$, Fifth and sixth Iteration values are same.

$\therefore x=2, y=2, z=2.$

3. Solve the given system of equations by using Gauss-Seidal iteration method, correct to '4' decimal places.

$\textcircled{20}x + y - 2z = 17$

$3x + \textcircled{20}y - z = -18$

$2x - 3y + \textcircled{20}z = 25$

Soln:

As the co-efficient matrix is diagonally dominant solving for x, y, z we get,

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

First Iteration [Let the Initial values be $y = 0, z = 0$]

$$x^{(1)} = \frac{1}{20}(17 - 0 + 0)$$

$$\boxed{x^{(1)} = 0.85}$$

$$y^{(1)} = \frac{1}{20}(-18 - 3(0.85) + 0)$$

$$\boxed{y^{(1)} = -1.0275}$$

$$z^{(1)} = \frac{1}{20}(25 - 2(0.85) + 3(-1.0275))$$

$$\boxed{z^{(1)} = 1.0109}$$

Second Iteration

$$x^{(2)} = \frac{1}{20}(17 - (-1.0275) + 2(1.0109))$$

$$\boxed{x^{(2)} = 1.0025}$$

$$y^{(2)} = \frac{1}{20}(-18 - 3(1.0025) + 1.0109)$$

$$\boxed{y^{(2)} = -0.9998}$$

$$z^{(2)} = \frac{1}{20} 25 - 2(1.0025) + 3(-0.9998)$$

$$\boxed{z^{(2)} = 0.9998}$$

Third Iteration

$$x^{(3)} = \frac{1}{20}\left(17-(-0.9998)+2(0.9998)\right)$$

$$\boxed{x^{(3)} = 1.0000}$$

$$y^{(3)} = \frac{1}{20}\left(-18-3(1.000)+0.9998\right)$$

$$\boxed{y^{(3)} = -1.0000}$$

$$z^{(3)} = \frac{1}{20}\left(25-2(1.0000)+3(-1.0000)\right)$$

$$\boxed{z^{(3)} = 1.0000}$$

Fourth Iteration

$$x^{(4)} = \frac{1}{20}\left(17-(-1)+2(1)\right)$$

$$\boxed{x^{(4)} = 1.0000}$$

$$y^{(4)} = \frac{1}{20}\left(-18-3(1)+1\right)$$

$$\boxed{y^{(4)} = -1.0000}$$

$$z^{(4)} = \frac{1}{20}\left(25-2(1)+3(-1)\right)$$

$$\boxed{z^{(4)} = 1.0000}$$

Third and Fourth iteration values are same. Hence $x=1$, $y=-1$ & $z=1$.

4. Using Gauss-Seidal method, solve the following system start with $x=1$, $y=-2$ & $z=3$.

$$\textcircled{x}+3y+52z = 173.61$$
$$x-\textcircled{27}y+2z = 71.31$$
$$41x-2y+\textcircled{3}z = 65.46$$

Soln:

As the co-efficient matrix is not diagonally dominant, we rewrite the given equation.

$(41)x - 2y + 3z = 65.46$ Now, the diagonal elements are

$x - (27)y + 2z = 71.31$ Dominant in the co-efficient matrix.

$x + 3y + (52)z = 173.61$

$\therefore x = \frac{1}{41}(65.46 + 2y - 3z)$

$y = \frac{-1}{27}(71.31 - x - 2z)$

$z = \frac{1}{52}(173.61 - x - 3y)$

Given: The Initial values be $x=1, y=-2, z=3$.

First Iteration

$x^{(1)} = \frac{1}{41}(65.46 + 2(-2) - 3(3))$

$\boxed{x^{(1)} = 1.28}$

$y^{(1)} = \frac{-1}{27}(71.31 - 1.28 - 2(3))$

$\boxed{y^{(1)} = -2.37}$

$z^{(1)} = \frac{1}{52}(173.61 - 1.28 - 3(-2.37))$

$\boxed{z^{(1)} = 3.45}$

Second Iteration

$x^{(2)} = \frac{1}{41}(65.46 + 2(1.28) - 3(3.45))$

$\boxed{x^{(2)} = 1.23}$

$y^{(2)} = \frac{-1}{27}(71.31 - 1.23 - 2(3.45))$

$\boxed{y^{(2)} = -2.34}$

$z^{(2)} = \frac{1}{52}(173.61 - 1.23 - 3(2.34))$

$\boxed{z^{(2)} = 3.45}$

Third Iteration

$$x^{(3)} = \frac{1}{41}\left(65.46 + 2y^{(2)} - 3z^{(2)}\right)$$

$$= \frac{1}{41}\left(65.46 + 2(-2.34) - 3(3.45)\right)$$

$$\boxed{x^{(3)} = 1.23}$$

$$y^{(3)} = \frac{-1}{27}\left(71.31 - 1.23 - 2(3.45)\right)$$

$$\boxed{y^{(3)} = -2.34}$$

$$z^{(3)} = \frac{1}{52}\left(173.61 - 1.23 - 3(-2.34)\right)$$

$$\boxed{z^{(3)} = 3.45}$$

Second and third Iteration values are same, Hence the result $x = 1.23$, $y = -2.34$, $z = 3.45$.

5. Solve the following system of equations by Gauss-Jacobi method. $④x_1 + x_2 + x_3 = 6$; $x_1 + ④x_2 + x_3 = 6$; $x_1 + x_2 + ④x_3 = 6$

Soln:

As the co-efficient matrix is diagonally dominant we write the given equation as,

$$x_1 = \frac{1}{4}\left(6 - x_2 - x_3\right)$$

$$x_2 = \frac{1}{4}\left(6 - x_1 - x_3\right)$$

$$x_3 = \frac{1}{4}\left(6 - x_1 - x_2\right)$$

Put $x=0$, $y=0$, $z=0$

First Iteration

$$x_1 = \frac{1}{4}(6)$$

$$\boxed{x_1 = 1.5}$$

$$x_2 = \frac{1}{4}(6)$$

$$\boxed{x_2 = 1.5}$$

$x_3 = \frac{1}{4}(6)$

$\boxed{x_3 = 1.5}$

Second Iteration:

$x_1 = \frac{1}{4}(6 - 1.5 - 1.5)$

$\boxed{x_1 = 0.75}$

$x_2 = \frac{1}{4}(6 - 1.5 - 1.5)$

$\boxed{x_2 = 0.75}$

$x_3 = \frac{1}{4}(6 - 1.5 - 1.5)$

$\boxed{x_3 = 0.75}$

Third Iteration

$x_1 = \frac{1}{4}(6 - 0.75 - 0.75)$

$\boxed{x_1 = 1.125}$

$x_2 = \frac{1}{4}(6 - 0.75 - 0.75)$

$\boxed{x_2 = 1.125}$

$x_3 = \frac{1}{4}(6 - 0.75 - 0.75)$

$\boxed{x_3 = 1.125}$

Fourth Iteration

$x_1 = \frac{1}{4}(6 - 1.125 - 1.125)$

$\boxed{x_1 = 0.9375}$

$x_2 = \frac{1}{4}(6 - 1.125 - 1.125)$

$\boxed{x_2 = 0.9375}$

$x_3 = \frac{1}{4}(6 - 1.125 - 1.125)$

$\boxed{x_3 = 0.9375}$

Fifth Iteration

$x_1 = \frac{1}{4}(6 - 0.9375 - 0.9375)$

$\boxed{x_1 = 1.031}$

$x_2 = \frac{1}{4}(6 - 0.9375 - 0.9375)$

$\boxed{x_2 = 1.031}$

$x_3 = \frac{1}{4}(6 - 0.9375 - 0.9375)$

$\boxed{x_3 = 1.031}$

Sixth Iteration

$x_1 = \frac{1}{4}(6 - 1.031 - 1.031)$

$\boxed{x_1 = 1.0075}$

$x_2 = \frac{1}{4}(6 - 1.031 - 1.031)$

$\boxed{x_2 = 1.0075}$

$x_3 = \frac{1}{4}(6 - 1.031 - 1.031)$

$\boxed{x_3 = 1.0075}$

Seventh Iteration

$x_1 = \frac{1}{4}(6 - 1.0075 - 1.0075)$

$\boxed{x_1 = 1}$

$x_2 = \frac{1}{4}(6 - 1.0075 - 1.0075)$

$\boxed{x_2 = 1}$

$x_3 = \frac{1}{4}(6 - 1.0075 - 1.0075)$

$\boxed{x_3 = 1}$

Hence, the required solution is $x=1$, $y=1$ & $z=1$.

6. Solve the following equations using Gauss-Jacobi method:

$⑶⓪x - 2y + 3z = 75;$ $x + ⑰y - 2z = 48;$ $x + y + ⑨z = 15$

Soln:

As the co-efficient matrix is diagonally dominant, we get,

$$x = \frac{1}{30}(75 + 2y - 3z)$$

$$y = \frac{1}{17}(48 - x + 2z)$$

$$z = \frac{1}{9}(15 - x - y)$$

Let the initial values be, $x=0, y=0, z=0$

First Iteration

$$x^{(1)} = \frac{1}{30}(75)$$

$$\boxed{x^{(1)} = 2.5}$$

$$y^{(1)} = \frac{1}{17}(48)$$

$$\boxed{y^{(1)} = 2.824}$$

$$z^{(1)} = \frac{1}{9}(15)$$

$$\boxed{z^{(1)} = 1.667}$$

Second Iteration

$$x^{(2)} = \frac{1}{30}(75 + 2y^{(1)} - 3(z)^{1})$$

$$= \frac{1}{30}(75 + 2(2.824) - 3(1.667))$$

$$\boxed{x^{(2)} = 2.522}$$

$$y^{(2)} = \frac{1}{17}(48 - x^{(1)} + 2z^{(1)})$$

$$= \frac{1}{17}(48 - 2.5 + 2\ 1.667)$$

$$\boxed{y^{(2)} = 2.873}$$

$$z^{(2)} = \frac{1}{9}\left(15 - x^{(1)} - y^{(1)}\right)$$

$$= \frac{1}{9}\left(15 - 2.5 - 2.824\right)$$

$$\boxed{z^{(2)} = 1.075}$$

Third Iteration

$$x^{(3)} = \frac{1}{30}\left(75 + 2y^{(2)} - 3z^{(2)}\right)$$

$$= \frac{1}{30}\left(75 + 2.(2.873) - 3(1.075)\right)$$

$$\boxed{x^{(3)} = 2.584}$$

$$y^{(3)} = \frac{1}{17}\left(48 - x^{(2)} + 2z^{(2)}\right)$$

$$= \frac{1}{17}\left(48 - 2.522 + 2(1.075)\right)$$

$$\boxed{y^{(3)} = 2.802}$$

$$z^{(3)} = \frac{1}{9}\left(15 - x^{(2)} - y^{(2)}\right)$$

$$= \frac{1}{9}\left(15 - 2.522 - 2.873\right)$$

$$\boxed{z^{(3)} = 1.067}$$

Fourth Iteration

$$x^{(4)} = \frac{1}{30}\left(75 + 2y^{(3)} - 3z^{(3)}\right)$$

$$= \frac{1}{30}\left(75 + 2(2.802) - 3(1.067)\right)$$

$$\boxed{x^{(4)} = 2.580}$$

$$y^{(4)} = \frac{1}{17}\left(48 - x^{(3)} + 2z^{(3)}\right)$$

$$= \frac{1}{17}\left(48 - 2.584 + 2(1.067)\right)$$

$$\boxed{y^{(4)} = 2.797}$$

$$z^{(4)} = \frac{1}{9}\left(15 - x^{(3)} - y^{(3)}\right)$$

$$= \frac{1}{9}\left(15 - 2.584 - 2.802\right)$$

$$\boxed{z^{(4)} = 1.068}$$

Fifth Iteration

$$x^{(5)} = \frac{1}{30}\left(75 + 2y^{(4)} - 3z^{(4)}\right)$$

$$= \frac{1}{30}\left(75 + 2(2.797) - 3(1.068)\right)$$

$$\boxed{x^{(5)} = 2.580}$$

$$y^{(5)} = \frac{1}{17}\left(48 - x^{(4)} + 2z^{(4)}\right)$$

$$= \frac{1}{17}\left(48 - 2.580 + 2(1.068)\right)$$

$$\boxed{y^{(5)} = 2.797}$$

$$z^{(5)} = \frac{1}{9}\left(15 - x^{(4)} - y^{(4)}\right)$$

$$= \frac{1}{9}\left(15 - 2.580 - 2.797\right)$$

$$\boxed{z^{(5)} = 1.069}$$

Sixth Iteration

$$x^{(6)} = \frac{1}{30}\left(75 + 2y^{(5)} - 3z^{(5)}\right)$$

$$= \frac{1}{30}\left(75 + 2(2.797) - 3(1.069)\right)$$

$$\boxed{x^{(6)} = 2.580}$$

$$y^{(6)} = \frac{1}{17}\left(48 - x^{(5)} + 2z^{(5)}\right)$$

$$= \frac{1}{17}\left(48 - 2.580 + 2(1.069)\right)$$

$$\boxed{y^{(6)} = 2.798}$$

$$z^{(6)} = \frac{1}{9}\left(15 - x^{(5)} - y^{(5)}\right)$$

$$= \frac{1}{9}\left(15 - 2.580 - 2.797\right)$$

$$\boxed{z^{(6)} = 1.069}$$

Seventh Iteration

$$x(7) = \frac{1}{30}\left(75 + 2y^{(6)} - 3z^{(6)}\right)$$

$$x^{(7)} = \frac{1}{30}\left(75 + 2(2.798) - 3(1.069)\right)$$

$$\boxed{x^{(7)} = 2.580}$$

$$y^{(7)} = \frac{1}{17}\left(48 - x^{(6)} + 2z^{(6)}\right)$$

$$= \frac{1}{17}\left(48 - 2.580 + 2(1.069)\right)$$

$$\boxed{y^{(7)} = 2.798}$$

$$z^{(7)} = \frac{1}{9}\left(15 - x^{(6)} - y^{(6)}\right)$$

$$= \frac{1}{9}\left(15 - 2.580 - 2.798\right)$$

$$\boxed{z^{(7)} = 1.069}$$

Sixth and seventh iteration values are same.

Hence $x = 2.580$, $y = 2.798$, $z = 1.069$.

Eigenvalues of a matrix by power method.

* If A is of order 'n', then its characteristic equation is of n^{th} degree. If 'n' is large, it is very difficult to find the exact roots of the charateristic equation and hence the eigen values are difficult to find.

* But there are numerical methods available for such cases. We list below two such methods are called

- * Power method
- * Jacobi method

* The second method can be applied only for symmetric matrices.

Problems

1. Determine the largest eigen value and the corresponding Eigen vector of the matrix using the power method for $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Let, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the approximate Eigenvector.

$$\therefore \quad AX_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 X_2$$

$$AX_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 2X_3$$

$$AX_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = 4 X_4$$

$$AX_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -3.5 \\ 2.5 \end{bmatrix} = 3.5 \begin{bmatrix} 0.71428 \\ -1 \\ 0.71428 \end{bmatrix} = 3.5 X_5$$

$$AX_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.71428 \\ -1 \\ 0.71428 \end{bmatrix} = \begin{bmatrix} 2.42855 \\ -3.42856 \\ 2.42856 \end{bmatrix}$$

$$= 3.42856 \begin{bmatrix} 0.70833 \\ -1 \\ 0.70833 \end{bmatrix}$$

$$= 3.42856 \; X_6$$

$$AX_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.70833 \\ -1 \\ 0.70833 \end{bmatrix} = \begin{bmatrix} 2.41666 \\ -3.41666 \\ 2.41666 \end{bmatrix}$$

$$= 3.4166 \begin{bmatrix} 0.70731 \\ -1 \\ 0.70731 \end{bmatrix} = 3.41666 \; X_7$$

$$AX_7 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.70731 \\ -1 \\ 0.70731 \end{bmatrix} = \begin{bmatrix} 2.41462 \\ -3.41462 \\ 2.41462 \end{bmatrix} = 3.41462 \begin{bmatrix} 0.70714 \\ -1 \\ 0.70714 \end{bmatrix}$$

Hence the largest eigenvalue = 3.41462 and the corresponding eigen vector = $\begin{bmatrix} 0.70714 \\ -1 \\ 0.70714 \end{bmatrix}$

Q. Determine the largest Eigen value and the corresponding Eigenvector of the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with $[1\ 1\ 0]^T$ upto 8th Iterations, by using power method.

Soln:

Given $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = 5 \begin{bmatrix} 0.8 \\ 1 \\ 0.6 \end{bmatrix} = 5X_2$$

$$AX_2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 3.2 \\ 6.8 \\ 9.2 \end{bmatrix} = 9.2 \begin{bmatrix} 0.347 \\ 0.739 \\ 1 \end{bmatrix} = 9.2X_3$$

$$AX_3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.347 \\ 0.739 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.564 \\ 6.519 \\ 12.609 \end{bmatrix} = 12.609 \begin{bmatrix} 0.124 \\ 0.517 \\ 1 \end{bmatrix}$$

$$= 12.609\,X_4$$

$$AX_4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.124 \\ 0.517 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.675 \\ 5.406 \\ 11.944 \end{bmatrix} = 11.944 \begin{bmatrix} 0.0565 \\ 0.452 \\ 1 \end{bmatrix}$$

$$= 11.944\,X_5$$

$$AX_5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.0565 \\ 0.452 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.412 \\ 5.072 \\ 11.752 \end{bmatrix} = 11.752 \begin{bmatrix} 0.035 \\ 0.431 \\ 1 \end{bmatrix}$$

$$= 11.752\,X_6$$

$$AX_6 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.035 \\ 0.431 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.328 \\ 4.967 \\ 11.689 \end{bmatrix} = 11.689 \begin{bmatrix} 0.028 \\ 0.424 \\ 1 \end{bmatrix}$$

$$= 11.689\, X_7$$

$$AX_7 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.028 \\ 0.424 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 5.688 \\ 11.668 \end{bmatrix} = 11.668 \begin{bmatrix} 0.025 \\ 0.487 \\ 1 \end{bmatrix}$$

$$= 11.668\, X_8$$

Hence the dominant eigenvalue $= 11.66$

Corresponding eigenvector $= \begin{bmatrix} 0.025 \\ 0.487 \\ 1 \end{bmatrix}$

8. Find the dominant Eigen value and the corresponding Eigenvector of $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ using power method. Also find the least latent root and hence find the 3rd eigen value.

Soln:

Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$Ax_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\, x_2$$

$$Ax_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7\, x_3$$

$$Ax_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix}$$

$$= 3.5714\, x_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12 X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix}$$

$$= 3.9706 X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix}$$

$$= 4.0072 X_7$$

$$AX_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix}$$

$$= 3.9982 X_8$$

$$AX_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4 X_9$$

$$AX_9 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4 X_{10}$$

∴ Dominant eigenvalue = 4 and the corresponding eigenvector is $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$. Let us find the least eigenvalue,

Let $B = A - 4I$ as $\lambda_1 = 4$

$$\therefore B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let us find the dominant eigenvalue of B.

Let $Y_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$$BY_1 = \begin{bmatrix}-3 & 6 & 1\\1 & -2 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-3\\1\\0\end{bmatrix} = -3\begin{bmatrix}1\\-0.3333\\0\end{bmatrix} = -3Y_2$$

$$BY_2 = \begin{bmatrix}-3 & 6 & 1\\1 & -2 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\-0.3333\\0\end{bmatrix} = \begin{bmatrix}-5\\1.6666\\0\end{bmatrix} = -5\begin{bmatrix}1\\-0.3333\\0\end{bmatrix}$$

$$= -5Y_3$$

$$BY_3 = \begin{bmatrix}-3 & 6 & 1\\1 & -2 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\-0.3333\\0\end{bmatrix} = \begin{bmatrix}-5\\1.6666\\0\end{bmatrix} = -5\begin{bmatrix}1\\-0.3333\\0\end{bmatrix}$$

∴ The Dominant eigenvalues of B is -5.

Adding 4; Smallest eigenvalue of $A = -5+4 = -1$

Sum of Eigenvalues = Trace $A = 1+2+3 = 6$

$$4 + (-1) + \lambda_3 = 6$$

$$\boxed{\lambda_3 = 3}$$

∴ Three eigen values of A are $4, 3, -1$.

4. Find the largest Eigenvalue of $\begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}$ and the corresponding Eigen vector by using power method.

Soln:

Let $x_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$$Ax_1 = \begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}25\\1\\2\end{bmatrix} = 25\begin{bmatrix}1\\0.04\\0.08\end{bmatrix} = 25X_2$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2 \begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix}$$

$$= 25.2\, X_3$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25.1778 \\ 1.1332 \\ 1.7337 \end{bmatrix} = 25.1778 \begin{bmatrix} 1 \\ 0.0450 \\ 0.06888 \end{bmatrix}$$

$$= 25.1778\, X_4$$

$$AX_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0450 \\ 0.06888 \end{bmatrix} = 25.1826 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

$$= 25.1826\, X_5$$

$$AX_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{bmatrix}$$

$$= 25.1821 \begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$

$$= 25.1821\, X_6$$

$\lambda_1 = 25.1821$ and the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}.$$

DEPARTMENT OF SCIENCE AND HUMANITIES

SUBJECT CODE : MA3251

SUBJECT NAME : STATISTICS AND NUMERICAL METHODS

DEPARTMENT : COMMON TO ALL BRANCES

YEAR/SEM : I / II

ACEDEMIC YEAR : 2021-2022

BATCH : XII

HAND WRITTEN MATERIAL

UNIT IV

INTERPOLATION, NUMERICAL DIFFERENTIATION AND NUMERICAL INTEGRATION

- Lagrange’s and Newton’s divided difference interpolations
- Newton’s forward and backward difference interpolation
- Approximation of derivates using interpolation polynomials
- Numerical single and double integrations using Trapezoidal and Simpson’s 1/3 rules

①

Subject Code/Title: MA3251- Statistics & Numerical Methods **Unit:** IV INTERPOLATION, NUMERICAL DIFFERENTI-TION & NUMERICAL INTEGRATION.

- ⊛ Lagrange's and Newton's divided difference interpolations for unequal intervals.
- ⊛ Newton's forward and backward difference interpolations for equal intervals.
- ⊛ Approximation of derivatives using interpolation polynomials
- ⊛ Numerical single and double integrations (i) Trapezoidal and (ii) Simpson's 1/3 rules.

Lagrange's Interpolation formula for unequal Intervals

Let us consider the set of arguments (x) and the corresponding entry (y) as.

x :	x_0	x_1	x_2	....	x_n
y :	y_0	y_1	y_2	....	y_n

Assume f(x) as a polynomial of degree n as

$$f(x) = A_0(x-x_1)(x-x_2)\dots(x-x_n) + A_1(x-x_0)(x-x_2)\dots(x-x_n) + A_2(x-x_0)(x-x_1)\dots(x-x_n) + \dots + A_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \quad \text{—①}$$

from the given values

$$y_0 = y(x_0) = A_0(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$$

$$\Rightarrow A_0 = y_0 / (x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$$

$$y_1 = y(x_1) = A_1(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$$

$$\Rightarrow A_1 = y_1 / (x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$$

In general $A_n = y_n / (x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})$

$$\therefore ① \Rightarrow y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times y_1$$

$$+ \cdots\cdots\cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \times y_n$$

This is Lagrange's formula.

---

① Find the equation of the cubic curve that passes through the points $(-1,-8)$, $(0,3)$, $(2,1)$ and $(3,2)$ using Lagrange's interpolation formula.

Sol

	x_0	x_1	x_2	x_3
x	-1	0	2	3
y	-8	3	1	2
	y_0	y_1	y_2	y_3

By Lagrange's formula, we've

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x-0)(x-2)(x-3)}{(-1)(-3)(-4)}(-8) + \frac{(x+1)(x-2)(x-3)}{(1)(-2)(-3)}(3)$$

$$+ \frac{(x+1)(x-0)(x-3)}{(3)(2)(-1)}(1) + \frac{(x+1)(x-0)(x-2)}{(4)(3)(1)}(2)$$

$$\therefore\ y = \frac{2}{3}(x^3-5x^2+6x) + \frac{1}{2}(x^3-4x^2+x+6) - \frac{1}{6}(x^3-2x^2-3x) + \frac{1}{6}(x^3-x^2-2x)$$

$$\Rightarrow y = \frac{7}{6}x^3 - \frac{31}{6}x^2 + \frac{14}{3}x + 3 = 1{\cdot}16x^3 - 5{\cdot}16x^2 + 4{\cdot}66x + 3$$

② Use Lagrange's method to find $\log_{10} 656$, given that $\log_{10} 654 = 2{\cdot}8156$, $\log_{10} 658 = 2{\cdot}8182$, $\log_{10} 659 = 2{\cdot}8189$ and $\log_{10} 661 = 2{\cdot}8202$.

Sol

	x_0	x_1	x_2	x_3
x :	654	658	659	661
y :	2·8156	2·8182	2·8189	2·8202
	y_0	y_1	y_2	y_3

Lagrange's formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$\therefore\ y\Big|_{656} = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2{\cdot}8156$$

$$+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2{\cdot}8182$$

$$+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2{\cdot}8189$$

$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times 2{\cdot}8202$$

$$y = \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)}(2.8156) + \frac{(2)(-3)(-5)}{(4)(-1)(-3)}(2.8182)$$

$$+ \frac{(2)(-2)(-5)}{(5)(1)(-2)}(2.8189) + \frac{(2)(-2)(-3)}{(7)(3)(2)}(2.8202)$$

$$= 0.60334 + 7.04558 - 5.6378 + 0.80577$$

$\therefore y(656) = 2.8168$ (approximately)

③ Construct the polynomial for the following data, using Lagrange's method and hence evaluate $f(2.5)$ & $f(3.5)$

x	0	1	3	4
$f(x)$	-12	0	6	12

Sol Given data

	x_0	x_1	x_2	x_3
x	0	1	3	4
y	-12	0	6	12
	y_0	y_1	y_2	y_3

[To obtain the polynomial, in the Lagrange's formula, keep x as it is]

By Lagrange's formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_1)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

⑤

$$y = f(x) = \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12) + \frac{x(x-1)(x-4)}{3(2)(-1)}(6)$$

$$+ \frac{x(x-1)(x-3)}{(4)(3)(1)}(12)$$

$$= \left(\frac{-1}{12}\right)(x^3-8x^2+19x-12)(-12) + \left(\frac{-1}{6}\right)(x^3-5x^2+4x)(6)$$

$$+ \left(\frac{1}{12}\right)(x^3-4x+3)(12)$$

$$\therefore f(x) = x^3 - 7x^2 + 18x - 12$$

$$f(2.5) = (2.5)^3 - 7(2.5)^2 + 18(2.5) - 12 = 4.875$$

$$f(3.5) = (3.5)^3 - 7(3.5)^2 + 18(3.5) - 12 = 8.125$$

④ Using Lagrange's formula, prove $y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$ nearly.

Sol

y_{-5}, y_{-3}, y_3, y_5 occurs in the answer, so we can have the following table

	x_0	x_1	x_2	x_3.
x:	-5	-3	3	5
y:	y_{-5}	y_{-3}	y_3	y_5
	y_0	y_1	y_2	y_3.

By Lagrange's formula,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x+3)(x-3)(x-5)}{(-5+3)(-5-3)(-5-5)} \times y_{-5} + \frac{(x+5)(x-3)(x-5)}{(-3+5)(-3-3)(-3-5)} \times y_{-3}$$

$$+ \frac{(x+5)(x+3)(x-5)}{(3+5)(3+3)(3-5)} \times y_3 + \frac{(x+5)(x+3)(x-3)}{(5+5)(5+3)(5-3)} \times y_5$$

at $x=1$, $y = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_3$

$+ \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$ ...

$\therefore y = -0.2 y_{-5} + 0.5 y_{-3} + y_3 - 0.3 y_5$.

$\therefore y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$

⑤ Inverse interpolation

Apply Lagrange's formula to find a root of the equation $f(x)=0$ given that $f(30)=-30$, $f(34)=-13$, $f(38)=3$, $f(42)=18$.

Sol To find the root of $f(x)=0$ it is enough to find x at $y=0$

	y_0	y_1	y_2	y_3
y:	−30	−13	3	18
x:	30	34	38	42
	x_0	x_1	x_2	x_3

By Lagrange's Inversion formula,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} \times x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} \times x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} \times x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} \times x_3$$

$$\therefore \text{at } y=0,\ x = \frac{(13)(-3)(-18)}{(-17)(-33)(-48)}(30) + \frac{(30)(-3)(-18)}{(17)(-16)(-31)}(34)$$

$$+ \frac{(30)(13)(-18)}{(33)(16)(-15)}(38) + \frac{(30)(13)(-3)}{(48)(31)(15)}(42)$$

$x = 37.2303$.

⑥ Find x for $f(x)=15$.

x	5	6	9	11
y	12	13	14	16

Sol

	x_0	x_1	x_2	x_3
x	5	6	9	11
y	12	13	14	16
	y_0	y_1	y_2	y_3

By Inverse Lagrange's formula,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} \times x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)} \times x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} \times x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} \times x_3$$

at $y=15$, $$x = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)} \times 5 + \frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)} \times 6$$

$$+ \frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)} \times 9 + \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)} \times 11$$

$$x = \frac{(2)(1)(-1)}{(-1)(-2)(-4)} \times 5 + \frac{(3)(1)(-1)}{(1)(-1)(-3)} \times 6 + \frac{(3)(2)(-1)}{(2)(1)(-2)} \times 9$$

$$+ \frac{(3)(2)(1)}{(4)(3)(2)} \times 11$$

$\therefore$ at $y=15$, $x = 11.5$

Newton's Interpolation formula for Unequal Intervals

Newton's Divided Difference formula.

Let $y=f(x)$ takes values $f(x_0), f(x_1), \dots, f(x_n)$ Corresponding to the arguments $x_0, x_1, \dots, x_n$

By defn. $f(x,x_0) = \dfrac{f(x)-f(x_0)}{x-x_0}$

$$\Rightarrow f(x) = f(x_0) + (x-x_0) f(x,x_0) \quad \text{—— ①}$$

Similarly $f(x,x_0,x_1) = \dfrac{f(x,x_0)-f(x_0,x_1)}{x-x_1}$

$$\therefore f(x,x_0) = f(x_0,x_1) + (x-x_1) f(x,x_0,x_1)$$

$$① \Rightarrow f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x,x_0,x_1) \quad \text{②}$$

Also $f(x,x_0,x_1,x_2) = \dfrac{f(x,x_0,x_1)-f(x,x_0,x_2)}{x-x_2}$

$$\Rightarrow f(x,x_0,x_1) = f(x,x_0,x_2) + (x-x_2) f(x,x_0,x_1,x_2)$$

$$② \Rightarrow f(x) = (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x,x_0,x_1,x_2) \quad \text{—— ③}$$

Proceeding in this manner for n observations, we've

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + \dots + (x-x_0)(x-x_1)\dots(x-x_{n-1}) f(x_0,x_1,\dots x_n)$$

This is Newton's divided difference formula for unequal intervals

① Using Newton's divided difference formula, find the values of f(2), f(8) and f(15) from the given table.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Sol we form the divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48 ✓				
		$\frac{100-48}{5-4} = 52$ ✓			
5	100		$\frac{97-52}{7-4} = 15$ ✓		
		$\frac{294-100}{7-5} = 97$		$\frac{21-15}{10-4} = 1$ ✓	
7	294		$\frac{202-97}{10-5} = 21$		0
		$\frac{900-294}{10-7} = 202$		$\frac{27-21}{11-5} = 1$	
10	900		$\frac{310-202}{11-7} = 27$		0
		$\frac{1210-900}{11-10} = 310$		$\frac{33-27}{13-7} = 1$	
11	1210		$\frac{409-310}{13-10} = 33$		
		$\frac{2028-1210}{13-11} = 409$			
13	2028				

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x-x_0)\, f(x_0,x_1) + (x-x_0)(x-x_1)\, f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)\, f(x_0,x_1,x_2,x_3) + \cdots \quad ①$$

where $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, $x_4 = 11$, $x_5 = 13$

and $f(x_0) = 48$, $f(x_0,x_1) = 52$, $f(x_0,x_1,x_2) = 15$, $f(x_0,x_1,x_2,x_3) = 1$

Using all this in ①, we get,

$f(x) = 48 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)(1)$

$\therefore f(2) = 48 - 104 + 90 - 30 = 4$; $f(8) = 48 + 4\times52 + 4\times3\times15 + 4\times3\times1\times1$

$\therefore f(8) = 448.$

$f(15) = 48 + 11\times52 + 11\times10\times15 + 11\times10\times8 = 3150.$

② From the following table find $f(x)$ and hence find $f(6)$ using Newton's interpolation formula.

x	1	2	7	8
$f(x)$	1	5	5	4

Sol

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1 x_0	1 ✓			
		4 ✓		
2 x_1	5		$-2/3$ ✓	
		0		$1/14$ ✓
7 x_2	5		$-1/6$	
		-1		
8 x_3	4			

By newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$

$$= 1 + (x-1)4 + (x-1)(x-2)\left(\frac{-2}{3}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$

$$f(x) = \frac{1}{42}(3x^3 - 58x^2 + 321x - 224)$$

$$f(x) = 0.071x^3 - 1.38x^2 + 7.642x - 5.333$$

$$\therefore f(6) = \frac{1}{42}(3 \times 216 - 36 \times 58 + 1926 - 224)$$

$$\Rightarrow f(6) = 6.2380$$

③ Find the fourth degree curve $y = f(x)$ passing through the points $(2,3)$, $(4,43)$, $(5,138)$, $(7,778)$ and $(8,1515)$ using Newton's divided difference formula.

Sol

given data

x	2	4	5	7	8
$f(x)$	3	43	138	778	1515

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0 2	3 ✓				
		20 ✓	25 ✓		
x_1 4	43			10 ✓	
		95			1 ✓
x_2 5	138		75	16	
		320			
x_3 7	778		139		
		737			
x_4. 8	1515				

By newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \cdots$$

$$= 3 + 20(x-2) + 25(x-2)(x-4) + 10(x-2)(x-4)(x-5) + 1(x-2)(x-4)(x-5)(x-7)$$

$$f(x) = 3 + (20x-40) + (25x^2 - 150x + 200) + (10x^3 - 110x^2 + 380x - 400) + (x^4 - 18x^3 + 115x^2 - 306x + 280)$$

$$f(x) = x^4 - 8x^3 + 30x^2 - 56x + 43$$

④ Compute $f(4)$ and $f(7.5)$ from the data; given below by obtaining the cubic function of x.

x	0	1	2	5
$f(x)$	2	3	12	147

Sol

x	$f(x)$	Δ	Δ^2	Δ^3
x_0 0	2 ✓			
		1 ✓		
x_1 1	3		4 ✓	1 ✓
		9		
x_2 2	12		9	
		45		
x_3 5	147			

By newton's divided difference ~~table~~ formula,

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$$f(x) = 2 + (x-0)(1) + (x-0)(x-1)\,4 + (x-0)(x-1)(x-2)(1)$$

$$\Rightarrow f(x) = x^3 + x^2 - x + 2 \Rightarrow f(4) = 78 \; ; \; f(7.5) \neq 472.625$$

Interpolation with ~~un~~equal intervals.

Defn.

Interpolation is the process of finding the intermediate values of function from a set of its values at specific points given in a tabulated form. Let us suppose that the following table represents a set of corresponding values of x and $y = f(x)$.

$x: x_0 \quad x_1 \quad x_2 \cdots\cdots x_n$

$y: y_0 \quad y_1 \quad y_2 \cdots\cdots y_n$

The process of computing y corresponding to x where $x_i < x < x_{i+1}$, $i = 0, 1, 2, \ldots, n-1$ is interpolation.

Gregory - Newton's forward interpolation formula for equal intervals.

If $y_0, y_1, y_2, \ldots, y_n$ are the values of $y = f(x)$ corresponding to equidistant values of $x = x_0, x_1, \ldots, x_n$ where $x_i - x_{i-1} = h$ for $i = 1, 2, 3, \ldots, n$ then

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots + \frac{u(u-1)\cdots\cdots(u-(\overset{n-1}{\cancel{n+1}}))}{n!}\Delta^n y_0$$

where $u = \dfrac{x - x_0}{h}$

Gregory - Newton's ~~fo~~ Backward interpolation formula for equal intervals

If $y_0, y_1, y_2, \ldots\ldots, y_n$ are the values of $y = f(x)$ corresponding to equidistant values of $x = x_0, x_1, \ldots, x_n$ where $x_i - x_{i-1} = h$ for $i = 1, 2, 3, \ldots, n$ then

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \cdots\cdots + \frac{v(v+1)(v+2)\cdots\cdots(v+(n-1))}{n!}\nabla^n y_n$$

where $v = \dfrac{x - x_n}{h}$

① Construct Newton's forward interpolation polynomial for the following data

x	4	6	8	10
y	1	3	8	16

Sol

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0 4	1 y_0	Δy_0 3-1=2	$\Delta^2 y_0$ 5-2=3	$\Delta^3 y_0$ 3-3=0
x_1 6	3	8-3=5	8-5=3	
x_2 8	8	16-8=8		
x_3 10	16			

where $u = \frac{x-x_0}{h} = \frac{x-4}{2} = 0.5x-2$

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \frac{(0.5x-2)}{1!}(2) + \frac{(0.5x-2)(0.5x-3)}{2!}(3) + 0$$

$$y = 0.375x^2 - 2.75x + 6.$$

② The population of a town is as follows.

Year x:	1941	1951	1961	1971	1981	1991
Pop. in Lakhs y:	20	24	29	36	46	51

Estimate the population increase during the period 1946 and 1976

Sol at 1946, we use Newtons forward interpolation formula.

$$y = y_0 + \cancel{A}\frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{2!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{3!}\Delta^4 y_0 + \cdots\cdots$$

where $u = \frac{x-x_0}{h} = \frac{1946-1941}{10} = 1/2.$

$\therefore y(1946) = ?$ & $y(1976) = ?$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0 1941	20 y_0					
		4 Δy_0	$\Delta^2 y_0$			
1951	24		1	$\Delta^3 y_0$	$\Delta^4 y_0$	
		5		1		$\Delta^5 y_0$
1961	29		2		0	
		7		1		−9
1971	36		3		−9	
		10		−8		
1981	46		−5			
		5				
1991	51					

at $x=1946$, $y = 20 + \frac{1}{2}(4) + \frac{(-\frac{1}{2})(-\frac{1}{2})}{2}(1) + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}(1)$

$$+ \frac{(-\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{24}(0) + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{120}(-9)$$

$$y = 20 + 2 - 0.125 + 0.0625 - 0.24609$$

$\therefore y(1946) = 21.69.$

at $x=1976$, we use newton's backward interpolation formula

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots\cdots$$

where $v = \frac{y-y_n}{h} = \frac{1976-1991}{10} = -3/2$

$\therefore$ ~~at~~ $x=1976$, $y = 51 - \frac{3}{2}(5) + \frac{(-3/2)(-1/2)}{2}(-5)\ \frac{(-\frac{3}{2})(-\frac{1}{2})(\frac{1}{2})}{6}(-8)$

$$+ \frac{(-3/2)(-1/2)(1/2)(3/2)}{24}(-9) + \frac{(-3/2)(-1/2)(\frac{1}{2})(3/2)(5/2)}{120} \times (-9)$$

$\therefore y(1976) = 40.808.$

$\therefore$ Increase in population during the period 1946 to 1976 is $40.808 - 21.690 = 19.4$ lakhs.

② From the data given below, find the number of Students whose weight is between 60 and 70.

weight	0-40	40-60	60-80	80-100	100-120
No. of Students	250	120	100	70	50

Sol

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below40	$250 \; y_0$	~~120~~ Δy_0			
		120	$\Delta^2 y_0$		
Below60	370	~~100~~	-20	$\Delta^3 y_0$	
		100		-10	$\Delta^4 y_0$
Below80	~~540~~ ~~600~~ 470	~~70~~	-30		20
		70		10	
Below100	540		-20		
		50.			
Below120	590				

We calculate the number of Students whose weight is less than 70. (ie: $x=70$)

$$\therefore \; u = \frac{x-x_0}{h} = \frac{70-40}{20} = 3/2 = 1.5$$

By Newton's forward interpolation formula, we've

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

at $x=70$, $$y = 250 + (1.5)(120) + \frac{(1.5)(0.5)}{2}(-20) + \frac{(1.5)(0.5)(-0.5)}{6}(-10) + \frac{(1.5)(0.5)(-0.5)(-1.5)}{24}(20)$$

$$y = 250 + 180 - 7.5 + 0.625 + 0.46875$$

$$y = 423.59 \Rightarrow y \approx 424 \text{ at } x=70.$$

at $x=60$, $y = 250+120 = 370$

∴ No. of Students whose weight is b/w 60 & 70 $= y(70) - y(60) = 424 - 370 = 54$

④

% of lead in the alloy	40	50	60	70	80	90
temperature in degrees.	184	204	226	250	276	304

Using Newton's interpolation formula find the melting point of the alloy containing 84% of lead and 42% of lead.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
40	184				
50	204	Δy_0			
		22	$\Delta^2 y_0$		
60	226		2	$\Delta^3 y_0$	
		24		0	$\Delta^4 y_0$
70	250		2		0
					$\nabla^4 y_n$
80	276	26		0	
		28	2	$\nabla^3 y_n$	
90	304	∇y_n	$\nabla^2 y_n$		
	y_n				

(i)

As $x=42$ is in the begining, use Newtons forward interpolation formula.

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Where $u = \frac{x-x_0}{h} = \frac{42-40}{10} = 0.2$

$$\therefore y = 184 + \frac{0.2}{1!}(20) + \frac{(0.2)(0.2-1)}{2!}(2) + 0 + 0.$$

$$\therefore y(42) = 187.84$$

(ii) As $x=84$ lies in the end of the table, we use Newton's backward interpolation formula

$$y = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

Where $v = \frac{x-x_n}{h} = \frac{84-90}{10} = -0.6$

$$\therefore y = 304 + \frac{(-0.6)(28)}{(1)} + \frac{(-0.6)(-0.6+1)}{2}(2) + 0 + 0$$

$$\therefore y(84) = 287.$$

⑤ Find a polynomial of degree four which takes the values

x	2	4	6	8	10
y	0	0	1	0	0

and hence compute f(7).

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2 (x_0)	0 (y_0)				
		0 Δy_0			
4	0		1 $\Delta^2 y_0$		
				-3 $\Delta^3 y_0$	
6	1	1			6 $\Delta^4 y_0$
			-2		
		-1		3	
8	0		1		
		0			
10	0				

By the Newton's forward interpolation formula,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots\cdots$$

where $u = \frac{x-x_0}{h} = \frac{x-2}{2}$

$$\therefore y = 0+0+\frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)}{2}(1) + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)}{6}(-3) + \frac{\left(\frac{x-2}{2}\right)\left(\frac{x-4}{2}\right)\left(\frac{x-6}{2}\right)\left(\frac{x-8}{2}\right)}{24}(6)$$

$$= \frac{(x-2)(x-4)}{8}\left[1 - \frac{(x-6)}{2} + \frac{1}{8}(x-6)(x-8)\right]$$

$$= \frac{1}{64}\left[(x-2)(x-4)\right]\left[8-4x+24+x^2-14x+48\right]$$

$$= \frac{1}{64}\left[(x-2)(x-4)(x-8)(x-10)\right]$$

$$= \frac{1}{64}\left[x^4 - 24x^3 + 196x^2 - 624x - 640\right]$$

$$y = 0.01x^4 - 0.375x^3 + 3.062x^2 - 9.75x - 10$$

$$\therefore y(7) = 0.7031$$

⑥ From the table given below, find $\sin 52^\circ$ by using Newtons interpolation formula

x	45°	50°	55°	60°
$\sin x$	0.7071	0.7660	0.8192	0.8660

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		-0.0057	
		0.0532		-0.0007
55°	0.8192		-0.0064	
		0.0468		
60°	0.8660			

By Newton's forward interpolation formula,

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\cdots$$

Where $u = \frac{x-x_0}{h} = \frac{52-45}{5} = 1.4$

$$y = 0.7071 + (1.4)(0.0589) + \frac{(1.4)(0.4)}{2}(-0.0057) + \frac{(1.4)(0.4)(-0.6)}{6}(-0.0007)$$

$\therefore \sin 52^\circ = 0.7880032$

⑦ Find the value of ~~Cos51°42'~~ $f(12)$ & $f(27)$ by using ~~Gra~~ Newton's backward, forward formula.

x	10	15	20	25	30
y	35.4	32.2	29.1	26	23.1

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	35.4				
		-3.2			
15	32.2		0.1		
		-3.1		-0.1	
20	29.1		0		0.3
		-3.1		0.2	
25	26		0.2		
		-2.9			
30	23.1				

Using Newton's forward interpolation formula

$$u=\frac{x-x_0}{h}=\frac{12-10}{5}=0.4$$

$$y=y_0+\frac{u}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0+\cdots$$

$$y=35.4+(0.4)(-3.2)+\frac{(0.4)(0.4-1)}{2}(0.1)+\frac{(0.4)(0.4-1)(0.4-2)}{6}(0.1)$$
$$+\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(0.3)$$

$$y(12)=33.95184$$

Using Newton's backward interpolation formula

$$y=y_n+\frac{v}{1!}\nabla y_n+\frac{v(v+1)}{2!}\nabla^2 y_n+\frac{v(v+1)(v+2)}{3!}\nabla^3 y_n+\cdots\cdots$$

where $v=\frac{x-x_n}{h}=\frac{27-30}{5}=-0.6$

$$y=23.1+(-0.6)(-2.9)+\frac{(-0.6+1)}{2}(0.2)+\frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(0.2)$$
$$+\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(0.3)$$

$$y(27)=24.8$$

Numerical Differentiation

Newton's forward formula for derivatives

When $x\neq x_0$, $u=\frac{x-x_0}{h}$

① $$y'=\left.\frac{dy}{dx}\right]_{x\neq x_0}=\frac{1}{h}\left[\Delta y_0+\frac{1}{2}(2u-1)\Delta^2 y_0+\frac{1}{6}(3u^2-6u+2)\Delta^3 y_0+\frac{1}{12}(2u^3-9u^2+11u-3)\Delta^4 y_0+\cdots\cdots\right]$$

② $$y''=\left.\frac{d^2y}{dx^2}\right]_{x\neq x_0}=\frac{1}{h^2}\left[\Delta^2 y_0+(u-1)\Delta^3 y_0+\frac{1}{12}(6u^2-18u+11)\Delta^4 y_0+\cdots\cdots\right]$$

③ $$y'''=\left.\frac{d^3y}{dx^3}\right]_{x\neq x_0}=\frac{1}{h^3}\left[\Delta^3 y_0+\frac{1}{2}(2u-3)\Delta^4 y_0+\cdots\cdots\right]$$

when $x=x_0$, $u=\frac{x-x_0}{h}=0$.

④ $y'=\left.\frac{dy}{dx}\right]_{x=x_0}=\frac{1}{h}\left[\Delta y^{0}-\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0-\frac{1}{4}\Delta^4 y_0+\cdots\cdots\right]$

⑤ $y''=\left.\frac{d^2y}{dx^2}\right]_{x=x_0}=\frac{1}{h^2}\left[\Delta^2 y_0-\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0-\cdots\cdots\right]$

⑥ $y'''=\left.\frac{d^3y}{dx^3}\right]_{x=x_0}=\frac{1}{h^3}\left[\Delta^3 y_0-\frac{3}{2}\Delta^4 y_0+\frac{7}{4}\Delta^5 y_0-\cdots\cdots\right]$

newton's backward formula for derivatives

when $x\neq x_n$, $v=\frac{x-x_n}{h}$.

⑦ $y'=\left.\frac{dy}{dx}\right]_{x\neq x_n}=\frac{1}{h}\left[\nabla y_n+\frac{1}{2}(2v+1)\nabla^2 y_n+\frac{1}{6}(3v^2+6v+2)\right.$
$\left.+\frac{1}{12}(2v^3+9v^2+11v+3)\nabla^4 y_n+\cdots\cdots\right]$

⑧ $y''=\left.\frac{d^2y}{dx^2}\right]_{x\neq x_n}=\frac{1}{h^2}\left[\nabla^2 y_n+(v+1)\nabla^3 y_n+\frac{1}{12}(6v^2+18v+11)\nabla^4 y_n+\cdots\cdots\right]$

⑨ $y'''=\left.\frac{d^3y}{dx^3}\right]_{x\neq x_n}=\frac{1}{h^3}\left[\nabla^3 y_n+\frac{1}{2}(2v+3)\nabla^4 y_n+\cdots\cdots\right]$

when $x=x_n$, $v=\frac{x-x_n}{h}=0$

⑩ $y'=\left.\frac{dy}{dx}\right]_{x=x_n}=\frac{1}{h}\left[\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\cdots\cdots\right]$

⑪ $y''=\left.\frac{d^2y}{dx^2}\right]_{x=x_n}=\frac{1}{h^2}\left[\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n+\cdots\cdots\right]$

⑫ $y'''=\left.\frac{d^3y}{dx^3}\right]_{x=x_n}=\frac{1}{h^3}\left[\nabla^3 y_n+\frac{3}{2}\nabla^4 y_n+\cdots\cdots\right]$

① Find the first two derivatives of $x^{1/3}$ at $x=50$ and $x=56$ given the table below.

x	50	51	52	53	54	55	56
$y=f(x)$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Sol

(i) We've $x=50$ & $x_0=50$ $h=1$; ie: $x=x_0$, $y'\Big]_{x=x_0} = ?$
(from question) (from the table) $(u=0)$

By Newton's forward interpolation formula for derivatives

$$y'\Big]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\cdots\right] \quad ①$$

$$y''\Big]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\cdots\right] \quad ②$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0 50	3.6840 y_0					
		Δy_0 → 0.0244				
51	3.7084		$\Delta^2 y_0$ → −0.0003			
		→ 0.0241		$\Delta^3 y_0$ → 0		
52	3.7325		→ −0.0003		$\Delta^4 y_0$ → 0	
		→ 0.0238		→ 0		$\Delta^5 y_0$ → 0
53	3.7563		→ −0.0003		→ 0	
		→ 0.0235		→ 0		→ 0 $\nabla^5 y_n$
54	3.7798		→ −0.0003		→ 0 $\nabla^4 y_n$	
		→ 0.0232		→ 0 $\nabla^3 y_n$		
55	3.8030		→ −0.0003 $\nabla^2 y_n$			
		→ 0.0229				
56 x_n	3.8259 y_n	∇y_n				

$$① \Rightarrow y'\Big]_{x=x_0} = \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right] = 0.02455$$

$$② \Rightarrow y''\Big]_{x=x_0} = \frac{1}{1^2}\left[-0.0003\right] = -0.0003.$$

(ii) We've $x = 56$ & $x_n = 56$, $h = 1$, ie: $x = x_n \Rightarrow v = 0$

By Newton's backward formula for derivatives,

$$y'\Big]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\cdots\right]$$

$$= \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + \frac{1}{3}(0) + \frac{1}{4}(0)\right]$$

$$y' = 0.02275$$

$$y''\Big]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{1^2}\left[-0.0003\right]$$

$$y'' = -0.0003.$$

② From the following table, find y' & y'' at $x = 1.1$.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Sol

We've $x = 1.1$ & $x_0 = 1$ ie: $x \neq x_0$, $u = \frac{x - x_0}{h} = \frac{\cancel{x-1}}{\cdot} = \frac{1.1-1}{0.1}$

$\Rightarrow u = 1$

$$\therefore y'\Big]_{x \neq x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2 - 6u + 2)}{6}\Delta^3 y_0 + \frac{1}{12}(2u^3 - 9u^2 + 11u - 3)\Delta^4 y_0) + \cdots\right] \quad ①$$

$$y''\Big]_{x \neq x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{1}{12}(6u^2 - 18u + 11)\Delta^4 y_0 + \cdots\right] \quad ②$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	
1.0	7.989							
		0.414						
1.1	8.403		−0.036					
		0.378		0.006				
1.2	8.781		−0.03		−0.002			
		0.348		0.004		0.001		
1.3	9.129		−0.026		−0.001		0.002	
		0.322		0.003		0.003		
1.4	9.451		−0.023		0.002			
		0.299		0.005				
1.5	9.750		−0.018					
		0.281						
1.6	10.031							

① ⇒

$$y'\Big]_{x \neq x_0} = \frac{1}{1}\Big[0.414 + \frac{(2(1)-1)}{2}(-0.036) + \frac{(3(1^2)-6(1)+2)}{6}(0.006)$$

$$+\frac{1}{12}(2(1^3)-9(1^2)+11(1)-3)(-0.002)$$

$$+0+0\Big]$$

$\therefore y' = 3.9484$

Consider: $\nabla^5 y_0 = 0.001 \approx 0$

$\nabla^6 y_0 = 0.002 \approx 0$

② ⇒ $$y''\Big]_{x \neq x_0} = \frac{1}{1^2}\Big[-0.036 + 0 + \frac{1}{12}(6(1^2) - 18(1) + 11)(-0.002) + 0 + 0\Big]$$

$\therefore y'' = -3.584$

③ The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year x:	1931	1941	1951	1961	1971
Population in thousands y:	40.62	60.80	79.95	103.56	132.65

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62 y_0				
		20.18 Δy_0			
1941	60.80		-1.03 $\Delta^2 y_0$		
		19.15		5.49 $\Delta^3 y_0$	
1951	79.95		4.46		$\Delta^4 y_0$ -4.47 $\nabla^4 y_n$
		23.61		1.02	
1961	103.56		5.48	$\nabla^3 y_n$	
		29.09 ∇y_n	$\nabla^2 y_n$		
1971	132.65 y_n				

(i) at $x=1931$, $x=1931$, $x_0=1931$ $\Rightarrow u=0$, $h=10$ (use NFIF)

$$y'\Big|_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{10}\left[20.18 - \frac{1}{2}(-1.03) + \frac{1}{3}(5.49) - \frac{1}{4}(-4.47)\right]$$

$$y' = 2.3642$$

(ii) at $x=1941$, $x=1941$, $x_0=1931$, $u=\frac{x-x_0}{h}=1$ (use NFIF)

$$y'\Big|_{x\neq x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}(2u-1)\Delta^2 y_0 + \frac{1}{6}(3u^2-6u+2)\Delta^3 y_0 + \frac{1}{12}(2u^3-9u^2+11u-6)\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{10}\left[20.18 + \frac{1}{2}(-1.03) + \frac{1}{6}(-1)(5.49) + \frac{1}{12}(-2)(-4.47)\right]$$

$$y' = 1.9495$$

(iii) when $x = 1961$, $x = 1961, x_n = 1971$ $v = \frac{x - x_n}{h} = -1$ (use NBIF)

$$y'|_{x \neq x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}(2v+1)\nabla^2 y_n + \frac{1}{6}(3v^2+6v+2)\nabla^3 y_n + \frac{1}{12}(2v^3+9v^2+11v+3)\nabla^4 y_n + \cdots\cdots\right]$$

$$= \frac{1}{10}\left[29.09 + \frac{1}{2}(-2+1)(5.48) + \frac{1}{6}(3(-1)^2+6(-1)+2)(1.02) + \frac{1}{12}(2(-1)^3+9(-1)^2+11(-1)+3)(-4.47)\right]$$

$$= \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) - \frac{1}{6}(1.02) - \frac{1}{12}(-4.47)\right]$$

$$y' = 2.6552$$

(iv) when $x = 1971$, $x_n = 1971$. ie: $x = x_n$. $v = \frac{x - x_n}{h} = 0$ (use NBIF)

$$y'|_{x = x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\cdots\right]$$

$$= \frac{1}{10}\left[29.09 + \frac{1}{2}(5.48) + \frac{1}{3}(1.02) + \frac{1}{4}(-4.47)\right]$$

$$y' = 3.1052$$

④ Find the values of sin 18° and sin 45° from the following table. (use Newton's forward formula)

$x°$	0°	10°	20°	30°	40°
$\cos x°$	1	0.9848	0.9397	0.866	0.766

Sol: Since the values of x are in degrees, h should be expressed as a number,

Hence $h = 10°$ has been taken as π radians $= 180°$.

ie: $h = {}^{10\pi}/_{180} \Rightarrow h = {}^{\pi}/_{18}$

(i) $\sin(18°) = ?$ $x = 18$ $x_0 = 0$ $u = \frac{x - x_0}{h} = \frac{18 - 0}{10} = 1.8$

$$y'|_{x \neq x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}(2u-1)\Delta^2 y_0 + \frac{1}{6}(3u^2-6u+2)\Delta^3 y_0 + \frac{1}{12}(2u^3-9u^2+11u-3)\Delta^4 y_0 + \cdots\cdots\right] \text{—①}$$

$$y'|_{x\neq x_0} = \frac{18}{\pi}\left[-0.0152 + 1.3(-0.0299) + \frac{1}{6}(0.92)(0.0013) + \frac{1}{24}(-1392)(0.001)\right]$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		-0.0152			
10	0.9848		-0.0299		
		-0.0451		0.0013	
20	0.9397		-0.0286		0.0010
		-0.0737		0.0023	
30	0.866		-0.0263		
		-0.1000			
40	0.766				

$$\therefore y' = \frac{18}{\pi}(-0.0544) = -0.3120.$$

Since $y = \cos x$

$y' = -\sin x$

put $x = 18°$, $-0.3120 = -\sin 18° \Rightarrow \sin 18° = 0.3120$

when $x = 45°$, $x = 45$, $x_0 = 0$ $x \neq x_0$ $u = \frac{x - x_0}{h}$.

$u = 4.5$

$$y'|_{x\neq x_0} = \frac{18}{\pi}\left[-0.0152 + 4(-0.0299) + \frac{1}{6}(35.75)(0.0013) + \frac{1}{24}(93)(0.001)\right]$$

$y' = -0.7058$

$y = \cos x \Rightarrow y' = -\sin x.$

i.e. $-0.7058 = -\sin 45°$

$\Rightarrow \sin 45° = 0.7058$

⑤ The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the data.

Time (sec)	0	5	10	15	20
velocity (m/sec)	0	3	14	69	228

Sol

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
			8		
5	3				
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

To find initial acceleration, ie: at $t=0$, $a=?$

$$a = \frac{dv}{dt} = v'\Big]_{t=0} = ?$$ ie: $t=0$ & $t_0=0$ $u=\frac{t-t_0}{h}=0$

$h=5$

$$v'\Big]_{t=t_0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\cdots\right]$$

$$= \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right]$$

$$v' = 1 \text{ m/sec}^2$$

⑥ A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of time t (seconds)

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

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Calculate the angular velocity and the angular acceleration of the rod when t=0.6 seconds.

Sol

t =x	θ =y	Δy	Δ²y	Δ³y	Δ⁴y .	Δ⁵y
0	0					
		0.12				
0.2	0.12		0.25			
		0.37		0.01		
0.4	0.49		0.26		0	
		0.63		0.01		0
0.6	1.12		0.27			
		0.9		0.01	0	
0.8	2.02		0.28			
		1.18				
1.0	3.20					

when $x=0.6$, $x_n=1$ (ie: $x \neq x_n$) $v=\frac{x-x_n}{h}=\frac{0.6-1}{0.2}=-3$.

$$y'\big|_{x\neq x_n}=\frac{1}{h}\left[\nabla y_n+\frac{1}{2}(2v+1)\nabla^2 y_n+\frac{1}{6}(3v^2+6v+2)\nabla^3 y_n + \frac{1}{12}(2v^3-9v^2+11v+3)\nabla^4 y_n+\cdots\right]$$

$$=\frac{1}{0.2}\left[1.18+\frac{1}{2}(-6+1)(0.28)+\frac{1}{6}(27-18+2)(0.01)\right]$$

$$=\frac{+1}{0.2}\left[1.18-\frac{5}{2}(0.28)-\frac{1}{6}(11)(0.01)\right]$$

$$=\frac{+1}{0.2}\left[1.18-0.7-0.0183\right]$$

$y' = 2.3085$ rad/sec

$$y''\big|_{x\neq x_n}=\frac{1}{h}\left[\nabla^2 y_n+(v+1)\nabla^3 y_n+\frac{(6v^2+18v+11)}{12}\nabla^4 y_n+\cdots\cdots\right]$$

$$=\frac{1}{0.2}\left[0.28+(-2)(0.01)+0\right]=1.3 \quad \text{ie: } y''=1.3 \text{ rad/sec}^2.$$

⑦ Find the first, second and third derivatives of the function below at the point $x=1.5$ and $x=4.0$.

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.37	7.0	13.625	24.0	38.875	59.0

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.5	3.37					
		3.626				
2.0	7		3			
		6.625		0.75		
2.5	13.625		3.75		0	
		10.375		0.75		0
3.0	24		4.5		0	
		14.875		0.75		
3.5	38.875		5.25			
		20.125				
4.0	59					

When $x=1.5$, $y', y'', y''' = ?$ $\quad x=1.5 \quad x_0=1.5 \quad u=\frac{x-x_0}{h}=0.$

(use NFIF) $\quad h=0.5$

$$y'\Big]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{0.5}\left[3.626 - \frac{1}{2}(3) + \frac{1}{3}(0.75) - 0\right]$$

$$y' = 4.75$$

$$y''\Big]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

$$= \frac{1}{0.5^2}\left[3 - 0.75 + \frac{11}{12}(0)\right]$$

$$y'' = 9$$

$$y'''\Big]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] = \frac{1}{0.5^3}\left[0.75 - 0\right]$$

$$\therefore y''' = 6$$

when $x = 4.0$, $y', y'', y''' = ?$ $x = 4.0$ $x_n = 4.0$ ie: $x = x_n \Rightarrow v = 0.$
$h = 0.5$

$$y'\Big]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\cdots\right]$$

$$= \frac{1}{0.5}\left[20.125 + \frac{5.25}{2} + \frac{0.75}{3} + 0\right]$$

$$y' = 46.$$

$$y''\Big]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\cdots\right]$$

$$= \frac{1}{0.5^2}\left[5.25 + 0.75 + 0\right]$$

$$y'' = 24.$$

$$y'''\Big]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\cdots\right]$$

$$= \frac{1}{0.5^3}\left[0.75 + 0\right]$$

$$y''' = 6.$$

Numerical Integration

Trapezoidal rule for single integral

$$\int_a^b f(x)\,dx = \frac{h}{2}\Big[(\text{sum of the first and last ordinates}) + 2(\text{sum of the remaining ordinates})\Big]$$

ie: $$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots\cdots)\right]$$

where $h = \frac{b-a}{n}$

Simpson's $\frac{1}{3}$ rule for single integral

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[(\text{sum of the first and last ordinates}) + 2(\text{sum of the remaining even ordinates}) + 4(\text{sum of the remaining odd ordinates})\Big]$$

ie: $$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots) + 4(y_1 + y_3 + \cdots\cdots)\right]$$

① Dividing the range into 10 equal parts, find the approximate value of $\int_0^{\pi} \sin x \, dx$ by Trapezoidal & Simpson's 1/3 rule.

Sol

$a = 0, \quad b = \pi \qquad n = 10, \qquad h = \frac{b-a}{n} = \pi/10 \qquad f(x) = \sin x$

x	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$\frac{4\pi}{10}$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
$y = \sin x$	0	0.3090	0.5878	0.8090	0.9511	1	0.9511	0.8090	0.5878	0.3090	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

1. Trapezoidal rule.

$$I = \int_0^{\pi} \sin x \, dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + \cdots + y_9)\right]$$

$$= \frac{\pi}{20}\left[(0+0) + 2(0.3090 + 0.5878 + 0.8090 + 0.9511 + 1 + 0.9511 + 0.8090 + 0.5878 + 0.3090)\right]$$

$I = 1.9843.$

2. Simpson's $\frac{1}{3}$ rule

$$I = \int_0^{\pi} \sin x \, dx = \frac{h}{3}\left[(y_0 + y_{10}) + 2(y_2 + y_4 + \cdots + y_8) + 4(y_1 + y_3 + \cdots + y_9)\right]$$

$$= \frac{\pi}{30}\left[(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090)\right]$$

$I = 2.0009$

Actual value.

$$I = \int_0^{\pi} \sin x \, dx.$$

$$= \left[-\cos x\right]_0^{\pi}$$

$$= -\left[\cos\pi - \cos 0\right]$$

$$= -\left[-1-1\right]$$

$$= 2$$

② Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Trapezoidal and Simpson's $\frac{1}{3}$ rule taking $h=0.25$

Sol $a=0$ $b=2$ $h=0.25$ $f(x)=\frac{1}{x^2+4}$

x	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0
$y=\frac{1}{x^2+4}$	0.25	0.2462	0.2353	0.2192	0.2	0.1798	0.16	0.1416	0.125
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

1) Trapezoidal rule

$$I=\int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+\cdots+y_7)\right]$$

$$= \frac{0.25}{2}\left[(0.25+0.125)+2(0.2462+0.2353+0.2192+0.2 +0.1798+0.16+0.1416)\right]$$

$I=0.3924$

2) Simpson's $\frac{1}{3}$ rule.

$$I=\int_0^2 \frac{dx}{x^2+4} = \frac{h}{3}\left[(y_0+y_8)+2(y_2+y_4+y_6\cancel{+y_8})+4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{0.25}{3}\left[(0.25+0.125)+2(0.2353+0.2+0.16) +4(0.2462+0.2192+0.1798+0.1416)\right]$$

$I = 0.3927$

Actual value.

$$I=\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big]_0^2 = \frac{1}{2}\left[\tan^{-1}(1)-\tan^{-1}(0)\right]$$

$$\therefore\ I=\frac{1}{2}\left[\frac{\pi}{4}-0\right]$$

$I=\pi/8 = 0.3926$

③ Evaluate $\int_0^1 e^x\,dx$, $h=0.1$ by Simpson's $\frac{1}{3}$ rule; verify with exact value.

Sol.

$f(x)=e^x$, $h=0.1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$y=e^x$	1	1.1051	1.2214	1.3498	1.4918	1.6487	1.8221	2.0137	2.2255	2.4596	2.7182

y_0 y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 y_{10}.

Simpson's 1/3 rule.

$$I=\int_0^1 e^x\,dx=\frac{h}{3}\left[(y_0+y_{10})+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$=\frac{0.1}{3}\left[(1+2.7182)+2(1.2214+1.4918+1.8221+2.2255)+4(1.1051+1.3498+1.6487+2.0137+2.4596)\right]$$

$$I=1.7182$$

Actual value.

$$I=\int_0^1 e^x dx = e^x\Big]_0^1 = e^1-e^0=1.7182$$

$\therefore$ Error $=|\text{Exact value} - \text{approximate value}|=0$.

④ When a train moving at 30 m/s. Steam is shut off and brakes are applied. The speed of the train (v) in m/s after (t) second is given below. Using Simpson's rule determine the distance moved by the train in 40 seconds

Time t	0	5	10	15	20	25	30	35	40
velocity v	30	24	19.5	16	13.6	11.7	10	8.5	7

Sol

We know that $v=\frac{ds}{dt} \Rightarrow ds=v\,dt$.

Integrating on both sides, $\int ds=\int v\,dt$

$\Rightarrow s=\int v\,dt$

where $f(x)=v$; $h=5$

By Simpson's 1/3 rule

$$S = \int_0^{40} v\,dt = \frac{h}{3}\left[(y_0+y_9) + 2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)\right]$$

$$S = \frac{.5}{3}\left[(30+7) + 2(19.5+13.6+10) + 4(24+16+11.7+8.5)\right]$$

$$S = 606.67 \text{ m.}$$

⑤ The velocity v of a particle 's' from a point on its path is given by the table. Estimate the time taken to travel 60 feet by using Simpson's 1/3 rule.

S (m)	0	10	20	30	40	50	60
V (m/s)	47	58	64	65	61	52	38

Sol

WKT $v = \frac{dS}{dt} \Rightarrow dt = \frac{dS}{v}$

Integrating on both sides, we get $t = \int \frac{dS}{v}$

Let $f(x) = \frac{1}{v}$. $h = 10$.

Given data are in v, so we find 1/v

v :	47	58	64	65	61	52	38
$\frac{1}{v}$:	0.0212	0.0172	0.0156	0.0154	0.0164	0.0192	0.026

By Simpson's rule,

$$t = \frac{h}{3}\left[(y_0+y_7) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{10}{3}\left[(0.0212 + 0.026) + 2(0.0156 + 0.0164) + 4(0.0172 + 0.0154 + 0.0192)\right]$$

$t = 1.060$ seconds.

Trapezoidal rule for double integration

$I = \frac{hk}{4}$ [(Sum of the values of f at four corner points)

+ 2(Sum of the values of f at remaining points on the boundary)

+ 4(Sum of the values of f at the interior points)]

Simpson's $\frac{1}{3}$ rule for double integration

$I = \frac{hk}{9}$ [(Sum of the values of f at four corners)

+ 2(sum of the values of f at the odd positions on the boundary)

+ 4(Sum of the values of f at the even positions on the boundary)

+ { 4(Sum of the values of f at odd position)
+ 8(Sum of the values of f at even position) on the odd row of the matrix except boundary }

+ { 8(sum of the values of f at odd positions)
+16(Sum of the values of f at even positions) on the even rows of the matrix except boundary }

① Evaluate $\int_0^1 \int_0^1 e^{x+y}\, dx\, dy$ using Trapezoidal & Simpson's rule.

Sol

$f(x,y) = e^{x+y}$ x varies from 0 to 1

y varies from 0 to 1

Let $h = k = 0.5$ &

$x \rightarrow$ $y \downarrow$	0	0.5	1
0	1	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1	2.7183	4.4817	7.3891

Trapezoidal rule

$$I = \int_0^1 \int_0^1 e^{x+y}\,dx\,dy = \frac{(0.5)(0.5)}{4}\Big[(1+2.7183+7.3891+2.7183) + 2(1.6487+4.4817+4.4817+1.6487) + 4(2.7183)\Big]$$

$$I = 3.0763$$

Using Simpsons rule,

$$I = \frac{(0.5)(0.5)}{9}\Big[(1+2.7183+7.3891+2.7183) + 4(1.6487+1.6487+4.4817+4.4817) + 16(2.7183)\Big]$$

$$I_1 = 2.9545$$

Actual value

$$I = \int_0^1 \int_0^1 e^{(x+y)}\,dx\,dy = \int_0^1 e^x\,dx \int_0^1 e^y\,dy = (e^1-1)(e^1-1) = (e^1-1)^2$$

$$\therefore I = 2.9524$$

$$\text{Error} = |\text{Exact} - \text{app}| = 0.0021.$$

② Evaluate $\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\,dx\,dy$ using Trapezoida & Simsons rule.

Sol

$f(x,y) = \frac{1}{xy}$ $\quad x:$ 2 to 2.4 ; $y:$ 1 to 1.4

$h = \frac{2.4-2}{4} = 0.1$; $k = \frac{1.4-1}{4} = 0.1$

			④	②	④		
	x → y ↓	2	2.1	2.2	2.3	2.4	
	1	① 0.5	0.4762	0.4545	0.4348	① 0.4167	
④	1.1	0.4545	0.4329 4×4=⑯	0.4132 4×2=⑧	0.3953 4×4=⑯	0.3788	④
②	1.2	0.4167	0.3698 4×2=⑧	0.3788 2×2=④	0.3623 4×2=⑧	0.3472	②
④	1.3	0.3846	0.3663 4×4=⑯	0.3497 4×2=⑧	0.3344 4×4=⑯	0.3205	④
	1.4	0.3571 ①	0.3401	0.3247	0.3106	0.2976 ①	
			④	②	④		

Trapezoidal rule.

$$\begin{aligned} I = \frac{(0.1)(0.1)}{4} \Big[&(0.5 + 0.4167 + 0.3571 + 0.2976) \\ &+ 2(0.3846 + 0.4167 + 0.4545 + 0.4762 + 0.4545 \\ &\quad + 0.4348 + 0.3788 + 0.3472 + 0.3205 \\ &\quad + 0.3106 + 0.3247 + 0.3401 + 0.3571) \\ &+ 4(0.4329 + 0.4132 + 0.3953 + 0.3968 + 0.3788 \\ &\quad + 0.3623 + 0.3663 + 0.3497 + 0.3344)\Big] \end{aligned}$$

$I = 0.0614$

Simpson's rule.

$$\begin{aligned} I = \frac{(0.1)(0.1)}{9} \Big[&(0.5 + 0.4167 + 0.2976 + 0.3571) \\ &+ 2(0.4167 + 0.4545 + 0.3472 + 0.3247) \\ &+ 4(0.3846 + 0.4545 + 0.4762 + 0.4348 \\ &\quad + 0.3788 + 0.3205 + 0.3106 + 0.3401 \\ &\quad + 0.3788) \\ &+ 8(0.3497 + 0.4132) + 16(0.3663 + 0.3344 \\ &\quad + 0.4329 + 0.3953)\Big] \end{aligned}$$

$I = 0.0613$

Remark: For Simpson's 1/3 rule, collected all the encircled data from the above table.

③ Evaluate $\int_1^2 \int_1^2 \frac{dx\,dy}{x^2+y^2}$ with $h=0.2$, $k=0.25$ along x, y direction.

Sol

$f(x,y) = \frac{1}{x^2+y^2}$ $x:$ 1 to 2 $y =$ 1 to 2

$h = 0.2$ $k = 0.25$

$x \rightarrow$ $y \downarrow$	1	1.2	1.4	1.6	1.8	2
1	0.5	0.4098	0.3378	0.2809	0.2359	0.2
1.25	1.25	0.3902	0.3331	0.2839	0.2436	0.2082
1.5	0.3077	0.2710	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.2221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1679	0.1524	0.1381	0.125

Since no. of rows ≠ no. of columns, so only Trapezoidal rule can be applied.

$$I = \int_1^2 \int_1^2 \frac{dx\,dy}{x^2+y^2} = \cancel{hk}\,\frac{(0.2)(0.25)}{4} \Big\{ (0.5 + 0.2 + 0.2 + 0.125)$$

$$+ 2(0.4098 + 0.3378 + 0.2809 + 0.2359 + 1.25 + 0.3077 + 0.2462 + 0.1838 + 0.1679 + 0.1524 + 0.1381 + 0.1416 + 0.16 + 0.2082)$$

$$+ 4(0.3092 + 0.3331 + 0.2839 + 0.2436 + 0.2710 + 0.2375 + 0.2079 + 0.1821 + 0.2221 + 0.1991 + 0.1779 + 0.1587)\Big]$$

$I \equiv 0.2323$

CHENNAI INSTITUTE OF TECHNOLOGY
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DEPARTMENT OF
SCIENCE AND HUMANITIES

SUBJECT CODE : MA3251
SUBJECT NAME : STATISTICS AND NUMERICAL METHODS
DEPARTMENT : COMMON TO ALL BRANCES
YEAR/SEM : I/II
ACEDEMIC YEAR : 2021-2022
BATCH : XII

HANDWRITTEN NOTES

UNIT V-NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

- Taylors series method
- Eulers method and Modified Eulers Method
- Fourth order Runge Kutta for solving first order differential equations.
- Milne's and Adams-Bash Forth predictor and corrector method for solving first order differential equations.

★ TAYLOR'S SERIES METHOD

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \cdots$$

PROBLEM - 1 :

Using Taylor's series method find y at $x = 0.1$ correct to four decimal places from $\frac{dy}{dx} = x^2 - y$, $y(0)=1$, with $h = 0.1$. Compute terms upto x^4.

[A.U. May 2000, Trichy A/M - 2010]
[A.U M/J - 2012] [A.U N/D 2016] [A.U N/D - 2017]

Solution:

Given $y' = x^2 - y$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = 0.1$.

$y' = x^2 - y$	$y_0' = x_0^2 - y_0 = 0 - 1 = -1$
$y'' = 2x - y'$	$y_0'' = 2x_0 - y_0' = (0) - (-1) = 1$
$y''' = 2 - y''$	$y_0''' = 2 - y_0'' = 2 - (1) = 1$
$y^{IV} = -y'''$	$y_0^{IV} = -y_0''' = -(1) = -1$

By Taylor's series formula

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \cdots$$

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!} + \cdots$$

$$= 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24} + \cdots$$

$$= 0.905163.$$

PROBLEM - 2:

Using Taylor's series method, find y at $x = 0.1$ if $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$.

Solution:

Given $y' = x^2y - 1$ and $x_0 = 0, y_0 = 1, h = 0.1$

$y' = x^2y - 1$	$y_0' = x_0^2 y_0 - 1 = 0 - 1 = -1$
$y'' = 2xy + x^2y'$	$y_0'' = 2x_0y_0 + x_0^2 y_0'$ $= 0 + 0 = 0$
$y''' = 2[xy' + y] + x^2y'' + 2xy'$ $= 2xy' + 2y + x^2y'' + 2xy'$ $= 2y + 4xy' + x^2y''$	$y_0''' = 2y_0 + 4x_0y_0' + x_0^2 y_0''$ $= 2(1) + 4(0)(-1) + (0)^2(0)$ $= 2.$
$y^{iv} = 2y' + 4[xy'' + y'] + x^2y''' + y''2x$ $= 2y' + 4xy'' + 4y' + x^2y''' + y''2x.$ $= 6y' + 6xy'' + xy'''.$	$y_0^{iv} = 6y_0' + 6x_0y_0'' + x_0y_0'''$ $= 6(-1) + 6(0)(0) + (0)(2)$ $= -6.$

By Taylor's series formula

$$\boxed{y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots}$$

$$= 1 + \frac{(0.1)}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(-6) + \cdots$$

$$= 1 - 0.1 + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{4} + \cdots$$

$$= 1 - 0.1 + 0.00033 - 0.000025$$

$$= 0.900305 \,//$$

PROBLEM - 3

By means of Taylor's series expansion, find y at $x = 0.1, 0.2$ correct to three significant digits, given $\frac{dy}{dx} - 2y = 3e^{x}$, $y(0) = 0$.

[N/D-2011, N/D-2014, A/M-15]

Solution:

Here $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $h = 0.1$

~~$y' =$~~ $\frac{dy}{dx} - 2y = 3e^{x}$

$y' = \frac{dy}{dx} = 3e^{x} + 2y$

$y' = 3e^{x} + 2y$	$y_0' = 3e^{x_0} + 2y_0$ $= 3 + 0 = 3$
$y'' = 3e^{x} + 2y'$	$y_0'' = 3e^{x_0} + 2y_0'$ $= 3 + 6 = 9$
$y''' = 3e^{x} + 2y''$	$y_0''' = 3e^{x_0} + 2y_0''$ $= 3 + 18 = 21$
$y^{iv} = 3e^{x} + 2y'''$	$y_0^{iv} = 3e^{x_0} + 2y_0'''$ $= 3 + 42 = 45$

By Taylor's series Formula

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots$$

$$= 0 + (0.1)(3) + \frac{(0.01)}{2}(9) + \frac{(0.001)}{6}(21) + \frac{(0.0001)}{24}(45) + \cdots$$

$$= 0.3 + 0.045 + 0.0035 + 0.0001875 + \cdots = 0.349$$

$y_1' = 2y_1 + 3e^{x_1} = 0.3486875 \times 2 + 3e^{0.1} = 4.012887.$

$y_1'' = 2y_1' + 3e^{x_1} = 11.34$; $y_1''' = 2y_1'' + 3e^{x_1} = 25.996.$

$$y_2 = y(0.2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \cdots$$

$$= 0.3486875 + (0.1)(4.012887) + \frac{0.01}{2}(11.34) + \left(\frac{0.001}{6}\right)(25.996) + \cdots$$

$= 0.8110156 = 0.811$ (correct to 3 decimals)

$$\boxed{\begin{aligned} y(0.1) &= 0.3486955 \\ y(0.2 &= 0.8112658 \end{aligned}}$$

PROBLEM - 4

Using Taylor's series method, compute the values of $y(0.1)$ and $y(0.2)$ for $\frac{dy}{dx} = 1 - 2xy$, given that $y(0) = 0$.
[A.U N/D-2009, APR-2017, M/J-2016]

Solution:

Given $\frac{dy}{dx} = y' = 1 - 2xy$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$

We know the Taylor's Series formula for y_1

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

we find

$y' = 1 - 2xy$	$y_0' = 1 - 2x_0y_0 = 1 - 2(0)(0) = 1$
$y'' = -2[xy' + y]$	$y_0'' = -2[x_0y_0' + y_0]$ $= -2[0(1) + 0] = 0$
$y''' = -2[xy'' + y' + y']$ $= -2[xy'' + 2y']$	$y_0''' = -2[x_0y_0'' + 2y_0]$ $= -2[0(0) + 2(1)] = -4$

Applying in formula,

$$y_1 = 0 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(-4) + \cdots$$

$= 0 + 0.1 + 0 - 0.0006$

$= 0.0993$

$$\boxed{y_1 = 0.0993 \Rightarrow y(0.1) = 0.0993}$$

Now here, $x_1 = 0.1$, $y_1 = 0.0993$

Taylor's series formula for y_2 is

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$y' = 1 - 2xy$	$y_1' = y'(x_1, y_1)$ $= 1 - 2x_1 y_1$ $= 1 - 2(0.1)(0.0993)$ $= 0.9801$
$y'' = -2[xy' + y]$	$y_1'' = -2[x_1 y_1' + y_1]$ $= -2[(0.1)(0.9801) + 0.09931]$ $= -0.3946$
$y''' = -2[xy'' + 2y']$	$y_1''' = -2x_1 y_1'' - 4y_1'$ $= -2(0.1)(-0.3946) - 4(0.9801)$ $= -3.8414$

$$y_2 = 0.0993 + \frac{(0.1)}{1!}(0.9801) + \frac{(0.1)^2}{2!}(-0.3946) + \frac{(0.1)^3}{3!}(-3.8414)$$

$$\boxed{y_2 = 0.1947 \Rightarrow y(0.2) = 0.1947}$$

PROBLEM-5:

Apply the Taylor's series method to find the value of $y(1.1)$, $y(1.2)$ and $y(1.3)$ correct to three decimal places given that $y' = xy^{1/3}$, $y(1) = 1$, taking the first three terms of the Taylor's series expansion get the closed form solution of the differential equation and compare the actual value of y to the approximated

Value calculated.

[A.U A/M 2019 - R-17] (16-Marks)

Solution

Take $x_0 = 1$, $y_0 = 1$, $h = 0.1$

(i) To find $y(1.1)$:

By Taylor's series formula.

We first find

$y' = xy^{1/3}$	$y_0' = x_0 y_0^{1/3} = (1)(1)^{1/3}$ $= 1$
$y'' = \frac{1}{3} x y^{-2/3} y' + y^{1/3}$	$y_0'' = \frac{1}{3} x_0 y_0^{-2/3} y_0' + y_0^{1/3}$ $= \left(\frac{1}{3}\right)(1)(1)^{-2/3}(1) + (1)^{1/3}$ $= 4/3$

∴ By formula

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2!}\left(\frac{4}{3}\right) + \cdots$$

$$\boxed{y(1.1) = 1.107}$$

(ii) To find $y(1.2)$:

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \cdots$$

$y_1 = 1.107$, $x_1 = 1.1$

$y_1' = x_1 y_1^{1/3} = (\cancel{1.1})(1.107)^{\cancel{1/3}} = 1.\cancel{13}8\ (1.1)(1.107)^{1/3}$

$= 1.138$

$y_1'' = \frac{1}{3} x_1 y_1^{-2/3} y_1' + y_1^{1/3}$

$= \frac{1}{3}(1.1)(1.107)^{-2/3}(1.138) + (1.107)^{1/3}$

$= 0.390 + 1.034 = 1.424$

$\therefore y_2 = 1.107 + (0.1)(1.138) + \frac{(0.1)^2}{2!}(1.424) + \cdots$

$= 1.228$

$\Rightarrow \boxed{y_2(1.2) = 1.228}$

(iii) To find $y(1.3)$:

$y_3 = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \cdots$

$y_2 = 1.228, \quad x_2 = 1.2$

$y_2' = x_2 y_2^{1/3} = (1.2)(1.228)^{1/3} = 1.285$

$y_2'' = \frac{1}{3}x_2 y_2^{-2/3} y_2' + y_2^{1/3}$

$= \frac{1}{3}(1.2)(1.228)^{-2/3}(1.285) + (1.228)^{1/3} = 1.519$

$y_3 = 1.228 + (0.1)(1.285) + \frac{(0.1)^2}{2!}(1.519) + \cdots = 1.364$

$\Rightarrow \boxed{y(1.3) = 1.364}$

Taylor's result:

y_0	y_1	y_2	y_3
1	1.107	1.228	1.364

To get exact result.

$y' = xy^{1/3}$

$\frac{dy}{dx} = xy^{1/3}$

$y^{-1/3}\,dy = x\,dx.$

$\int y^{-1/3}\,dy = \int x\,dx$

$$\frac{y^{-1/3+1}}{-1/3+1} = \frac{x^2}{2} + \frac{c}{2}.$$

$$\frac{y^{2/3}}{2/3} = \frac{x^2}{2} + \frac{c}{2}$$

$$3y^{2/3} = x^2 + C.$$

$$\Rightarrow 3 = 1 + C \Rightarrow C = 2 \quad [\text{since } x=1, y=1]$$

$$\Rightarrow 3y^{2/3} = x^2 + 2$$

$$\Rightarrow y^{2/3} = \left[\frac{x^2+2}{3}\right]$$

$$y = \left[\frac{x^2+2}{3}\right]^{3/2}.$$

$$x_1 = 1.1 \Rightarrow y_1 = \left[\frac{(1.1)^2+2}{3}\right]^{3/2} = 1.107$$

$$x_2 = 1.2 \Rightarrow y_2 = \left[\frac{(1.2)^2+2}{3}\right]^{3/2} = 1.228.$$

$$x_3 = 1.3 \Rightarrow y_3 = \left[\frac{(1.3)^2+2}{3}\right]^{3/2} = 1.364$$

Result:

	y_0	y_1	y_2	y_3
Taylor's Series	1	1.107	1.228	1.364
Exact value	1	1.107	1.228	1.364

PROBLEM – 6

Solve, $\frac{dy}{dx} = y^2+x^2$ with $y(0)=1$, Use Taylor's Series at $x=0.2$ and 0.4. Find $x=0.1$ [M/J – 2010].

Solution:

Given $y' = y^2+x^2$, $x_0=0$, $y_0=1$.

$y' = x^2+y^2$	$y_0' = x_0^2 + y_0^2 = 1+0 = 1$
$y'' = 2yy' + 2x$	$y_0'' = 2y_0y_0' + 2x_0$ $= 2(1)(1)+0 = 2$.
$y''' = 2[yy''+y'y']+2$ $= 2yy''+2(y')^2+2$	$y_0''' = 2y_0y_0''' + 2(y_0')^2+2$ $= 2(1)(2) + 2(1)^2+2$ $= 2(2)+2+2 = 8$.

By Taylor's series

$$y = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$= 1 + 0.2 + (0.2)^2 + \frac{4}{3}(0.2)^3 + \cdots$$

$$= 1 + 0.2 + 0.04 + 0.01067$$

$$= 1.25067.$$

PROBLEM – 7:

Using Taylor's series method, find (1.1), given $y' = x+y$, $y(1)=0$.

Solution:

Given $y' = x+y$, $x_0=1$, $y_0=0$, $h=0.1$

$y' = x+y$	$y_0' = x_0 + y_0 = 1+0 = 1$
$y'' = 1+y'$	$y_0'' = 1+y_0' = 1+1 = 2$
$y''' = y''$	$y_0''' = y_0'' = 2$
$y^{iv} = y'''$	$y_0^{iv} = y_0''' = 2$

By Taylor's series method

$$\boxed{y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots}$$

$$y(1.1) = 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) +$$

$$= 0.1 + 0.01 + 0.0003 + 0.0000083$$

$$= 0.1103083$$

$$= 0.1103$$

★ EULER AND MODIFIED EULER METHOD:

Euler Method:

$$\boxed{y_{n+1} = y_n + hf(x_n, y_n),} \quad n = 0, 1, 2$$

Modified Euler Method:

$$\boxed{y_{n+1} = y(x+h) = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}hf(x_n, y_n\right)\right]}$$

PROBLEM - 1:

Using Euler's method find $y(0.2)$ and $y(0.4)$ from $\frac{dy}{dx} = x+y$, $y(0) = 1$ with $h = 0.2$

[M/J - 2012, A/M - 2015]

Solution:

$$f(x,y) = x+y, \quad x_0 = 0, \; y_0 = 1, \; x_1 = 0.2, \; x_2 = 0.4$$

By Euler's algorithm

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 1 + 0.2\,[x_0 + y_0] = 1 + (0.2)[0+1] = 1.2.$$

$$\boxed{y(0.2) = 1.2}$$

$$y_2 = y_1 + h\,f(x_1, y_1)$$

$$= 1.2 + 0.2\,[x_1 + y_1] = 1.2 + 0.2\,[0.2 + 1.2] = 1.2 + 0.28$$

$$\boxed{y(0.4) = 1.48}$$

PROBLEM - 2:

Using modified Euler's method compute $y(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}$, $y(0) = 1$ [N/D - 2010]

Solution:

By Modified Euler's method

$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]\right.$$

$$= 1 + (0.1)\left[f\left[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\left[y_0 - \frac{2x_0}{y_0}\right]\right]\right)$$

$$= 1 + (0.1)\,f\,[0.05, 1 + (0.05)(1-0)]$$

$$= 1 + (0.1)\,f\,[0.05, 1.05]$$

$$= 1 + (0.1)\left[1.058 - \frac{2(0.005)}{1.05}\right]$$

$$= 1 + (0.1)[1.05 - 0.0952]$$

$$= 1 + (0.1)(0.9548)$$

$$= 1 + 0.09548$$

$$\boxed{y_1 = 1.09548}$$

PROBLEM - 3

Evaluate $y(1.2)$ correct to three decimal places by the modified Euler's method, given that $\frac{dy}{dx} = (y - x^2)^3$, $y(1) = 0$ taking $h = 0.2$.

[M/J - 2014]

Solution:

Given $f(x, y) = (y - x^2)^3$, $x_0 = 1$, $y_0 = 0$, $h = 0.2$, $x_1 = 1.2$

By Modified Eulers method

$$y_1 = y_0 + h \left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]\right.$$

$$= 0 + (0.2) f \left[\left[1 + \frac{0.2}{2}\right], 0 + \frac{0.2}{2}\left[y_0 - x_0^2\right]^3\right]$$

$$= (0.2) f\left[1.1, (0.1)(0-1)^3\right]$$

$$= (0.2) f(1.1, -0.1)$$

$$= 0.2\left[-0.1 - (1.1)^2\right]^3$$

$$= 0.2\left[-0.1 - 1.21\right]^3 = (0.2)(-2.25)$$

$$= -0.45$$

$$\boxed{y(1.2) = -0.45}$$

PROBLEM - 4:

Consider the initial value problem $\frac{dy}{dx} = y - x^2 + 1$, $y(0) = 0.5$ using the modified Euler's method find $y(0.2)$.

[N/D - 2013, 14, M/J - 2012]

Solution:

$f(x, y) = y - x^2 + 1$, $x_0 = 0$, $y_0 = 0.5$, $h = 0.2$, $x_1 = 0.2$

By Euler's modified method

$$\boxed{y_{n+1} = y_n + hf\left[x_n + h/_2 , y_n + h/_2 f(x_n , y_n)\right]}$$

$$y_1 = y_0 + hf\left[x_0 + h/_2 , y_0 + h/_2 f(x_0 , y_0)\right]$$

$$y_1 = 0.5 + (0.2) f\left[0 + \frac{0.2}{2} , 0.5 + \frac{0.2}{2} f(0, 0.5)\right]$$

$$= 0.5 + (0.2) f\left[(0.1 , 0.5 + (0.1)[0.5 - 0 + 1]\right]$$

$$= 0.5 + (0.2) f\left[0.1 , 0.5 + 0.1 (1.5)\right]$$

$$= 0.5 + (0.2) f\left[0.1 , 0.65\right]$$

$$= 0.5 + (0.2)\left[0.65 - 0.01 + 1\right]$$

$$= 0.5 + (0.2)(1.64)$$

$$\boxed{y_1 = 0.828}$$

PROBLEM - 5:

Using modified Euler's method find $y(0.1)$ if $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.

[A.U N/D - 2020, A/M - 2021, A.U N/D-2024]

Solution:

Given $f(x,y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = 0.1$

By modified Euler's method,

$$\boxed{y_{n+1} = y_n + hf\left[x_n + h/_2 , y_n + h/_2 f(x_n , y_n)\right]}$$

$$y_1 = y_0 + hf\left[x_0 + h/_2 , y_0 + \frac{h}{2} f(x_0 , y_0)\right]$$

$$f(x_0 , y_0) = x_0^2 + y_0^2 = 0 + 1 = 1.$$

$$y_1 = 1 + (0.1) f\left[0 + \frac{0.1}{2} , 1 + \frac{0.1}{2}(1)\right]$$

$$= 1 + (0.1) f\left[0.05 , 1.05\right]$$

$$= 1 + (0.1)\left[(0.05)^2 + (1.05)^2\right] = 1.1105$$

(i.e) $\boxed{y(0.1) = 1.1105}$

PROBLEM - 6

Solve $y' = 1-y$, $y(0)=0$ by modified Euler's method. Find $y(0.1)$, $y(0.2)$ and $y(0.3)$.

[A.U. April 2005, CBT A/M - 2011]

Solution:

Given $f(x,y) = 1-y$, $x_0 = 0$, $y_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $h = 0.1$

By modified Eulers method.

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{1}{2} h f(x_n, y_n)\right],$$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$f(x_0, y_0) = 1 - y_0 = 1 - 0 = 1.$$

$$\Rightarrow y_1 = 0 + (0.1) f\left[0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}(1)\right]$$

$$= (0.1) f[0.05, 0.05] = (0.1)[1 - 0.05]$$

$$= 0.095$$

$$\boxed{y(0.1) = 0.095}$$

$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right]$$

$$f(x_1, y_1) = 1 - y_1 = 1 - 0.095 = 0.905.$$

$$\Rightarrow y_2 = (0.095) + (0.1) f\left[(0.1) + \frac{0.1}{2}, 0.095 + \frac{0.1}{2}(0.905)\right]$$

$$= (0.095) + (0.1) f[0.15, 0.14025]$$

$$= 0.095 + (0.1)[1 - 0.14025]$$

$$= 0.095 + (0.1)(0.85975) = 0.18098.$$

$$\Rightarrow \boxed{y(0.2) = 0.18098}$$

$$y_3 = y_2 + hf\left[x_2 + \frac{h}{2},\ y_2 + \frac{h}{2} f(x_2, y_2)\right]$$

$$f(x_2, y_2) = 1 - y_2 = 1 - 0.18098$$
$$= 0.81902$$

$$y_3 = 0.18098 + (0.1) f\left[0.2 + \frac{0.1}{2},\ 0.18098 + \frac{0.1}{2}(0.81902)\right]$$
$$= 0.18098 + (0.1) f\left[0.25,\ 0.18098 + 0.040951\right]$$
$$= 0.18098 + (0.1) f(0.25,\ 0.221931)$$
$$= 0.18098 + (0.1)\left[1 - 0.221931\right]$$
$$= 0.258787$$

$$\Rightarrow \boxed{y(0.3) = 0.258787}$$

☆ FOURTH ORDER RUNGE-KUTTA METHOD FOR SOLVING FIRST ORDER EQUATIONS:

Fourth order R-K Method:

$$K_1 = hf(x, y)$$
$$K_2 = hf\left[x + \frac{h}{2},\ y + \frac{K_1}{2}\right]$$
$$K_3 = hf\left[x + \frac{h}{2},\ y + \frac{K_2}{2}\right]$$
$$K_4 = hf\left[x + h,\ y + K_3\right]$$

and $\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$$y(x+h) = y(x) + \Delta y$$

PROBLEM-1:

Using R-K method of fourth order, Solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$

with $y(0)=1$ at $x=0.2$, $x=0.4$

[N/D - 2011, 2013, 2017, A/M - 2013, 2015, 2017]

Solution:

Given $y' = f(x,y) = \dfrac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $h = 0.2$

$$K_1 = h f(x_0, y_0) = (0.2)\left[\frac{y_0^2 - x_0^2}{y_0^2 + x_0^2}\right] = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$$

$$\boxed{K_1 = 0.2}$$

$$K_2 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right]$$

$$= (0.2) f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$$

$$= (0.2) f(0.1, 1.1)$$

$$= (0.2)\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$$

$$= (0.2)\left(\frac{1.2}{1.22}\right) = (0.2)\left(\frac{1.2}{1.22}\right)$$

$$= (0.2)(0.9836)$$

$$\boxed{K_2 = 0.19672}$$

$$K_3 = h f\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$$

$$= (0.2) f\left[0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2}\right]$$

$$= (0.2) f[0.1, 1.0983606]$$

$$= (0.2)\left[\frac{(1.0983806)^2 - (0.1)^2}{(1.0983606)^2 + (0.1)^2}\right]$$

$$\boxed{K_3 = 0.1967}$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$

$$= (0.2) f(0.2, 1.1967)$$

$$=(0.2)\left[\frac{(0.2)^2-(1.1967)^2}{(0.2)^2+(1.1967)^2}\right]$$

$$\boxed{K_4= 0.1891}$$

$$\therefore \Delta y = \frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$$

$$=\frac{1}{6}\left[0.2+2(0.19672)+2(0.1967)+0.1891\right]$$

$$=0.19598.$$

$$\therefore y_1 = y_0+\Delta y = 1+0.19598.$$

$$\boxed{y(0.2) = 1.19598}.$$

To find $y(0.4)$

Here $x_1=0.2$, $y_1=1.1959$, $h=0.2$.

Now $K_1 = hf(x_1, y_1)$

$$=(0.2)\left[\frac{(1.1959)^2-(0.2)^2}{(1.1959)^2+(0.2)^2}\right]$$

$$=(0.2)\left[\frac{1.4301-0.04}{1.4301+0.04}\right] = 0.1891 ; \boxed{K_1=0.1891}$$

$$K_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{K_1}{2}\right)$$

$$=(0.2)\left[\frac{\left(1.1959+\frac{0.1891}{2}\right)^2-\left(0.2+\frac{0.2}{2}\right)^2}{\left(1.1959+\frac{0.1891}{2}\right)^2+\left(0.2+\frac{0.2}{2}\right)^2}\right]$$

$$=(0.2)\left\{\frac{1.6651-0.09}{1.6651+0.09}\right\} = 0.1794.$$

$$\boxed{K_2 = 0.1794}$$

$$K_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right]$$

$$= \overset{(0.2)}{\cancel{h}}\left[\frac{\left(1.1959 + \frac{0.1794}{2}\right)^2 - \left(0.2 + \frac{0.2}{2}\right)^2}{\left(1.1959 + \frac{0.1794}{2}\right)^2 + \left(0.2 + \frac{0.2}{2}\right)^2}\right]$$

$$= (0.2)\left[\frac{1.6529 - 0.09}{1.6529 + 0.09}\right]$$

$$\boxed{K_3 = 0.1793}$$

$$K_4 = h \cdot f(x_1 + h, y_1 + k_3)$$

$$= h\left[\frac{(1.1959 + 0.1793)^2 - (0.2 + 0.2)^2}{(1.1959 + 0.1793)^2 + (0.2 + 0.2)^2}\right]$$

$$= (0.2)\left[\frac{1.8911 - 0.16}{1.8911 + 0.16}\right]$$

$$\boxed{K_4 = 0.1687}$$

$$\therefore \Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= \frac{1}{6}\left[0.1891 + 2(0.1794 + 2(0.1793) + 0.1687\right]$$

$$= \frac{1}{6}\left[0.1891 + 0.3588 + 0.3586 + 0.1687\right]$$

$$\boxed{\Delta y = 0.1792.}$$

$$\therefore y_2 = y_1 + \Delta y = 1.1959 + 0.1792$$

$$\boxed{y_2 = 1.3751}$$

$$y(0.4) = 1.3751$$

x	1	0.2	0.4
y	1	1.19598	1.3751

PROBLEM - 2

Use Runge-kutta method of the fourth order to find $y(0.2)$ and $y(0.4)$. given that $y\frac{dy}{dx} = y^2 - x, y(0) = 2$ by taking $h = 0.2$ (upto 4 decimal places.

[A.U N/D 2020] [A.U A/M 2021 R-17]

Solution

Given $y\frac{dy}{dx} = y^2 - x$.

$$\Rightarrow yy' = y^2 - x$$

$$y' = y - \frac{x}{y} = f(x, y), \quad x = 0, y_0 = 2,$$

$$x_1 = 0.2, \; x_2 = 0.4$$

$$h = 0.2.$$

By Fourth order R-K algorithm

$k_1 = hf(x, y)$	$k_2 = hf\left[x + \frac{h}{2}, y + \frac{k_1}{2}\right]$
$k_3 = hf\left[x + \frac{h}{2}, y + \frac{k_2}{2}\right]$	$k_4 = hf\left[x + h, y + k_3\right]$

$$\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$y(x+h) = y(x) + \Delta y.$$

(i) To find $y(0.2)$

$k_1 = hf(x_0, y_0)$ $= (0.2)f(0, 2)$ $= (0.2)\left(2 - \frac{0}{2}\right)$ $= 0.4$	$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$ $= (0.2)f\left[0 + \frac{0.2}{2}, 2 + \frac{0.4}{2}\right]$ $= (0.2)f[0.1, 2.2]$ $= (0.2)\left[2.2 - \frac{0.1}{2.2}\right] = 0.4309$

$K_3 = hf\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right]$

$=(0.2)f\left[0+\frac{0.2}{2}, 2+\frac{0.4309}{2}\right]$

$=(0.2)f[0.1, 2.2135]$

$=(0.2)\left[2.2155-\frac{0.1}{2.2155}\right]$

$=0.4341$

$K_4 = hf[x_0+h, y_0+K_3]$

$=(0.2)f[0+0.2, 2+0.4341]$

$=(0.2)f[0.2, 2.4341]$

$=(0.2)\left[2.4341-\frac{0.2}{2.4341}\right]$

$=0.4698.$

$\Delta y = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$=\frac{1}{6}[0.4+2(0.4309)+2(0.4341)+0.4698]$

$\boxed{\Delta y = 0.4333.}$

$y(0.2) = y_1 = y_0+\Delta y = 2+0.4333 = 2.4333.$

(i.e) $\boxed{y(0.2) = 2.4333.}$

Problem-3

Apply fourth order Runge-Kutta method to determine $y(0.1)$ and $y(0.2)$ with $h=0.1$ from $\frac{dy}{dx} = x^2+y^2$, $y(0)=1$.

[A.U. May 2000, CBT M/J 2010, A/M - 2011]
[A.U N/D - 2019]

Solution:

Given $y' = f(x,y) = x^2+y^2$, $x_0=0$, $x_1=0.1$, $x_2=0.2$, $h=0.1$

(1) To find $y(0.1)$:

$K_1 = hf(x_0, y_0)$

$=(0.1)f[0, 1]$

$=(0.1)[0+1]=0.1$

$K_2 = hf\left[x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right]$

$=(0.1)f\left[0+\frac{0.1}{2}, 1+\frac{0.1}{2}\right]$

$=(0.1)f[0.05, 1.05]$

$=(0.1)[(0.05)^2+(1.05)^2]$

$=0.1105$

$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= (0.1)f\left[0.05, 1 + \frac{0.1105}{2}\right]$

$= (0.1)f[0.05, 1.05525]$

$= (0.1)[(0.05)^2 + (1.05525)^2]$

$= 0.1116$

$K_4 = hf[x_0 + h, y_0 + k_3]$

$= (0.1)f[0 + 0.1, 1 + 0.1116]$

$= (0.1)f[0.1, 1.1116]$

$= (0.1)[(0.1)^2 + (1.1116)^2]$

$= 0.1246.$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.1 + 2(0.1105) + 2(0.1116) + 0.1246]$

$\boxed{\Delta y = 0.11147}$

$y_1 = y(0.1) = y_0 + \Delta y = 1 + 0.11147 = 1.11147.$

$= 1.1115$ (correct to 4 decimals)

$\Rightarrow \boxed{y_1 = y(0.1) = 1.115}$

(ii) To find $y(0.2)$:

Again apply R.k method.

$x_1 = 0.1, \quad y_1 = 1.1115.$

$K_1 = hf(x_1, y_1)$

$= (0.1)f(0.1, 1.1115)$

$= (0.1)[(0.1)^2 + (1.1115)^2]$

$= 0.1245.$

$K_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right]$

$= (0.1)f\left[0.1 + \frac{0.1}{2}, 1.1115 + \frac{0.1245}{2}\right]$

$= (0.1)f[0.15, 1.17375]$

$= (0.1)[(0.15)^2 + (1.17375)^2]$

$= 0.14.$

$K_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right]$

$= (0.1)f\left[0.15, 1.1115 + \frac{0.14}{2}\right]$

$= (0.1)f[0.15, 1.1815]$

$= 0.1418$

$K_4 = hf[x_1 + h, y_1 + k_3]$

$= (0.1)f[0.1 + 0.1, 1.1115 + 0.1418]$

$= (0.1)f[0.2, 1.2533]$

$= (0.1)[(0.2)^2 + (1.2533)^2] = 0.1611$

$$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= \frac{1}{6}[0.1245 + 2(0.14) + 2(0.1418 + 0.1611]$$

$$\boxed{\Delta y = 0.1415}$$

$$y_2 = y(0.2) = y_1 + \Delta y = 1.1115 + 0.1415 = 1.253$$

$$\boxed{y_2 = y(0.2) = 1.253}$$

Problem-4

Find $y(0.7)$ and $y(0.8)$ given that $y' = y - x^2$, $y(0.6) = 1.7379$ by using Runge kutta method of fourth order. Take $h = 0.1$

[M/J-2012, N/D-2016]

Solution:

Given $y' = y - x^2$, $x_0 = 0.6$, $y_0 = 1.7379$, $x_1 = 0.7$,
$x_2 = 0.8$, $h = 0.1$

$$K_1 = hf(x_0, y_0)$$
$$= (0.1)[y_0 - x_0^2]$$
$$= (0.1)[1.7379 - (0.6)^2]$$
$$K_1 = 0.13779$$

$$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$$
$$= (0.1)f[0.6 + \frac{0.1}{2}, 1.7379 + \frac{0.13779}{2}]$$
$$= (0.1)f[0.65, 1.806795]$$
$$= (0.1)[1.806795 - (0.65)^2]$$
$$K_2 = 0.13843$$

$$K_3 = hf[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}]$$
$$= (0.1)f[0.65 ~~+0.~~, 1.7379 + \frac{0.13843}{2}]$$
$$= (0.1)f[0.65, 1.807115]$$
$$= (0.1)[1.807115 - (0.65)^2]$$
$$= 0.13846.$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$
$$= (0.1)f(0.6 + 0.1, 1.7379 + 0.13846)$$
$$= (0.1)f[0.7, 1.87636]$$
$$= (0.1)f[0.7, [1.87636 - (0.7)^2]$$
$$K_4 = 0.13864$$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.13779 + 2(0.13843) + 2(0.13846) + 0.13864]$

$= 0.13837$

$y_1 = y(0.7) = y_0 + \Delta y$

$= 1.7379 + 0.13837$

$\boxed{y_1 = 1.87627}$

$= 1.876$ (app)

$\Longrightarrow \boxed{y(0.7) = 1.876}$

(ii) To find $y(0.8)$

Again apply R-K method.

$\therefore x_1 = 0.7, \quad y_1 = 1.876.$

$K_1 = hf(x_1, y_1) = (0.1) f[0.7, 1.876]$

$= (0.1)[1.876 - (0.7)^2]$

$= 0.1386$

$K_2 = h[x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}]$

$= (0.1)[0.7 + \frac{0.1}{2}, 1.876 + \frac{0.1386}{2}]$

$= (0.1) f[0.75, 1.9453]$

$= (0.1)[1.9453 - (0.75)^2]$

$= 0.13828.$

$K_3 = hf[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}]$

$= (0.1) f[0.75, 1.876 + \frac{0.13828}{2}]$

$= (0.1) f[0.75, 1.94514]$

$= (0.1)[1.94514, 0.75)^2$

$= 0.138264.$

$K_4 = hf[x_1 + h, y_1 + k_3]$

$= (0.1) f[0.7 + 0.1, 1.876 + 0138264]$

$= (0.1) f[0.8, 2.014264]$

$= (0.1)[2.014264 - (0.8)^2]$

$= 0.1374264.$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + K_4]$

(94)

$= \frac{1}{6}[0.1386 + 2(0.13828) + 2(0.138264) + 0.1374264]$

$= 0.138186.$

$y_2 = y(0.8) = y_1 + \Delta y = 1.876 + 0.138186 = 2.014186.$

(correct to 4 decmal)

$= 2.0142.$

$$\boxed{y_2 = y(0.8) = 2.0142}$$

PROBLEM-5

Given $\frac{dy}{dx} = x^3 + y$ $y(0) = 2$. Compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ by Runge kutta method of fourth order.

Solution:

Given $y' = x^3 + y = f(x,y)$, $x_0 = 0$, $y_0 = 2$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

By Fourth order R-K algorithm.

(i) To find $y(0.2)$: $x_0 = 0$, $y_0 = 2$, $h = 0.2$.

$k_1 = hf(x_0, y_0)$
$= (0.2)[x_0^3 + y_0]$
$= (0.2)[0 + 2]$
$= 0.4.$

$k_2 = hf[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}]$
$= (0.2)f[0 + \frac{0.2}{2}, 2 + \frac{0.4}{2}]$
$= (0.2) f[0.1, 2.2]$
$= (0.2)[(0.1)^3 + 2.2]$
$= 0.4402.$

$k_3 = hf[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}]$
$= (0.2)f[0 + \frac{0.2}{2}, 2 + \frac{0.4402}{2}]$
$= (0.2)f[0.1, 2.2201]$
$= (0.2)[(0.1)^3 + 2.2201] = 0.4442.$

$k_4 = hf[x_0 + h, y_0 + k_3]$
$= (0.2)[0 + 0.2, 2 + 0.4442]$
$= (0.2)f[0.2, 2.4442]$
$= (0.2)[(0.2)^3 + 2.4442]$
$= 0.4904$

$\Delta y = \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$= \frac{1}{6}[0.4 + 2(0.4402) + 2(0.4442) + 0.4904]$

$= 0.4432.$

$y(0.2) = y_1 = y_0 + \Delta y = 2 + 0.4432$

$= 2.443$ (correct to 3 decimal)

(i.e) $y(0.2) = 2.443$

ii) To find $y(0.4)$:

$h = 0.2, \ x_1 = 0.2 \ \ y_1 = 2.443.$

Again apply R-k method.

$K_1 = hf(x_1, y_1)$

$= (0.2) f[0.2, 2.443]$

$= (0.2)[(0.2)^3 + 2.443]$

$= (0.2)[2.451]$

$= 0.4902.$

$K_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right]$

$= (0.2) f\left[0.2 + \frac{0.2}{2}, 2.443 + \frac{0.4902}{2}\right]$

$= (0.2) f[0.3, 2.6881]$

$= (0.2)[(0.3)^3 + 2.6881]$

$= (0.2)(2.7151) = 0.5430.$

$K_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right]$

$= 0.2 f\left[0.2 + \frac{0.2}{2}, 2.443 + \frac{0.543}{2}\right]$

$= (0.2) f[0.3, 2.7145]$

$= (0.2)[(0.3)^3 + 2.7145]$

$= (0.2)[2.7415]$

$= 0.5483$

$K_4 = hf[x_1 + h, y_1 + k_3]$

$= (0.2) f[0.2 + 0.2, 2.443 + 0.5483]$

$= (0.2) f[0.4, 2.9913]$

$= (0.2)[(0.4)^3 + 2.9913]$

$= 0.6111.$

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= \frac{1}{6}[0.4902 + 2(0.543) + 2(0.5483) + 0.6111]$

$$= \frac{1}{6}\left[3.2839\right]$$

$$= 0.5473.$$

$$y(0.4) = y_2 = y + \Delta y = 2.\cancel{4}43 + 0.5473 = 2.99$$

$$\boxed{y(0.4) = 2.99}$$

(ii) To find $y(0.6)$:

$x_2 = 0.4,\ y_2 = 2.99,\ h = 0.2.$

Here $x_2 = 04,\ y_2 = 2.99.$

$$K_1 = hf(x_2, y_2)$$
$$= (0.2)f(0.4, 2.99)$$
$$= (0.2)\left[(0.4)^3 + 2.99\right]$$
$$= (0.2)\left[3.054\right]$$
$$= 0.6108.$$

$$K_2 = h\left[x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right]$$
$$= (0.2)f\left[0.4 + \frac{0.2}{2}, 2.99 + \frac{0.6108}{2}\right]$$
$$= (0.2)f\left[0.5, 3.2954\right]$$
$$= (0.2)\left[(0.5)^3 + 3.2954\right]$$
$$= 0.6841$$

$$K_3 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right]$$
$$= (0.2)f\left[0.4 + \frac{0.2}{2}, 2.99 + \frac{0.6841}{2}\right]$$
$$= (0.2)f\left[0.5, 3.3321\right]$$
$$= (0.2)\left[(0.5)^3 + 3.3321\right]$$
$$= (0.2)\left[3.4571\right]$$
$$= 0.6914.$$

$$K_4 = hf\left[x_2 + h, y_2 + k_3\right]$$
$$= (0.2)f\left[0.4 + 0.2, 2.99 + 0.6914\right]$$
$$= (0.2)(3.8974]$$
$$= 0.7795$$

$$\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$
$$= \frac{1}{6}\left[0.6108, 2(0.6841) + 2(0.6914) + 0.7795\right]$$
$$= \frac{1}{6}\left[4.1413\right] = 0.6902$$

$y(0.6) = y_3 = y_2 + \Delta y = 2.99 + 06902$

$= 3.68$

Hence we get

x	0	0.2	0.4	0.6
y	2	2.443	2.99	3.68

✰ R-K METHOD FOR SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS :

$$\frac{dy}{dx} = f_1(x, y, z) \quad \& \quad \frac{dz}{dx} = f_2(x, y, z)$$

with the initial conditions

$$y(x_0) = y_0 , \quad z(x_0) = z_0 .$$

[Here x is independent variable, while y and z are dependent variable].

$K_1 = hf_1[x_0, y_0, z_0]$

$K_2 = hf_1[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}]$

$K_3 = hf_1[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}]$

$K_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + l_3]$

$\Delta y = \frac{1}{6}[K_1 + 2k_2 + 2k_3 + K_4]$

$y_1 = y_0 + \Delta y .$

$l_1 = hf_2(x_0, y_0, z_0)$

$l_2 = hf_2[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}]$

$l_3 = hf_2[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}]$

$l_4 = hf_2[x_0 + \frac{h}{2}, y_0 + \frac{K_3}{2}, z_0 + \frac{l_3}{2}]$

$\Delta z = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

$z_1 = z_0 + 4z .$

Problem-1

Solve for $y(0.1)$ and $z(0.1)$ from the simultaneou differential equation $\frac{dy}{dx} = 2y + z$; $\frac{dz}{dx} = y - 3z$; $y(0) = 0$ $z(0) = 0.5$ using Runge-Kutta method of fourth order.

[N/D - 2012]

Solution:

Given $x_0 = 0$, $y_0 = 0$, $z_0 = 0.5$, $h = 0.1$

$f_1(x, y, z) = 2y + z$	$f_2(x, y z) = y - 3z$
$K_1 = h f_1 (x_0, y_0, z_0)$ $= (0.1) f_1 (0, 0, 0.5)$ $= (0.1)[2(0) + 0.5]$ $\boxed{K_1 = 0.05}$	$l_1 = h f_2 (x_0, y_0, z_0)$ $= (0.1) f_2 (0, 0, 0.5)$ $= (0.1)[0 - 3(0.5)]$ $\boxed{l_1 = -0.15}$
$K_2 = h f_1 \left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right]$ $= (0.1) f_1 \left[0 + \frac{0.1}{2}, 0 + \frac{0.05}{2}, 0.5 - \frac{0.15}{2}\right]$ $= (0.1) f_1 [0.05, 0.025, 0.425]$ $= (0.1)[2(0.025) + 0.425]$ $\boxed{K_2 = 0.0475.}$	$l_2 = h f_2 \left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{h}{2}\right]$ $= (0.1) f [0.05, 0.025, 0.425]$ $= (0.1)[0.025 - 3(0.0425)]$ $\boxed{l_2 = -0.125}$
$K_3 = h f_1 \left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right]$ $= (0.1) f_1 \left[0 + \frac{0.5}{2}, 0 + \frac{0.6475}{2}, 0.5 - \frac{0.215}{2}\right]$ $= (0.1)[2(0.02375) + 0.4375]$ $\boxed{K_3 = 0.0485}$	$l_3 = h f_2 \left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right]$ $l_3 = (0.1) f_2 (0.05, 0.2375, 0.4375)$ $l_3 = (0.1)[0.02375 - 3(0.4375)]$ $\boxed{l_3 = 0.12888}$

$K_4 = hf_1[x_0+h, y_0+K_3, z_0+l_3]$

$=(0.1)f_1[0+0.1, 0+0.485, 0.5-0.12888]$

$=(0.1)f_1[0.1, 0.0485, 0.37112]$

$=(0.1)[2(0.0485)+0.37112]$

$\boxed{K_4 = 0.04681}$

$l_4 = hf_2(x_0+h, y_0+K_3, z_0+l_3]$

$= (0.1)f_2[0.1, 0.0485, 0.37112]$

$= (0.1)[0.0485 - 3(0.37112)]$

$\boxed{l_4 = -0.10649}$

$\Delta y = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= \frac{1}{6}[0.05+2(0.0475)+2(0.0485)+0.04681]$

$\boxed{\Delta y = 0.04814}$

$\Delta z = \frac{1}{6}[l_1+2l_2+2l_3+l_4]$

$= \frac{1}{6}[-0.15+2(-0.125)+2(-0.12888)-0.10649]$

$\boxed{\Delta z = -0.12738}$

$y(0.1) = y_0+\Delta y = 0+0.04814 = 0.04814,$

$z(0.1) = z_0+\Delta z = 05-0.12793$

$z(0.1) = 0.37262$

★ MILNE'S PREDICTOR AND CORRECTOR METHODS FOR SOLVING FIRST ORDER EQUATIONS:

Milne's Predictor formula:

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

Milne's corrector Formula:

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y_n' + y'_{n+1}\right]$$

Problem-1

Given $\frac{dy}{dx} = x^3+y$, $y(0)=2$. The values of $y(0.2)=2.073$ $y(0.4)=2.452$, $y(0.6)=3.023$ are got by R.k method of 4^{th} ~~mel~~ order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h=0.2$

[N/J-2014, N/D-2017]

Solution:

$x_0 = 0$	$y_0 = 2$
$x_1 = 0.2$	$y_1 = 2.073$
$x_2 = 0.4$	$y_2 = 2.452$
$x_3 = 0.6$	$y_3 = 3.023$
$x_4 = 0.8$	$y_4 = ?$

$y' = f(x,y) = x^3+y$ — ①

By Milne's Predictor Formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y_n'\right]$$

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \quad \text{—②}$$

$① \Rightarrow \quad y' = x^3+y$

$y_1' = x_1^3 + y_1 = (0.2)^3 + 2.073 = 2.081$

$y_2' = x_2^3 + y_2 = (0.4)^3 + 2.452 = 2.516$

$y_3' = x_3^3 + y_3 = (0.6)^3 + 3.023 = 3.239.$

$② \Rightarrow y_{4,p} = 2 + \frac{4(0.2)}{3}\left[2(2.081) - 2.516 + 2(3.239)\right]$

$= 2 + \frac{0.8}{3}(8.124)$

$= 2 + 2.1664.$

$\boxed{y_{4,p} = 4.1664}$

Using Milne's corrector formula

$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right]$

$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$ — ③

$y_1' = x_1^3 + y_1 = (0.2)^3 + 2.073 = 2.081$

$y_2' = x_2^3 + y_2 = 2.516$

$y_3' = x_3^3 + y_3 = 3.239$

$y_4' = x_4^3 + y_4 = (0.8)^3 + 4.1664$

$= 4.6784.$

$(3) \Rightarrow y_{4,c} = 2.452 + \frac{0.2}{3}[2516 + 4(3.239 + 4.6784)$

$= 2.452 + \frac{0.2}{3}[20.1504]$

$y_{4,c} = 3.79536$

Problem-2

Using Milne's method find $y(4.4)$ given

$5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$.

& $y(4.3) = 1.0143$

[N/D-2014, 2017, M/J-2012, 2014]

Solution:

Given

$5xy' + y^2 - 2 = 0$

$5xy' = 2 - y^2$

$y' = \frac{2-y^2}{5x}$, $x_0 = 4$, $x_1 = 4.1$, $x_2 = 4.2$, $x_3 = 4.3$, $x_4 = 4.4$.

$y_0 =$

$x_0 = 4$ $y_0 = 1$

$x_1 = 4.1$ $y_1 = 1.0049$

$x_2 = 4.2$ $y_2 = 1.0097$

$x_3 = 4.3$ $y_3 = 1.0143$

$x_4 = 4.4$ $y_4 = ?$

$$y_1' = \frac{2-y_1^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = 0.0493$$

$$y_2' = \frac{2-y_2^2}{5x_2} = \frac{2-(1.0097)^2}{5(42)} = 0.0467.$$

$$y_3' = \frac{2-y_3^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = 0.0452.$$

By Milne's Prediction formula,

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3\right]$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0493) - 0.0467 + 2(0.0452)\right]$$

$$y_{4,p} = 1.01897.$$

$$y_4' = \frac{2-y_4^2}{5x_4} = \frac{2-(1.01897)^2}{5(4.4)} = 0.0437.$$

Using Milne's corrector formula.

$$y_{4,c} = y_0 + \frac{h}{3}(h_2' + 4y_3' + y_4)$$

$$= 1.0097 + \frac{0.1}{3}[0.0467 + 4(0.0452) + 0.0437]$$

$$y_{4,c} = 1.01874.$$

PROBLEM - 3

Using Milne's method find $y(2)$, if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$ given $y(0)=2$, $y(0.5)=2.636$, $y(0.1)=3.595$ and $y(1.5)=4.968$.

[N/D-2011, 2015].

Solution:

$x_0 = 0$	$y_0 = 2$
$x_1 = 0.5$	$y_1 = 2.636$
$x_2 = 1$	$y_2 = 3.595$
$x_3 = 1.5$	$y_3 = 4.968$
$x_4 = 2$	$y_4 = ?$

$y' = \frac{1}{2}(x+y)$.

By Milnes Predictor formula.

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3'] \quad — ①$$

$$y_1' = \frac{1}{2}[x_1 + y_1] = \frac{1}{2}[0.5 + 2.636] = 1.568.$$

$$y_2' = \frac{1}{2}[x_2 + y_2] = \frac{1}{2}[1 + 3.595] = 2.2975$$

$$y_3' = \frac{1}{2}[x_3 + y_3] = \frac{1}{2}[1.5 + 4.968] = 3.234$$

$$① \Rightarrow y_{4,p} = 2 + \frac{4(0.5)}{3}(2(1.568) - 2.2975) + 2(3.234)])$$

$$\boxed{y_{4,p} = 6.871}$$

MILNE'S CORRECTOR FORMULA :

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \quad -①$$

$$y_4' = \frac{1}{2}(x_4 + y_4) = \frac{1}{2}(2 + 6.871) = 4.4355.$$

$$y_{4,c} = 3.595 + \frac{0.5}{3}\left[2.2975 + 4(3.234) + 4.4355\right]$$

$$= 3.595 + \frac{0.5}{3}\left[19.669\right]$$

$$\boxed{y_{4,c} = 6.8732}$$

PROBLEM - 4 :

Determine the value of $y(0.4)$ using Milne's method given $y' = xy + y^2$, $y(0) = 1$; use Taylor series to get the value of $y(0.1)$, $y(0.2)$ and $y(0.3)$.

[A/M - 2015, A/M - 2017]

Solution:

Here $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $x_4 = 0.4$.

$$y' = xy + y^2$$

$$y'' = xy' + y + 2yy'$$

$$y''' = xy'' + y' + y' + 2yy'' + 2(y')^2$$

$$= xy'' + 2y' + 2yy'' + 2y'^2$$

$$y_0' = x_0y_0 + y_0^2 = 1$$

$$y_0'' = x_0y_0' + y_0 + 2y_0y_0' = 3$$

$$y_0''' = x_0y_0'' + 2y_0' + 2y_0y_0'' + 2y_0'^2$$

$$= 10.$$

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$= 1 + (0.1)(1) + \frac{(0.01)}{2}(3) + \frac{0.001}{6}(10) + \cdots$$

$$= 1 + 0.1 + 0.015 + 0.001666$$

$$\boxed{y_1 = 1.1167.}$$

$$y_1 = y(0.1) = 1.1167$$

$$y_1' = x_1 y_1 + y_1^2 + 2y_1 y_1' = (0.1)(1.3587) + 1.1167 + 2(1.1167)(1.3587).$$

$$y_1'' = x_1 y_1' + + y_1 + 2y_1 y_1'$$

$$= (0.1)(1.3587) + 1.1167 + 2(1.1167)(1.3587)$$

$$y_1'' = 4.2871$$

$$y_1''' = x_1 y_1'' + 2y_1' + 2y_1 y_1'' + 2(y_1')^2$$

$$= 16.4131.$$

$$y_2 = y_1 + \frac{h y_1'}{1!} + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

$$= 1.1167 + \frac{0.1}{1}(1.3587) + \frac{0.01}{2}(4.2871) + \frac{0.001}{6}(16.4131)$$

$$y_2 = y(0.2) = 1.2767$$

$$y_2' = x_2 y_2 + y_2^2 = (0.2)(1.2767) + (1.2767)^3 = 1.8853.$$

$$y_2'' = x_2 y_2' + y_2 + 2y_2 y_2'$$

$$= (0.2)(1.8853) + 1.2767 + 2(1.2767)(1.8853)$$

$$y_2'' = 6.4677.$$

$$y_3''' = x_2 y_2'' + 2y_2' + 2[y_2 y_2'' + (y_2')^2] = 28.6875$$

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \cdots$$

$$= 1.2767 + (0.1)(1.8853) + \frac{0.01}{2}(6.4677) + \frac{0.001}{6}(28.6875)$$

$$= 1.5023.$$

by Milne's predictor method

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$y_1' = x_1y_1 + y_1^2 = (0.1)(1.1167) + (1.1167)^2 = 1.3587$$

$$y_2' = x_2y_2 + y_2^2 = (0.2)(1.2767) + (1.2767)^2 = 1.8853$$

$$y_3' = x_3y_3 + y_3^2 = (0.3)(1.5023) + (1.5023)^2 = 2.7076.$$

$$y_{4,p} = 1 + \frac{4(0.1)}{3}[2(1.3587) - 1.8853 + 2(2.7076)]$$

$$= 1.83297.$$

by Milne's corrector formula.

$$y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.2767 + \frac{0.1}{3}[1.8853 + 4(2.7076 + 4.09296]$$

$$\boxed{y_{4,c} = 1.83698}$$

Problem-6

Using R-K method of order 4. find y for $x=0.1$, 0.2, 0.3 given that $\frac{dy}{dx} = xy + y^2$, $y(0)=1$ and also using Milne's method find the solution at $x=0.4$ [N/D - 2014]

Solution:

Given $y' = xy + y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

Using R-K method

$$K_1 = hf(x_0, y_0)$$

$$= (0.1)[x_0y_0 + y_0^2]$$

$$= (0.1)(0+1)$$

$$K_1 = 0.1$$

$$K_2 = hf[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}]$$

$$= (0.1)f[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}]$$

$$= (0.1)f[0.05, 1.05]$$

$$= (0.1)[(0.05) + (1.05)^2]$$

$$K_2 = 0.1155$$

$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= (0.1) f[0.05, 1.05775]$

$= (0.1)\left[(0.1)(1.11717) + (1.24807)\right]$

$K_3 = 0.11717$

$K_4 = hf(x_0 + h, y_0 + k_3)$

$= (0.1) f[0 + 0.1, 1 + 0.11717]$

$= (0.1) f[0.1, 1.11717]$

$= (0.1)\left[(0.1)(1.11717) + 1.24807\right]$

$K_4 = 0.13598$

$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.1 + 2(0.1155) + 2(0.11717) + (0.13598)\right]$

$= \frac{1}{6}[0.70132]$

$\Delta y = 0.11689$

To find $y(0.2)$

$x_1 = 0.1, \quad y_1 = 1.11689,$

$K_1 = hf(x_1, y_1)$

$= (0.1) f(0.1, 1.11689)$

$= (0.1)\left[(0.1)(1.11689) + (1.11689)^2\right]$

$= (0.1)[0.111689 + 1.24744]$

$K_1 = 0.1359.$

$K_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right].$

$= (0.1) f\left[0.1 + \frac{0.1}{2}, 1.11689 + \frac{0.1359}{2}\right]$

$= (0.1) f[0.15, 1.18484]$

$= (0.1)\left[0.1(1.18484) + (1.18484)^2\right]$

$= (0.1)[0.177726 + 1.403846]$

$K_2 = 0.1582.$

$K_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right]$

$= (0.1) f\left[0.1 + \frac{0.1}{2}, 1.11689 + \frac{0.1582}{2}\right]$

~~$= (0.1)[0.177786 + 1.403846]$~~

$= (0.1)\left[(0.15)(1.19579) + (1.19599)^2\right]$

$= (0.1)[0.1798985 + 1.43039208]$

$K_3 = 0.16098$

$K_4 = hf[x_1 + h, y_1 + k_3]$

$= (0.1) f[0.2, 1.27787]$

$= (0.1)[0.255574 + 1.63295]$

$k_4 = 0.1889$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1359 + 2(0.1582) + 2(0.16098) + 0.1889]$$

$$= \frac{1}{6}[0.96316]$$

$$\Delta y = 0.16053$$

$$y(0.2) = y_1 + \Delta y = 1.11689 + 0.16053 = 1.2774$$

To find $y(0.3)$

$$x_2 = 0.2, \quad y_2 = 1.2774$$

$$K_1 = hf(x_2, y_2) = (0.1) f(0.2, 12774)$$

$$= (0.1)[(0.2)(1.2774) + (1.2774)^2]$$

$$\boxed{K_1 = 0.1887}$$

$$K_2 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right]$$

$$= (0.1) f\left[0.2 + \frac{0.1}{2}, 12774 + \frac{0.1887}{2}\right]$$

$$= (0.1) f[0.25, 1.37175]$$

$$= (0.1)[(0.25)(1.37175) + (1.37175)^2]$$

$$= (0.1)[0.3429375 + 1.881698]$$

$$\boxed{K_2 = 0.22246}$$

$$K_3 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}\right]$$

$$= (0.1) f\left[0.2 + \frac{0.1}{2}, 1.2774 + \frac{0.22246}{2}\right]$$

$$= (0.1) f(0.25, 1.38863)$$

$$= (0.1)[(0.25)(1.38863) + (1.38863)^2$$

$$\boxed{K_3 = 6.2275}$$

$$K_4 = hf(x_2 + h, y_2 + k_3)$$

$$= (0.1) f(0.3, 1.5049)$$

$$= (0.1)[(0.3)(1.5049) + (1.5049)^2]$$

$$\boxed{K_4 = 0.2716.}$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1887 + 2(0.2246) + 2(0.2275) + 0.2716]$$

$$\Delta y = 0.2267$$

$$y(0.3) = y_2 + \Delta y = 1.2774 + 0.2267.$$

$$y(0.3) = 1.5041$$

$x_0 = 0$	$y_0 = 1$
$x_1 = 0.1$	$y_1 = 1.11889$
$x_2 = 0.2$	$y_2 = 1.274$
$x_3 = 0.3$	$y_3 = 1.5041$
$x_4 = 0.4$	$y_4 = ?$

$y = xy + y^2$

$y_0' = 0$

$y_1' = x_1 y_1 + y_1^2$
$= (0.1)(1.11689) + (1.11689)^2$
$= 1.3591$

$y_2' = x_2 y_2 + y_2^2$
$= (0.2)(1.2774) + (1.2774)^2$
$= 1.8872.$

$y_3' = x_3 y_3 + y_3^2 = (0.3)(1.5041) + (1.5041)^2$
$= 2.7136.$

By Milne's Predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' + y_2' + 2y_3']$$

$$= 1 + \frac{4(0.1)}{3}[2(1.3591) - (1.8872) + 2(2.7136)]$$

$$y_{4,p} = 1.8344.$$

$x_4 = 0.4$, $y_4 = 1.8344$

$$y_4' = x_4 y_4 + y_4^2 = (0.4)(1.8344) + (1.8344)^2 = 4.09818.$$

Using Milne's corrector formula

$$y_{4,c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.2774 + \frac{0.1}{3}[1.8872 + 4(2.7136) + 4.09878]$$

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$\boxed{y_{4,c} = 1.8387}$

* Use Milne's Predictor - corrector formula to find $y(0.4)$ given $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}$, $y(0)=1$, $y(0.1)=1.06$, $y(0.2)=1.12$ and $y(0.3)=1.21$).

Ans : $y_{4,p} = 0.7979$
$y_{4,c} = 1.2797$

ADAMS - BASH FORTH PREDICTOR AND CORRECTOR METHODS :

Predictor formula

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

Corrector formula

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

PROBLEM - 1

Given $\frac{dy}{dx} = x^2(1+y)$, $y(1)=1$, $y(1.1)=1.233$, $y(1.2)=1.548$. $y(1.8)=1.979$, evaluate $y(1.4)$ by Adam - Bashforth method.

Solution :

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$ $x_3 = 1.3$, $x_4 = 1.4$

$y_0 = 1$, $y_1 = 1.233$ $y_2 = 1.548$ $y_3 = 1.979$, $y_4 = ?$

By Adam's method to find $y(1.4)$:

Predictor formula is

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right] \quad (1)$$

Here $y_0' = x_0^2(1+y_0) = (1)^2(1+1) = 2.$

$y_1' = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.70193$

$y_2' = x_2^2(1+y_2) = (1.2)^2(1+1.548) = 3.66912.$

$y_3' = x_3^2(1+y_3) = (1.3)^2(1+1.979) = 5.0345.$

$$(1) \Rightarrow y_{4,p} = 1.979 + \frac{0.1}{24}\left[55(5.0345) - 59(3.66912) + 37(2.70193) - 9(2)\right]$$

$$= 1.979 + \frac{0.1}{24}\left[276.8975 - 216.47808 + 99.97141 - 18\right]$$

$= 2.5721$

$$\boxed{y_{4,p} = 2.5721}$$

Corrector method:

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

$$y_4' = x_4^2(1+y_4) = (1.4)^2\left[1 + 2.5721\right]$$

$= 7.0013.$

$$② \Rightarrow y_{4,c} = 1.979 + \frac{0.1}{24}\left[9(7.0013) + 19(5.0345) - 5(3.66912) + 2.70193)\right]$$

$$= 2.5749 \Rightarrow y_{4,c} = 2.5749$$

$$y(1.4) = 2.5749$$

PROBLEM - 2

Using Adams - Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

Soln:

Given: $y' = \frac{2 - y^2}{5x}$. Let $h = 0.1$

Given $x_0 = 4$, $y_0 = 1$, $x_1 = 4.1$, $y_1 = 1.0049$, $x_2 = 4.2$, $y_2 = 1.0097$. $x_3 = 4.3$, $y_3 = 1.0143$.

Adam's predictor formula is

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}\right.$$

putting $n = 3$, we have.

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 52y_2' + 37y_1' - 9y_0'\right]$$

$$y_0' = (y')_{(x_0,y_0)} = \frac{2 - y_0^2}{5x_0} = 0.05.$$

$$y_1' = (y')_{(x_1,y_1)} = \frac{2 - y_1^2}{5x_1} = 0.0483$$

$$y_2' = (y')_{(x_2,y_2)} = \frac{2 - y_2^2}{5x_2} = 0.0467$$

$$y_3' = (y')(x_3, y_3) = \frac{2-y_3^2}{5x_3} = 0.0452$$

Using these values in (1), we get

$$y_{4,P} = 1.0143 + \frac{0.1}{24}\left[55(0.0452) - 59(0.0467) + 37(0.0483) - 9(0.05)\right]$$

$$= 1.01413 + \frac{0.1}{24}(4.2731 - 3.2053) = 1.0186$$

$$y(4.4) = 1.0186.$$

Adam's corrector formula is

$$y_{n+1,c} = y_n + \frac{h}{24}\left(9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right)$$

Putting n=3 we get

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right] \quad \text{②}$$

Now $y_4' = (y')(x_4, y_4) = \frac{2-y_4^2}{5x_4} = 0.0437.$

∴ (2) becomes

$$y_{4,c} = 1.0143 + \frac{0.1}{24}\left[9(0.0437) + 19(0.0452) - 5(0.0467) + 0.0483\right]$$

$$= 1.0143 + \frac{0.1}{24} \times 1.0669 = 1.0187.$$

$$\therefore y(4.4) = 1.0187.$$