UNIT- I
MATRICES

* Characteristic equation
* Eigen values & Eigen vectors
* Properties
* Cayley - Hamilton theorem
* Diagonalization of matrices
* Canonical form
* Nature of quadratic form
* Applications: Stretching of an elastic membrane.

CHARACTERISTIC EQUATION:-

The equation $|A-\lambda I| = 0$ is said to be the characteristic equation of the transformation or the characteristic equation of the matrix A.

Note:-

* characteristic eqn for 2x2 matrix is $\lambda^2 - S_1\lambda + S_2 = 0$,

where S_1 = Sum of the main diagonal elements

$S_2 = |A|$

* characteristic eqn for 3x3 matrix is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$,

where S_1 = Sum of the main diagonal elements

S_2 = Sum of the minors of the main diagonal elements

$S_3 = |A|$

Problem:- 1

Find the characteristic eqn of $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Soln:-

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

The characteristic eqn of A is $\lambda^2 - S_1\lambda + S_2 = 0$

S_1 = Sum of the main diagonal elements

$= 1 + 2 = 3$

$S_2 = |A| = 2 - 0 = 2$

$\therefore$ characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

Problem: 2

Find the characteristic eqn of $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$

Soln:-

The char. eqn of $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ 5 & 2 & -4 \end{bmatrix}$

is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements

$= 2+1-4 = -1$

S_2 = Sum of the minors of the main diagonal elements.

$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$

$= (-4-6) + (-8+5) + (2+9)$

$= 10-3+11 = -2$

$S_3 = \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix}$

$= 2(-4-6) + 3(-12+15) + 1(6+5)$

$= -20+9+11$

$= 0$

$\therefore$ Char. eqn is $\lambda^3 + \lambda^2 - 2\lambda = 0$

* Eigen values and Eigen vectors of a real matrix:-

Procedure:-

Step 1: Find the characteristic equation $|A-\lambda I| = 0$

Step 2: Solving the characteristic eqn, we gen Eigen values

Step 3: To find the eigen vectors, solve $(A-\lambda I)X = 0$

Non symmetric matrices with non repeated Eigen values:-

Problem: 1

Find the Eigen values & Eigen vectors of the matrix

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ (Ap-2019)

Soln:-

Let $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Step 1:- The char. eqn of A is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Sum of the main diagonal elements

$= 3+5+3 = 11$

S_2 = Sum of the minor of the main diagonal elements.

$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$

$= (15-1) + (9-1) + (15-1)$

$= 36$

$$S_3 = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$= 3(15-1)+1(-3+1)+1(1-5)$

$= 42-2-4 = 36$

$\therefore$ char eqn is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

Step II:-

Put $\lambda = 2$, $2^3 - 11(2^2) + 36(2) - 36 = 0$

$$\begin{array}{r|rrrr} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$\lambda^2 - 9\lambda + 18 = 0$

$(\lambda-6)(\lambda-3) = 0$

$\therefore \lambda = 3, 6$

$\therefore$ The Eigen values are $2, 3, 6$.

Step III:-

To find Eigen vector.

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (3-\lambda)x_1 - x_2 + x_3 &= 0 \\ -x_1 + (5-\lambda)x_2 - x_3 &= 0 \\ x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \text{— I}$$

Case (i) put $\lambda = 2$ in I

$x_1 - x_2 + x_3 = 0$ — (1)

$-x_1 + 3x_2 - x_3 = 0$ — (2)

$x_1 - x_2 + x_3 = 0$ — (3)

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$\therefore X_1 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Case (ii) put $\lambda = 3$ in I

$0x_1 - x_2 + x_3 = 0$ — (4)

$-x_1 + 2x_2 - x_3 = 0$ — (5)

$x_1 - x_2 + 0x_3 = 0$ — (6)

solve (4) & (5)

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii) put $\lambda=6$ in (I)

$$-3x_1 - x_2 + x_3 = 0 \quad \text{--- (7)}$$
$$-x_1 - x_2 - x_3 = 0 \quad \text{--- (8)}$$
$$x_1 - x_2 - 3x_3 = 0 \quad \text{--- (9)}$$

Solve (7) & (8)

$$\frac{x_1}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$$\therefore X_3 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \Rightarrow X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Result:-

Eigenvalues	Eigenvectors
2	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
3	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
6	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Problem 2:-

Find the Eigen values and Eigen vectors of the matrix

$$A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

(A/M-2018)

Soln:-

Given $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$

Step 1:-

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 11 - 2 - 6 = 3$$

$$S_2 = \begin{vmatrix} -2 & -5 \\ -4 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -7 \\ 10 & -6 \end{vmatrix} + \begin{vmatrix} 11 & -4 \\ 7 & -2 \end{vmatrix}$$

$$= (12-20) + (-66+70) + (-22+28)$$
$$= -8 + 4 + 6$$
$$= 2$$

$$S_3 = \begin{vmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{vmatrix}$$

$$= 11(12-20) + 4(-42+50) - 7(-28+20)$$
$$= -88 + 32 + 56$$
$$= 0$$

$\therefore$ Char eqn is $\lambda^3 - 3\lambda^2 + 2\lambda = 0$

Step-2:- $\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\lambda = 0$, $\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-2)(\lambda-1) = 0$

$\lambda = 2, 1$

$\therefore$ The eigen values are $0, 1, 2$.

Step 3:-

Solve $(A - \lambda I)X = 0$

$$\begin{pmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned}(11-\lambda)x_1-4x_2-7x_3&=0\\ 7x_1-(2+\lambda)x_2-5x_3&=0\\ 10x_1-4x_2-(6+\lambda)x_3&=0\end{aligned}\right\}\quad -\text{I}$$

case(i) put $\lambda=0$ in I

$11x_1-4x_2-7x_3=0$ —1

$7x_1-2x_2-5x_3=0$ —2

$10x_1-4x_2-6x_3=0$ —3

Solve 1 & 2

$$\frac{x_1}{\begin{vmatrix}-4 & -7\\ -2 & -5\end{vmatrix}}=\frac{x_2}{\begin{vmatrix}-7 & 11\\ -5 & 7\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}11 & -4\\ 7 & -2\end{vmatrix}}$$

$$\frac{x_1}{20-14}=\frac{x_2}{-49+55}=\frac{x_3}{-22+28}$$

$$\frac{x_1}{6}=\frac{x_2}{6}=\frac{x_3}{6}$$

$$\therefore X_1=\begin{bmatrix}6\\6\\6\end{bmatrix}\Rightarrow X_1=\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

case(ii) Put $\lambda=1$ in I

$10x_1-4x_2-7x_3=0$ —4

$7x_1-3x_2-5x_3=0$ —5

$10x_1-4x_2-7x_3=0$ —6

Solve 4 & 5

$$\frac{x_1}{\begin{vmatrix}-4 & -7\\ -3 & -5\end{vmatrix}}=\frac{x_2}{\begin{vmatrix}-7 & 10\\ -5 & 7\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}10 & -4\\ 7 & -3\end{vmatrix}}$$

$$\frac{x_1}{20-21}=\frac{x_2}{-49+50}=\frac{x_3}{-30+28}$$

$$\frac{x_1}{-1}=\frac{x_2}{1}=\frac{x_3}{-2}$$

$$\therefore X_2=\begin{bmatrix}-1\\1\\-2\end{bmatrix}\Rightarrow X_2=\begin{bmatrix}1\\-1\\2\end{bmatrix}.$$

Case(iii) put $\lambda=2$ in I

$9x_1-4x_2-7x_3=0$ —7

$7x_1-4x_2-5x_3=0$ —8

$10x_1-4x_2-8x_3=0$ —9

Solve 7 & 8

$$\frac{x_1}{\begin{vmatrix}-4 & -7\\ -4 & -5\end{vmatrix}}=\frac{x_2}{\begin{vmatrix}-7 & 9\\ -5 & 7\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}9 & -4\\ 7 & -4\end{vmatrix}}$$

$$\frac{x_1}{20-28}=\frac{x_2}{-49+45}=\frac{x_3}{-36+28}$$

$$\frac{x_1}{-8}=\frac{x_2}{-4}=\frac{x_3}{-8}$$

$$\therefore X_3=\begin{bmatrix}-8\\-4\\-8\end{bmatrix}\Rightarrow X_3=\begin{bmatrix}2\\1\\2\end{bmatrix}$$

Result:-

Eigenvalues	0	1	2
Eigen vector	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}-1\\1\\-2\end{bmatrix}$	$\begin{bmatrix}2\\1\\2\end{bmatrix}$

Problem: 3:-

Find the Eigen values and Eigen vectors of a matrix

$$A=\begin{bmatrix}-1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$$ (N/D-2018)

Soln:-

Given $A=\begin{bmatrix}-1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$

Step 1:-

Let the char. eqn be

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = -1+2+0 = 1$

$S_2 = \begin{vmatrix}2 & 1\\ -1 & 0\end{vmatrix} + \begin{vmatrix}-1 & -2\\ -1 & 0\end{vmatrix} + \begin{vmatrix}-1 & 2\\ 1 & 2\end{vmatrix}$

$= (0+1) + (0-2) + (-2-2)$

$= -5$

$S_3 = \begin{vmatrix}-1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{vmatrix}$

$= -1(0+1) - 2(0+1) - 2(-1+2)$

$= -5$

Step-2

put $\lambda=1 \Rightarrow 1^3 - 1^2 - 5 + 5 = 0$

$\therefore \lambda = 1$ is a root

$$\begin{array}{c|cccc} 1 & 1 & -1 & -5 & 5\\ & 0 & 1 & 0 & -5\\ \hline & 1 & 0 & -5 & \boxed{0} \end{array}$$

$\lambda^2 - 5 = 0$

$\lambda^2 = 5$

$\lambda = \pm\sqrt{5}$

$\therefore$ The eigen values are, $1, -\sqrt{5}, \sqrt{5}$.

Step-3:- Solve $(A-\lambda I)X = 0$

$$\begin{bmatrix}-1-\lambda & 2 & -2\\ 1 & 2-\lambda & 1\\ -1 & -1 & 0-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\left.\begin{aligned}(-1-\lambda)x_1 + 2x_2 - 2x_3 &= 0\\ x_1 + (2-\lambda)x_2 + x_3 &= 0\\ -x_1 - x_2 - \lambda x_3 &= 0\end{aligned}\right\} \text{(I)}$$

Case(i) put $\lambda = 1$ in (I)

$-2x_1 + 2x_2 - 2x_3 = 0$ — (1)

$x_1 + x_2 + x_3 = 0$ — (2)

$-x_1 - x_2 - x_3 = 0$ — (3)

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix}2 & -2\\ 1 & 1\end{vmatrix}} = \frac{x_2}{\begin{vmatrix}-2 & -2\\ 1 & 1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-2 & 2\\ 1 & 1\end{vmatrix}}$$

$$\frac{x_1}{2+2} = \frac{x_2}{-2+2} = \frac{x_3}{-2-2}$$

$$\frac{x_1}{4} = \frac{x_2}{0} = \frac{x_3}{-4}$$

$\therefore X_1 = \begin{bmatrix}4\\ 0\\ -4\end{bmatrix} = \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$

Case(ii) put $\lambda = \sqrt{5}$ in (I)

$(-1-\sqrt{5})x_1 + 2x_2 - 2x_3 = 0$ — (4)

$x_1 + (2-\sqrt{5})x_2 + x_3 = 0$ — (5)

$-x_1 - x_2 - \sqrt{5}x_3 = 0$ — (6)

Solve (4) & (5)

$$\frac{x_1}{\begin{vmatrix}2 & -2\\ 2-\sqrt{5} & 1\end{vmatrix}} = \frac{x_2}{\begin{vmatrix}-2 & -1-\sqrt{5}\\ 1 & 1\end{vmatrix}} = \frac{x_3}{\begin{vmatrix}-1-\sqrt{5} & 2\\ 1 & 2-\sqrt{5}\end{vmatrix}}$$

$$\frac{x_1}{2+4-2\sqrt{5}} = \frac{x_2}{-2+1\sqrt{5}} = \frac{x_3}{-2-2\sqrt{5}+\sqrt{5}+5-2}$$

$$\frac{x_1}{6-2\sqrt5} = \frac{x_2}{-1+\sqrt5} = \frac{x_3}{1-\sqrt5}$$

$$\frac{x_1}{(-1+\sqrt5)(-1+\sqrt5)} = \frac{x_2}{-1+\sqrt5} = \frac{x_3}{-(-1+\sqrt5)}$$

$$\frac{x_1}{-1+\sqrt5} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} -1+\sqrt5 \\ 1 \\ -1 \end{bmatrix}$$

case (iii) Put $\lambda = -\sqrt5$ in (1)

$(-1+\sqrt5)x_1 + 2x_2 - 2x_3 = 0 \rightarrow (7)$

$x_1 + (2+\sqrt5)x_2 + x_3 = 0 \rightarrow (8)$

$-x_1 - x_2 + \sqrt5 x_3 = 0 \rightarrow (9)$

Solve (8) & (9)

$$\frac{x_1}{\begin{vmatrix} 2+\sqrt5 & 1 \\ -1 & \sqrt5 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & 1 \\ \sqrt5 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 2+\sqrt5 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{2\sqrt5+5+1} = \frac{x_2}{-1-\sqrt5} = \frac{x_3}{-1+2+\sqrt5}$$

$$\frac{x_1}{2\sqrt5+6} = \frac{x_2}{-(1+\sqrt5)} = \frac{x_3}{1+\sqrt5}$$

$$\frac{x_1}{(1+\sqrt5)(1+\sqrt5)} = \frac{x_2}{-(1+\sqrt5)} = \frac{x_3}{1+\sqrt5}$$

$$\therefore X_3 = \begin{bmatrix} 1+\sqrt5 \\ -1 \\ 1 \end{bmatrix}.$$

Non Symmetric Matrices with Repeated Eigen Values

Problem 1:-

Find all the eigen values & Eigen vector of the matrix

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Soln:-

Let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Step-1:-

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = -2+1+0 = -1$

$$S_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= (0-12) + (0-3) + (2-4)$

$= -21$

$$S_3 = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

$\therefore$ Char. eqn is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

Step-2

Put $\lambda = -3$, $(-3)^3 + (-3)^2 - 21(-3) - 45 = 0$

-3	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

$\lambda^2 - 2\lambda - 15 = 0$

$(\lambda+3)(\lambda-5) = 0$

$\lambda = -3, 5$

$\therefore -3, -3, 5$ are eigen values.

Step 3:-

To find Eigen vectors,

Solve $(A-\lambda I)X = 0$

$$\begin{pmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (-2-\lambda)x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + (1-\lambda)x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 - \lambda x_3 &= 0 \end{aligned}\right\} \quad (I)$$

Case(i) Put $\lambda = 5$ in (I)

$-7x_1 + 2x_2 - 3x_3 = 0$ — (1)

$2x_1 - 4x_2 - 6x_3 = 0$ — (2)

$-x_1 - 2x_2 - 5x_3 = 0$ — (3)

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} -3 & -7 \\ -6 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\therefore X_1 = \begin{bmatrix} -24 \\ -48 \\ 24 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Case(ii) Put $\lambda = -3$ in (I)

$x_1 + 2x_2 - 3x_3 = 0 \rightarrow (4)$

$2x_1 + 4x_2 - 6x_3 = 0 \rightarrow (5)$

$-x_1 - 2x_2 + 3x_3 = 0 \rightarrow (6)$

All the equations are same,

Take (4)

Put $x_1 = 0$

$2x_2 - 3x_3 = 0$

$2x_2 = 3x_3$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Case(iii)

Put $x_2 = 0$

$x_1 - 3x_3 = 0$

$x_1 = 3x_3$

$$\frac{x_1}{3} = \frac{x_3}{1}.$$

$$\therefore X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Problem-2:-

Find the eigen values & Eigen vectors of $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Soln:-

Step 1:-

Let the Char. eqn be

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 3 - 3 + 7 = 7$

$$S_2 = \begin{vmatrix} -3 & -4 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 10 \\ -2 & -3 \end{vmatrix}$$

$= (-21+20) + (21-15) + (-9+20)$

$= -1+6+11$

$= 16$

$$S_3 = \begin{vmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{vmatrix}$$

$= 3(-21+20) - 10(-14+12) + 5(-10+9)$

$= 12$

$\therefore$ The char. eqn is $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

Step 2:-

$\lambda = 2 \Rightarrow 8 - 28 + 32 - 12 = 0$

$\therefore \lambda = 2$ is a one root

$$\begin{array}{c|cccc} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-3)(\lambda-2) = 0$

$\lambda = 3, 2$

$\therefore 2, 2, 3$ are the eigen values

Step-3:-

Solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (3-\lambda)x_1 + 10x_2 + 5x_3 &= 0 \\ -2x_1 - (3+\lambda)x_2 - 4x_3 &= 0 \\ 3x_1 + 5x_2 + (7-\lambda)x_3 &= 0 \end{aligned}\right\} \rightarrow \text{I}$$

Case(i) put $\lambda = 3$ in I

$0x_1 + 10x_2 + 5x_3 = 0 \rightarrow (1)$

$-2x_1 - 6x_2 - 4x_3 = 0 \rightarrow (2)$

$3x_1 + 5x_2 + 4x_3 = 0 \rightarrow (3)$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} 10 & 5 \\ -6 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 5 & 0 \\ -4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 10 \\ -2 & -6 \end{vmatrix}}$$

$$\frac{x_1}{-40+30} = \frac{x_2}{-10-0} = \frac{x_3}{0+20}$$

$$\frac{x_1}{-10} = \frac{x_2}{-10} = \frac{x_3}{20}$$

$$\therefore X_1 = \begin{bmatrix} -10 \\ -10 \\ 20 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} +1 \\ 1 \\ -2 \end{bmatrix}$$

case(ii) Put $\lambda = 2$ in I

$x_1 + 10x_2 + 5x_3 = 0 \rightarrow (4)$

$-2x_1 - 5x_2 - 4x_3 = 0 \rightarrow (5)$

$3x_1 + 5x_2 + 5x_3 = 0 \rightarrow (6)$

Solve (4) & (5)

$$\frac{x_1}{\begin{vmatrix} 10 & 5 \\ -5 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 5 & 1 \\ -4 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 10 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{-40+25} = \frac{x_2}{-10+4} = \frac{x_3}{-5+20}$$

$$\frac{x_1}{-15} = \frac{x_2}{-6} = \frac{x_3}{15}$$

$$\frac{x_1}{5} = \frac{x_2}{2} = \frac{x_3}{-5}$$

$$\therefore X_2 = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix} = X_3.$$

Problem 3:-

Find the Eigen values and Eigen vectors of $\begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$.

Soln:-

Let $A = \begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$

Step 1:-

Let the char. eqn be

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 6 - 13 + 4 = -3$

$S_2 = \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$

$= (-52+60) + (24-35) + (-78+84)$

$= 8 - 11 + 6 = 3$

$S_3 = \begin{vmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{vmatrix}$

$= 6(-52+60) + 6(56-70) + 5(-84+91)$

$= 6(8) + 6(-14) + 5(7)$

$= -1$

$\therefore$ The char. eqn is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$.

Step-2:-

Put $\lambda = -1 \Rightarrow -1 + 3 - 3 + 1 = 0$

$\therefore \lambda = -1$ is a root.

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$\therefore \lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)(\lambda+1) = 0$

$\lambda = -1, -1$

$\therefore \lambda = -1, -1, -1$ are the eigen values.

Step-3:-

Solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 6x_2 + 5x_3 &= 0 \\ 14x_1 - (13+\lambda)x_2 + 10x_3 &= 0 \\ 7x_1 - 6x_2 + (4-\lambda)x_3 &= 0 \end{aligned}\right\} \rightarrow (1)$$

Case (i) Put $\lambda = -1$ in (1)

$7x_1 - 6x_2 + 5x_3 = 0 \rightarrow (1)$

$14x_1 - 12x_2 + 10x_3 = 0 \rightarrow (2)$

$7x_1 - 6x_2 + 5x_3 = 0 \rightarrow (3)$

All the three eqn. are same

Put $x_1 = 0$ in (1)

$-6x_2 + 5x_3 = 0$

$-6x_2 = -5x_3$

$\frac{x_2}{-5} = \frac{x_3}{-6}$

$\therefore X_1 = \begin{bmatrix} 0 \\ -5 \\ -6 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$

Put $x_2 = 0$ in (1)

$7x_1 + 5x_3 = 0$

$7x_1 = -5x_3$

$\frac{x_1}{-5} = \frac{x_3}{7}$ $\quad \therefore X_2 = \begin{bmatrix} -5 \\ 0 \\ 7 \end{bmatrix}$

Put $x_3 = 0$ in (1)

$7x_1 - 6x_2 = 0$

$7x_1 = 6x_2$

$\frac{x_1}{6} = \frac{x_2}{7}$ $\quad \therefore X_3 = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}$.

Problem: 4

Find the eigen values and Eigen vectors of $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Soln:-

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Given matrix is an upper triangular matrix. Hence 2, 2, 2 are the eigen values.

Solve $(A-\lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \text{(I)}$$

When $\lambda = 2$

$0x_1 + x_2 + 0x_3 = 0 \rightarrow (1)$

$0x_1 + 0x_2 + x_3 = 0 \rightarrow (2)$

$0x_1 + 0x_2 + 0x_3 = 0 \rightarrow (3)$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$\therefore X_1 = X_2 = X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Symmetric Matrices with Non-Repeated Eigen values:

Problem: 1

Find the eigen values and Eigen vectors of $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Soln:-

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Step-1

Let the char. eqn be

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 2+2+2 = 6$.

$S_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

$= (4-0) + (4-1) + (4-0)$

$= 11$

$S_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$

$= 2(4-0) - 0 + 1(0-2)$

$= 8-2 = 6$

$\therefore$ The char. eqn is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Step-2:-

$\lambda = 1 \Rightarrow 1-6+11-6 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-2)(\lambda-3) = 0$

$\lambda = 2, 3$

$\therefore$ 1, 2, 3 are eigen values.

Step-3 :- solve $(A-\lambda I)=0$

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1+0x_2+x_3&=0 \\ 0x_1+(2-\lambda)x_2+0x_3&=0 \\ x_1+0x_2+(2-\lambda)x_3&=0 \end{aligned}\right\} \rightarrow \text{I}$$

case(i) put $\lambda=1$ in I

$x_1+0x_2+x_3=0 \rightarrow (1)$

$0x_1+x_2+0x_3=0 \rightarrow (2)$

$x_1+0x_2+x_3=0 \rightarrow (3)$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-1}=\frac{x_2}{0}=\frac{x_3}{1}$$

$$\therefore X_1=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

case(ii) put $\lambda=2$ in I

$0x_1+0x_2+x_3=0 \rightarrow (4)$

$0x_1+0x_2+0x_3=0 \rightarrow (5)$

$x_1+0x_2+0x_3=0 \rightarrow (6)$

Solve (4) & (6)

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{x_1}{0}=\frac{x_2}{1}=\frac{x_3}{0} \therefore X_2=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

case(iii) put $\lambda=3$ in I

$-x_1+0x_2+x_3=0 \rightarrow (7)$

$0x_1-x_2+0x_3=0 \rightarrow (8)$

$x_1+0x_2-x_3=0 \rightarrow (9)$

Solve (7) & (8)

$$\frac{x_1}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x_1}{1}=\frac{x_2}{0}=\frac{x_3}{1}$$

$$\therefore X_3=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Problem : 2

Find the Eigen values and Eigen vectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

Soln :-

$$\text{Let } A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Step 1 :-

Let the char eqn be

$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

$S_1=8+7+3=18$

$$S_2=\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix}+\begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix}+\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$=(21-16)+(24-4)+(56-36)$

$=45$

$$S_3=\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$= 8(21-16) + 6(-18+8) + 2(24-14)$

$= 0$

$\therefore$ The char eqn is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

Step-2:-

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\lambda = 0, (\lambda-3)(\lambda-15) = 0$

$\lambda = 0, 3, 15$

$\therefore$ The eigen values are 0, 3, 15.

Step 3:-

Solve $(A-\lambda I)x = 0$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (8-\lambda)x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + (7-\lambda)x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \rightarrow (I)$$

case (i) Put $\lambda = 0$ in (I)

$8x_1 - 6x_2 + 2x_3 = 0 \rightarrow (1)$

$-6x_1 + 7x_2 - 4x_3 = 0 \rightarrow (2)$

$2x_1 - 4x_2 + 3x_3 = 0 \rightarrow (3)$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 8 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} \quad \therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case (ii) put $\lambda = 3$ in (I)

$5x_1 - 6x_2 + 2x_3 = 0$ — (4)

$-6x_1 + 4x_2 - 4x_3 = 0$ — (5)

$2x_1 - 4x_2 + 0x_3 = 0$ — (6)

Solve (4) & (5)

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & 5 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\therefore X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

case (iii) put $\lambda = 15$ in (I)

$-7x_1 - 6x_2 + 2x_3 = 0$ — (7)

$-6x_1 - 8x_2 - 4x_3 = 0 \rightarrow (8)$

$2x_1 - 4x_2 - 12x_3 = 0 \rightarrow (9)$

Solve (7) & (8)

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -7 \\ -4 & -6 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$

$$\therefore X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Symmetric Matrices with Repeated Eigen values :-

Problem 1:-

Find the Eigen values and Eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

soln:-

Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Step 1:-

Let the char. eqn. be

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 0$

$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$= -1 - 1 - 1 = -3$

$S_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 - 1(0-1) + 1(1-0) = 2.$

$\therefore$ char. eqn is $\lambda^3 - 3\lambda - 2 = 0$

Step 2:- $\lambda = -1 \Rightarrow -1 - 3(-1) - 2 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda+1)(\lambda-2) = 0$

$\lambda = -1, +2$

$\therefore$ The eigen values are $2, -1, -1$

Step-3:- Solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -\lambda x_1 + x_2 + x_3 &= 0 \\ x_1 - \lambda x_2 + x_3 &= 0 \\ x_1 + x_2 - \lambda x_3 &= 0 \end{aligned}\right\} \rightarrow \text{I}$$

Case(i) $\lambda = 2$ in I

$-2x_1 + x_2 + x_3 = 0 \rightarrow (1)$

$x_1 - 2x_2 + x_3 = 0 \rightarrow (2)$

$x_1 + x_2 - 2x_3 = 0 \rightarrow (3)$

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \quad \therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

case(ii) $\lambda = -1$ in I.

$x_1 + x_2 + x_3 = 0$ — (4)

$x_1 + x_2 + x_3 = 0$ — (5)

$x_1 + x_2 + x_3 = 0$ — (6)

All the eqn. are same.

Put $x_1 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$$\frac{x_2}{-1} = \frac{x_3}{1} \quad \therefore X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Case(iii)

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 in orthogonal to x_1 & x_2.

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow l + m + n = 0$ — (7)

$\begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \Rightarrow 0l - m + n = 0$ — (8)

Solve (7) & (8)

$$\frac{l}{\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}} = \frac{m}{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{l}{2} = \frac{m}{-1} = \frac{n}{-1} \quad \therefore X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

Properties of Eigen values and Eigen vectors.

Prop 1:-

* The sum of the Eigen values of a matrix is the sum of the elements of the main diagonal.

(or)

The sum of the Eigen values of a matrix is equal to the trace of the matrix.

* Product of the Eigen values is equal to the determinant of the matrix

Prop-2:-

A square matrix A and its transpose A^T have the same Eigen values

Proof:-

Let A be a square matrix of order n.

The char. eqn of A & A^T are

$|A-\lambda I|=0$ — (I) & $|A^T-\lambda I|=0$ — (II)

Since, the determinant value is unaltered by the inter change of rows & columns

W.K.T $|A|=|A^T|$

Hence (I) and (II) are identical

$\therefore$ The Eigen values of A and A^T are the same.

Prop-3:-

The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Prop-4:-

If λ is an Eigen value of a matrix A, then $\frac{1}{\lambda}$, $(\lambda\neq 0)$ is the Eigen value of A^{-1}.

Proof:-

If X be the Eigen vector corresponding to λ, then

$$AX=\lambda x \quad \text{—(1)}$$

$\times$ by A^{-1}

$$A^{-1}AX=\lambda A^{-1}X$$

$$IX=\lambda A^{-1}X$$

$\div$ by λ

$$\frac{1}{\lambda}X=A^{-1}X$$

$$A^{-1}X=\frac{1}{\lambda}X.$$

This being of the same form as (1) shows that $\frac{1}{\lambda}$ is an Eigen value of the A^{-1}.

Prop-5:-

If λ is an Eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also its Eigen value.

Prop-6:-

If $\lambda_1, \lambda_2, \dots \lambda_n$ are the Eigen values of a matrix A, then A^m has the Eigen values $\lambda_1^m, \lambda_2^m, \dots \lambda_n^m$.

Proof:-

Let λ_i be the Eigen value of A and x_i the corresponding Eigen vector. Then

$$Ax_i = \lambda_i x_i \quad -① $$

We have
$$A^2 x_i = A(Ax_i)$$
$$= A(\lambda_i x_i) \quad \text{by } ①$$
$$= \lambda_i (A x_i)$$
$$= \lambda_i (\lambda_i x_i)$$
$$= \lambda_i^2 x_i$$

III^{ly} $A^3 x_i = \lambda_i^2 x_i$

In general,
$$A^m x_i = \lambda_i^m x_i \quad -②$$

① & ② are in same form.

Prop.7:-

The Eigen values of a real symmetric matrix are ~~equal~~ real numbers.

Prop-8:-

The Eigenvectors corresponding to distinct Eigenvalue of a real symmetric matrix are orthogonal.

Prop-9:-

The similar matrices have same Eigen values.

Reduction of Quadratic form to canonical form by orthogonal transformation:-

Quadratic form:-

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

Note:-

The matrix corresponding to the quadratic form is

$$\begin{bmatrix} \text{coeff } x_1^2 & \tfrac{1}{2}\text{coeff } x_1x_2 & \tfrac{1}{2}\text{coeff } x_1x_3 \\ \tfrac{1}{2}\text{coeff } x_2x_1 & \text{coeff } x_2^2 & \tfrac{1}{2}\text{coeff } x_2x_3 \\ \tfrac{1}{2}\text{coeff } x_3x_1 & \tfrac{1}{2}\text{coeff } x_3x_2 & \text{coeff } x_3^2 \end{bmatrix}$$

Index:- (S)

Number of positive square terms in the canonical form.

Signature:- (2S−r)

Difference of number of positive and negative square terms in the canonical form.

Rank:- (r)

Number of square terms in the canonical form.

Nature of the Quadratic form:-

* Positive definite ⇒ If all the Eigen values are positive.
* Positive Semi definite ⇒ If all the Eigen values are positive and atleast one zero.
* Negative definite ⇒ If all eigen values are negative.
* Negative semi definite ⇒ If all the Eigen values are negative and atleast one zero.
* Indefinite ⇒ In all other cases.

Test for Nature of a quadratic form through Principal minors:-

i) The Q.F is positive definite if $D_1, D_2, \ldots D_n$ are all positive

ie, $D_n > 0$ for all n,

where $D_1, D_2, \ldots, D_n$ are the Principal minors of A.

ii) The Q.F is negative definite if $D_1, D_3, D_5 \ldots$ are all negative and $D_2, D_4, \ldots D_6$ are all Positive.

ie, $(-1)^n D_n > 0$ for all n.

iii) The Q.F is positive semi definite if $D_n \geq 0$ and atleast one $D_i = 0$

iv) The Q.F is negative semi definite if $(-1)^n D_n \geq 0$ and atleast one $D_i = 0$.

v) The Q.F is indefinite in all other cases.

Working Producer:-

Step-1 : Write the matrix of the Quadratic form.

Step-2:- Find the char. eqn.

Step-3:- Find the Eigen Values

Step-4:- Find the Eigen vectors.

Step-5:- Form Normalized matrix N.

Step 6:- Find N^T.

Step 7:- Find AN

Step 8: Find $D = N^T AN$

Step 9: Canonical form.

$$[y_1 \; y_2 \; y_3] D \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Problem-1:-

Reduce the QF $Q = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$ into canonical form by an orthogonal transformation.

Soln:-

Given

$$Q = 6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$$

Step-1 Matrix form:-

$$A = \begin{bmatrix} \text{coeff } x_1^2 & \tfrac{1}{2}\text{coeff } x_1x_2 & \tfrac{1}{2}\text{coeff } x_1x_3 \\ \tfrac{1}{2}\text{coeff } x_1x_2 & \text{coeff } x_2^2 & \tfrac{1}{2}\text{coeff } x_2x_3 \\ \tfrac{1}{2}\text{coeff } x_3x_1 & \tfrac{1}{2}\text{coeff } x_3x_2 & \text{coeff } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Step 2:-

To find char. eqn.

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 6+3+3 = 12$$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= (9-1) + (18-4) + (18-4)$$

$$= 36$$

$$S_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 32$$

$\therefore$ The char. eqn

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Step 3:-

To find Eigen values:-

If $\lambda = 2 \Rightarrow 8 - 48 + 72 - 32 = 0$

$\therefore \lambda = 2$ is a root.

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\therefore \lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 8)(\lambda - 2) = 0$$

$$\lambda = 2, 8$$

$\therefore$ The Eigen values are $8, 2, 2$

Step 4:- To find Eigen vectors

Solve $(A - \lambda I)X = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\ -2x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 2x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \text{— (I)}$$

Case (i) Put $\lambda = 8$ in (I)

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \text{— (1)}$$
$$-2x_1 - 5x_2 - x_3 = 0 \quad \text{— (2)}$$
$$2x_1 - x_2 - 5x_3 = 0 \quad \text{— (3)}$$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 2 & -2 \\ -1 & -2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} \quad \therefore X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

case (ii) put $\lambda = 2$ in (1)

$4x_1 - 2x_2 + 2x_3 = 0 \rightarrow (4)$

$-2x_1 + x_2 - x_3 = 0 \rightarrow (5)$

$2x_1 - x_2 + x_3 = 0 \rightarrow (6)$

All the eqn are same,

Take $2x_1 - x_2 + x_3 = 0$

put $x_1 = 0$, we get

$-x_2 + x_3 = 0$

$x_2 = x_3 \quad \therefore X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

case (iii)

To find the third Eigen vector orthogonal to x_1 and x_2. since the matrix A is symmetric.

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 is orthogonal to X_1 and X_2.

$X_1^T X_3 = 0 \Rightarrow [2 \ -1 \ 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$

$\Rightarrow 2l - m + n = 0 - (7)$

$X_2^T X_3 = 0 \Rightarrow [0 \ 1 \ 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$

$\Rightarrow 0l + m + n = 0 - (8)$

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{m}{\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{l}{-2} = \frac{m}{-2} = \frac{n}{2} \quad \therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Step 5:- Normalised matrix N:-

$$N = \begin{bmatrix} \frac{2}{\sqrt6} & 0 & \frac{1}{\sqrt3} \\ \frac{-1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{-1}{\sqrt3} \end{bmatrix}$$

Step 6:- Find N^T

$$N^T = \begin{bmatrix} \frac{2}{\sqrt6} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt6} \\ 0 & \frac{1}{\sqrt2} & 1/\sqrt2 \\ 1/\sqrt3 & 1/\sqrt3 & -1/\sqrt3 \end{bmatrix}$$

Step 7:- Find AN

$$AN = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt6} & 0 & 1/\sqrt3 \\ \frac{-1}{\sqrt6} & 1/\sqrt2 & 1/\sqrt3 \\ 1/\sqrt6 & 1/\sqrt2 & -1/\sqrt3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12+2+2}{\sqrt6} & \frac{0-2+2}{\sqrt2} & \frac{6-2-2}{\sqrt3} \\ \frac{-4-3-1}{\sqrt6} & \frac{0+3-1}{\sqrt2} & \frac{-2+3+1}{\sqrt3} \\ \frac{4+1+3}{\sqrt6} & \frac{0-1+3}{\sqrt2} & \frac{2-1-3}{\sqrt3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{16}{\sqrt6} & 0 & 2/\sqrt3 \\ -8/\sqrt6 & 2/\sqrt2 & 2/\sqrt3 \\ 8/\sqrt6 & 2/\sqrt2 & -2/\sqrt3 \end{bmatrix}$$

Step-8:- Find $D = N^T AN$

$$D = \begin{bmatrix} \frac{2}{\sqrt6} & \frac{-1}{\sqrt6} & 1/\sqrt6 \\ 0 & \frac{1}{\sqrt2} & 1/\sqrt2 \\ +1/\sqrt3 & 1/\sqrt3 & -1/\sqrt3 \end{bmatrix} \begin{bmatrix} 16/\sqrt6 & 0 & 2/\sqrt3 \\ -8/\sqrt6 & 2/\sqrt2 & 2/\sqrt3 \\ 8/\sqrt6 & 2/\sqrt2 & -2/\sqrt3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32+8+8}{6} & \frac{0-2+2}{\sqrt{12}} & \frac{4-2-2}{\sqrt{18}} \\ \frac{0-8+8}{\sqrt{12}} & \frac{0+2+2}{2} & \frac{0+2-2}{\sqrt{6}} \\ \frac{16-8-8}{\sqrt{16}} & \frac{0+2-2}{\sqrt{6}} & \frac{2+2+2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 9: canonical form:-

$$[y_1\ y_2\ y_3] \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

$\therefore$ The canonical form is $8y_1^2 + 2y_2^2 + y_3^2$.

Problem: 2:-

Reduce the quadratic form $2x^2+5y^2+3z^2+4xy$ to a canonical form through an orthogonal transformation. Find also its nature. (APr-2018)

Soln:- Given $Q = 2x^2+5y^2+3z^2+4xy$

Step 1:- Matrix form

$$A = \begin{bmatrix} \text{co.eff } x_1^2 & \frac{1}{2}\text{co.eff } x_1x_2 & \frac{1}{2}\text{co.eff } x_1x_3 \\ \frac{1}{2}\text{co.eff } x_2x_1 & \text{co.eff } x_2^2 & \frac{1}{2}\text{coeff } x_2x_3 \\ \frac{1}{2}\text{co.eff } x_3x_1 & \frac{1}{2}\text{coeff } x_3x_2 & \text{co.eff } x_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Step 2:- To find the char. eqn:-

Let the char. eqn. be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 2+5+3 = 10$$

$$S_2 = \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix}$$

$$= 15 + 6 + (10-4)$$

$$= 27$$

$$S_3 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$= 2(15-0) - 2(6-0) = 18$$

$\therefore$ char eqn is $\lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

Step 3:- To find the Eigen values

Put $\lambda = 1 \Rightarrow 1 - 10 + 27 - 18 = 0$

$\therefore \lambda = 1$ is a root.

$$\begin{array}{c|cccc} 1 & 1 & -10 & 27 & -18 \\ & 0 & 1 & -9 & 18 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$(\lambda-3)(\lambda-6) = 0$$

$$\lambda = 3, 6$$

$\therefore$ The Eigen values are 1, 3, 6.

Step-4:- To find Eigenvectors:-

Solve $(A-\lambda I)X=0$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1+2x_2+0x_3&=0 \\ 2x_1+(5-\lambda)x_2+0x_3&=0 \\ 0x_1+0x_2+(3-\lambda)x_3&=0 \end{aligned}\right\} \text{(I)}$$

Case (i) Put $\lambda=1$ in (I)

$x_1+2x_2+0x_3=0 \rightarrow (1)$

$2x_1+4x_2+0x_3=0 \rightarrow (2)$

$0x_1+0x_2+2x_3=0 \rightarrow (3)$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{4}=\frac{x_2}{-2}=\frac{x_3}{0}$$

$$\therefore X_1=\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Case (ii) Put $\lambda=3$ in (I)

$-x_1+2x_2+0x_3=0 \;-(4)$

$2x_1+2x_2+0x_3=0 \;-(5)$

$0x_1+0x_2+0x_3=0 \;-(6)$

Solve (4) & (5)

$$\frac{x_1}{\begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix}}$$

$$\frac{x_1}{0}=\frac{x_2}{0}=\frac{x_3}{-2-4}$$

$$\therefore X_2=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Case (iii) Put $\lambda=6$ in (I)

$-4x_1+2x_2+0x_3=0 \rightarrow (7)$

$2x_1-x_2+0x_3=0 \rightarrow (8)$

$0x_1+0x_2-3x_3=0 \rightarrow (9)$

Solve (8) & (9)

$$\frac{x_1}{\begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 0 & 2 \\ -3 & 0 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{3}=\frac{x_2}{6}=\frac{x_3}{0}$$

$$X_3=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

Step 5:-

Normalised matrix N:-

$$N=\begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$$

Step 6:-

$$N^T=\begin{bmatrix} \frac{2}{5} & -1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & 2/\sqrt{5} & 0 \end{bmatrix}$$

Step 7:- Find AN

$$AN = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt5} & 0 & 1/\sqrt5 \\ -1/\sqrt5 & 0 & 2/\sqrt5 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4-2}{\sqrt5} & 0 & \frac{2+4}{\sqrt5} \\ \frac{4-5}{\sqrt5} & 0 & \frac{2+10}{\sqrt5} \\ 0 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt5} & 0 & \frac{6}{\sqrt5} \\ -1/\sqrt5 & 0 & 12/\sqrt5 \\ 0 & 3 & 0 \end{bmatrix}$$

Step 8:- Find $D = N^T AN$

$$D = \begin{bmatrix} \frac{2}{\sqrt5} & \frac{-1}{\sqrt5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt5 & 2/\sqrt5 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt5 & 0 & \frac{6}{\sqrt5} \\ -1/\sqrt5 & 0 & 12/\sqrt5 \\ 0 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4+1}{5} & 0 & \frac{12-12}{5} \\ 0 & 3 & 0 \\ \frac{2-2}{5} & 0 & \frac{6+24}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Step 9:- canonical form:-

$$(y_1\, y_2\, y_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= y_1^2 + 3y_2^2 + 6y_3^2.$$

Step 10:-

Nature of the Q.F is positive definite.

Problem 3:-

Reduce the quadratic form $2xy - 2yz + 2xz$ into canonical form by an orthogonal transformation & find its nature, index, rank, & signature.

Soln:- Given $Q = 2xy - 2yz + 2xz$

Step 1: Matrix form:-

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Step 2:- To find the char. eqn:-

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 0$

$$S_2 = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1) + (0-1) + (0-1) = -3$$

$$S_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 0(0-1) - 1(0+1) + 1(1-0)$$

$$= -2$$

$\therefore$ The char eqn is

$$\lambda^3 - 3\lambda + 2 = 0.$$

Step 3:- To find Eigen values:-

$\lambda=1 \Rightarrow 1-3+2=0$

$\therefore \lambda=1$ is a root

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$\lambda^2+\lambda-2=0$

$(\lambda+2)(\lambda-1)=0$

$\lambda=-2, 1$

$\therefore$ The Eigen values are $-2, 1, 1$.

Step 4:- To find Eigen vectors

solve $(A-\lambda I)X=0$

$$\begin{bmatrix} 0-\lambda & 1 & 1 \\ 1 & 0-\lambda & -1 \\ 1 & -1 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -\lambda x_1+x_2+x_3 &=0 \\ x_1-\lambda x_2-x_3 &=0 \\ x_1-x_2-\lambda x_3 &=0 \end{aligned}\right\} \rightarrow \text{I}$$

case(i) $\lambda=-2$ in I

$2x_1+x_2+x_3=0 \rightarrow (1)$

$x_1+2x_2-x_3=0 \rightarrow (2)$

$x_1-x_2+2x_3=0 \rightarrow (3)$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-3}=\frac{x_2}{3}=\frac{x_3}{3}$$

$$\therefore X_1=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

case(ii) put $\lambda=1$ in I

$-x_1+x_2+x_3=0 \rightarrow (4)$

$x_1-x_2-x_3=0 \rightarrow (5)$

$x_1-x_2-x_3=0 \rightarrow (6)$

All eqn are same.

Take $x_1-x_2-x_3=0$

Put $x_2=0 \Rightarrow x_1=x_3$

$$\therefore X_2=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

case (iii)

Let $X_3=\begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 is orthogonal to X_1 & X_2:

$-l+m+n=0 \rightarrow (7)$

$l+0m+n=0 \rightarrow (8)$

solve (7) & (8)

$$\frac{l}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}}=\frac{m}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}=\frac{n}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}}$$

$$\frac{l}{1}=\frac{m}{1+1}=\frac{n}{-1}$$

$$\therefore X_3=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Step 6:- Normalised matrix N:-

$$N=\begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

Step-6 Find N^T

$$N^T = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

Step 7:- Find AN

$$AN = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0+1+1}{\sqrt{3}} & \frac{0+0+1}{\sqrt{2}} & \frac{0+2-1}{\sqrt{6}} \\ \frac{-1+0-1}{\sqrt{3}} & \frac{1+0-1}{\sqrt{2}} & \frac{1+0+1}{\sqrt{6}} \\ \frac{-1-1+0}{\sqrt{3}} & \frac{1+0+0}{\sqrt{2}} & \frac{1-2+0}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{3}} & 1/\sqrt{2} & 1/\sqrt{6} \\ -2/\sqrt{3} & 0 & 2/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

Step 8:- Find $D = N^T A N$

$$D = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -2/\sqrt{3} & 0 & 2/\sqrt{6} \\ -2/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2-2-2}{3} & \frac{-1+0+1}{\sqrt{6}} & \frac{-1+2-1}{\sqrt{18}} \\ \frac{2+0-2}{\sqrt{6}} & \frac{1+0+1}{2} & \frac{1+0-1}{\sqrt{12}} \\ \frac{2-4+2}{\sqrt{18}} & \frac{1+0-1}{\sqrt{12}} & \frac{1+4+1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -6/3 & 0 & 0 \\ 0 & 2/2 & 0 \\ 0 & 0 & 6/6 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 9:- canonical form

$$(y_1\, y_2\, y_3) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -2y_1^2 + y_2^2 + y_3^2.$$

Step 10:- Nature of the Quadratic form is Indefinite

Index:-

No. of positive square terms in the canonical form = 2.

Rank:-

No. of square terms in the canonical form = 3

Signature:-

Difference between No. of positive & negative square terms

$= 2 - 1$

$= 1.$

Problem:4:-

Reduce the quadratic form $x_1^2+2x_2^2+x_3^2-2x_1x_2+2x_2x_3$ to the canonical form through an orthogonal transformation and hence show that it is positive semi definite. Also give a non-zero set of values (x_1, x_2, x_3) which makes this quadratic form zero.

soln:-

step 1: Matrix form

$$A=\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}$$

step-2:-

Let the char eqn be

$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$

$S_1 = 1+2+1 = 4$

$$S_2=\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix}+\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}+\begin{vmatrix}1 & -1\\ -1 & 2\end{vmatrix}$$

$=(2-1)+(1-0)+(2-1)$

$=3$

$$S_3=\begin{vmatrix}1 & -1 & 0\\ -1 & 2 & 1\\ 0 & 1 & 1\end{vmatrix}$$

$=1(2-1)+1(-1-0)+0$

$=0$

$\therefore$ The char. eqn. is

$\lambda^3-4\lambda^2+3\lambda=0$

step-3:- $\lambda^3-4\lambda^2+3\lambda=0$

$\lambda(\lambda^2-4\lambda+3)=0$

$\lambda(\lambda-3)(\lambda-1)=0$

$\lambda=0, 1, 3$

$\therefore$ Eigen values are $0, 1, 3$.

step-4:-

To find the Eigen vectors

Solve $(A-\lambda I)X=0$

$$\begin{bmatrix}1-\lambda & -1 & 0\\ -1 & 2-\lambda & 1\\ 0 & 1 & 1-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\left.\begin{aligned}(1-\lambda)x_1-x_2+0x_3&=0\\ -x_1+(2-\lambda)x_2+x_3&=0\\ 0x_1+x_2+(1-\lambda)x_3&=0\end{aligned}\right\}\rightarrow \text{(I)}$$

case(i) put $\lambda=0$ in (I)

$x_1-x_2+0x_3=0 \rightarrow (1)$

$-x_1+2x_2+x_3=0 \rightarrow (2)$

$0x_1+2x_2+x_3=0 \rightarrow (3)$

Solve (1) & (2)

$$\frac{x_1}{\begin{vmatrix}-1 & 0\\ 2 & 1\end{vmatrix}}=\frac{x_2}{\begin{vmatrix}0 & 1\\ 1 & -1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1 & -1\\ -1 & 2\end{vmatrix}}$$

$$\frac{x_1}{-1}=\frac{x_2}{-1}=\frac{x_3}{2-1}$$

$$\therefore X_1=\begin{bmatrix}1\\ 1\\ -1\end{bmatrix}$$

Case (ii) Put $\lambda = 1$ in (I)

$0x_1 - x_2 + 0x_3 = 0 \rightarrow (4)$

$-x_1 + x_2 + x_3 = 0 \rightarrow (5)$

$0x_1 + x_2 + 0x_3 = 0 \rightarrow (6)$

Solve (4) & (5)

$$\frac{x_1}{\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Case (iii) Put $\lambda = 3$ in (I)

$-2x_1 - x_2 + 0x_3 = 0 \rightarrow (7)$

$-x_1 - x_2 + x_3 = 0 \rightarrow (8)$

$0x_1 + x_2 - 2x_3 = 0 \rightarrow (9)$

$$\frac{x_1}{\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix}} = \frac{x_2}{\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Step-5:- Normalised Matrix N:-

$$N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Step-6:- Find N^T

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

Step-7:- Find AN

$$AN = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-1+0}{3} & \frac{1+0+0}{\sqrt{2}} & \frac{-1-2+0}{\sqrt{6}} \\ \frac{-1+2-1}{\sqrt{3}} & \frac{-1+0+1}{\sqrt{2}} & \frac{1+4+1}{\sqrt{6}} \\ \frac{0+1-1}{\sqrt{3}} & \frac{0+0+1}{\sqrt{2}} & \frac{0+2+1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/\sqrt{2} & -3/\sqrt{6} \\ 0 & 0 & 6/\sqrt{6} \\ 0 & 1/\sqrt{2} & 3/\sqrt{6} \end{bmatrix}$$

Step 8:- Find $D = N^T AN$

$$D = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & -3/\sqrt{6} \\ 0 & 0 & 6/\sqrt{6} \\ 0 & 1/\sqrt{2} & 3/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1+0-1}{\sqrt{6}} & \frac{-3+6-3}{\sqrt{18}} \\ 0 & \frac{1+0+1}{2} & \frac{-3+0+3}{\sqrt{12}} \\ 0 & \frac{-1+0+1}{\sqrt{12}} & \frac{3+12+3}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Step-9:- canonical form:-

$$(y_1\ y_2\ y_3)\ D \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0y_1^2 + y_2^2 + 3y_3^2$$

Step-10:-

Nature of the Q.F:-

canonical form contains both Positive and atleast one zero.

$\therefore$ It is positive definite.

To find non-zero setofvalues (x_1, x_2, x_3) which makes this quadratic form zero.

y_1 is when $y_2=0$, $y_3=0$ and y_1 is arbitrary, the orthogonal transformation $X = NY$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x_1 = \tfrac{1}{\sqrt{3}} y_1 + \tfrac{1}{\sqrt{2}} y_2 - \tfrac{1}{\sqrt{6}} y_3$$

$$x_2 = \tfrac{1}{\sqrt{3}} y_1 + \tfrac{2}{\sqrt{6}} y_3$$

$$x_3 = -\tfrac{1}{\sqrt{3}} y_1 + \tfrac{1}{\sqrt{2}} y_2 + \tfrac{1}{\sqrt{6}} y_3$$

Taking $y_1=\sqrt{3}$, $y_2=0$, and $y_3=0$ we get $x_1=1$, $x_2=1$, $x_3=-1$

These values x_1, x_2, x_3 makes the Q.F zero.

Problem: 5:-

The Eigen values of a 3×3 real symmetric matrices A corresponding to the Eigen values $2, 3, 6$ are $[1, 0, -1]^T$, $[1, 1, 1]^T$ and $[-1, 2, -1]^T$ respectively, find the matrix A.

Soln:-

Given: Eigen values are $2, 3, 6$

Eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

The Normalised Matrix

$$N = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

W.K.T $D = N^T A N$

$ND = NN^T A N$

$NDN^T = (NN^T) A (NN^T)$

$NDN^T = A$

$\Rightarrow A = NDN^T$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

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$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-6}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{3}} & \frac{12}{\sqrt{6}} \\ \frac{-2}{\sqrt{2}} & \frac{3}{\sqrt{3}} & \frac{-6}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{2}+\frac{3}{3}+\frac{6}{6} & 0+\frac{3}{3}-\frac{12}{6} & -\frac{2}{2}+\frac{3}{3}+\frac{6}{6} \\ 0+\frac{3}{3}-\frac{12}{6} & 0+\frac{3}{3}+\frac{24}{6} & 0+\frac{3}{3}-\frac{12}{6} \\ -\frac{2}{2}+\frac{3}{3}+\frac{6}{6} & 0+\frac{3}{3}-\frac{12}{6} & \frac{2}{2}+\frac{3}{3}+\frac{6}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Cayley Hamilton theorem:-

Every square matrix satisfies its own characterstic equation.

Uses:-

* To calculate
(i) the positive integral powers of A.
(ii) The inverse of a non-singular square matrix A.

Problem 1:

Using Cayley-Hamilton thm find A^{-1}. if $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Soln:-

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Let the char eqn be

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 2+1+2 = 5$

$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= (2-0) + (4-1) + (2-0)$

$= 7$

$S_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$

$= 2(2-0) - 1(0-0) + 1(0-1)$

$= 4 - 1 = 3$

$\therefore$ Char. eqn is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

"By cayley Hamilton thm, Every Square matrix satisfies its own characterstic equation."

Put $\lambda = A$

$\therefore A^3 - 5A^2 + 7A - 3I = 0$ —(1)

multiply by A^{-1}

$A^2 - 5A + 7I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 5A + 7I$

$$A^{-1} = \frac{1}{3}\left[A^2 - 5A + 7I\right] \quad \text{——} (2)$$

Now, $A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$\therefore A^2 - 5A + 7I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\therefore (2) \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Problem 2:-

Using cayley hamilton thm, find A^{-1} if $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

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Soln:-

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 1 + 3 - 4 = 0$$

$$S_2 = \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= (-12-12) + (-4-6) + (3-1)$$

$$= -20.$$

$$S_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{vmatrix}$$

$$= 1(-12-12) - 1(-4-6) + 3(-4+6)$$

$$= -24 + 10 + 6$$

$$= -8$$

$\therefore$ The char. eqn is $\lambda^3 - 0\lambda^2 - 20\lambda + 8 = 0$

By cayley Hamilton thm,

"Every square matrix satisfies its own characteristic eqn".

$$\therefore A^3 - 20A + 8I = 0$$

$\times$ by A^{-1}

$$\therefore A^2 - 20I + 8A^{-1} = 0$$

$$-8A^{-1} = A^2 - 20I$$

$$A^{-1} = -\frac{1}{8}\left[A^2 - 20I\right]$$

Now

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$\therefore A^2-20I = \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} + \begin{bmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{bmatrix}$$

$$= \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\therefore ① \Rightarrow A^{-1} = \frac{-1}{8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

Problem 3:-

Verify cayley Hamilton thm, find A^4 and A^{-1} when

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Soln:-

Given $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 2+2+2 = 6$$

$$S_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1)+(4-2)+(4-1)$$

$$= 8$$

$$S_3 = \begin{vmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1)+1(-2+1)+2(1-2)$$

$$= 6-1-2$$

$$= 3$$

$\therefore$ char eqn is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

$\therefore$ By cayley Hamilton theorem,

"Every square matrix satisfies its own characterstic equation."

$$\therefore A^3 - 6A^2 + 8A - 3I = 0 \rightarrow ①$$

Verification:-

$$A^2 = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+2 & -2-2-2 & 4+1+4 \\ -2-2-1 & 1+4+1 & -2-2-2 \\ 2+1+2 & -1-2-2 & 2+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14+6+9 & -7-12-9 & 14+6+18 \\ -10-6-6 & 5+12+6 & -10-6-12 \\ 10+5+7 & -5-10-7 & 10+5+14 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 8A - 3I$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 8\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} + \begin{bmatrix} -42 & 36 & -54 \\ 30 & -36 & 36 \\ -30 & 30 & -42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence verified.

To find A^4:-

① $\times A \Rightarrow A^4 - 6A^3 + 8A^2 - 3A = 0$

$A^4 = 6A^3 - 8A^2 + 3A$

$$= 6\begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 8\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 174 & -168 & 228 \\ -132 & 138 & -168 \\ 132 & -132 & 174 \end{bmatrix} + \begin{bmatrix} -56 & 48 & -72 \\ 40 & -48 & 48 \\ -40 & 40 & -56 \end{bmatrix} + \begin{bmatrix} 6 & -3 & 6 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}.$$

To find A^{-1},

① $\times A^{-1} \Rightarrow A^2 - 6A + 8I - 3A^{-1} = 0$

$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$

$\Rightarrow A^{-1} = \frac{1}{3}\left[A^2 - 6A + 8I\right] \rightarrow$ ②

Now $A^2 - 6A + 8I$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix} \therefore ② \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

Problem - 4:-

Use cayley Hamilton thm, to find the value of the matrix given by

(i) $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

(ii) $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

if $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Soln:-

Given

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 2 + 1 + 2 = 5$$

$$S_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0) + (4-1) + (2-0)$$

$$= 7$$

$$S_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - (0-1) + 1(0-1)$$

$$= 3$$

$\therefore$ Char eqn is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By cayley Hamilton thm,

"Every square matrix satisfies its own characteristic equation."

$$\therefore A^3 - 5A^2 + 7A - 3I = 0 \quad \rightarrow (1)$$

Let $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ & A^8 - 5A^7 + 7A^6 - 3A^5 \\ \hline & A^4 - 5A^3 + 8A^2 - 2A \\ & A^4 - 5A^3 + 7A^2 - 3A \\ \hline & A^2 + A + I \end{array}$$

$$\therefore f(A) = (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$$

$$= 0(A^5 + A) + A^2 + A + I$$

$$= A^2 + A + I$$

Now

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore f(A) = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Let $g(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

$$\begin{array}{r|l} & A^5 + 8A + 35I \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I \\ & (-)\ A^8 - 5A^7 + 7A^6 - 3A^5 \\ \hline & 8A^4 - 5A^3 + 8A^2 - 2A \\ & (-)\ 8A^4 - 40A^3 + 50A^2 - 24A \\ \hline & 35A^3 - 48A^2 + 22A + I \\ & (-)\ 35A^3 - 175A^2 + 245A - 105I \\ \hline & 127A^2 - 223A + 106I \end{array}$$

$$\therefore g(A) = (A^3 - 5A^2 + 7A - 3I)(A^5 + 8A + 35I) + 127A^2 - 223A + 106I$$

$$= 127A^2 - 223A + 106I$$

$$= 127\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 223\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 106\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 635 & 508 & 508 \\ 0 & 127 & 0 \\ 508 & 508 & 635 \end{bmatrix} + \begin{bmatrix} -446 & -223 & -233 \\ 0 & -223 & 0 \\ -223 & -223 & -446 \end{bmatrix} + \begin{bmatrix} 106 & 0 & 0 \\ 0 & 106 & 0 \\ 0 & 0 & 106 \end{bmatrix}$$

$$= \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}.$$

Problem-5 :-

Verify Cayley-Hamilton theorem and find its inverse of

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$ N/D-2019

Soln:-

Given $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Let the char. eqn be

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 3 + 5 + 3 = 11$$

$$S_2 = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15-1) + (9-1) + (15-1)$$

$$= 14 + 8 + 14 = 36$$

$$S_3 = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) + 1(-3+1) + 1(1-5)$$

$$= 3(14) + 1(-2) + 1(-4) = 36$$

$\therefore$ The char. eqn is
$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

By cayley-Hamilton theorem.

"Every square matrix satisfies its own characterstic equation"

$\therefore A^3 - 11A^2 + 36A - 36I = 0$

<u>Verification</u>

$$A^2 = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+1+1 & -3-5-1 & 3+1+3 \\ -3-5-1 & +1+25+1 & -1-5-3 \\ 3+1+3 & -1-5-3 & 1+1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 33+9+7 & -11-45-7 & 11+9+21 \\ -27-27-9 & 9+135+9 & -9-27-27 \\ 21+9+11 & -7-45-11 & 7+9+33 \end{bmatrix}$$

$$= \begin{bmatrix} 49 & -63 & 41 \\ -63 & 153 & -63 \\ 41 & -63 & 49 \end{bmatrix}$$

$\therefore A^3 - 11A^2 + 36A - 36I$

$$= \begin{bmatrix} 49 & -63 & 41 \\ -63 & 153 & -63 \\ 41 & -63 & 49 \end{bmatrix} - \begin{bmatrix} 121 & -99 & 77 \\ -99 & 297 & -99 \\ 77 & -99 & 121 \end{bmatrix}$$

$$+ \begin{bmatrix} 108 & -36 & 36 \\ -36 & 180 & -36 \\ 36 & -36 & 108 \end{bmatrix} - \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To find A^{-1}

$\times$ by $A^{-1} \Rightarrow A^2 - 11A + 36I - 36A^{-1} = 0$

$$36A^{-1} = A^2 - 11A + 36I$$

$$= \begin{bmatrix} 11 & -9 & 7 \\ -9 & 27 & -9 \\ 7 & -9 & 11 \end{bmatrix} - \begin{bmatrix} 33 & -11 & 11 \\ -11 & 55 & -11 \\ 11 & -11 & 33 \end{bmatrix}$$

$$+ \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 14 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{36} \begin{bmatrix} 14 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 14 \end{bmatrix} //$$

Application of Eigen values and Eigen vector-

Stretching of an elastic membrane

* For any system of differential equations of the form

$$\frac{dx}{dt} = AX$$

Where A is an $n \times n$ matrix with distinct eigen values $\lambda_1, \lambda_2, \dots \lambda_n$ and t is the independent variable.

$\therefore$ The solution is $X = \quad Y$

where is the ormalized matrix of A, and

$$Y = \left[c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}, \dots c_n e^{\lambda_n t}\right]^T$$

Problem:-

Solve the system of differential equation $\frac{dx}{dt} = 4x + 2y$

$$\frac{dy}{dt} = -x + y$$

with initial conditions $x(0) = 1$, $y(0) = 0$.

Soln:-

Let $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

The char. eqn is

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$S_1 = [4+1] = 5$

$S_2 = \begin{vmatrix} 4 & 2 \\ -1 & 1 \end{vmatrix} = 4 + 2 = 6$

$\therefore$ The char. eqn is

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = 3, \lambda = 2.$$

$\therefore$ The eigen values are 2, 3.

To find Eigen vector

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$(4-\lambda)x + 2y = 0$ — (1)

$-x + (1-\lambda)y = 0$ — (2)

case(i) put $\lambda = 2$

$2x + 2y = 0$ — (3)

$-x - y = 0$ — (4)

The eqn (3) & (4) are same

$2x = -2y$

$\frac{x}{-1} = \frac{y}{1}$ $\therefore X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

case(ii) put $\lambda = 3$

$x + 2y = 0$ — (5)

$-x - 2y = 0$ — (6)

$\therefore$ The eqn (5) & (6) are same

$x + 2y = 0$

$x = -2y$ $\therefore X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\frac{x}{-2} = \frac{y}{1}$

$$N = \begin{bmatrix} \frac{-1}{\sqrt{2}} & -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \end{bmatrix}, \quad N^{-1} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$D = N^{-1}AN = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -2/\sqrt{2} & -6/\sqrt{5} \\ 2/\sqrt{2} & 3/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\therefore r(t) = ce^{3t} \qquad s(t) = ke^{2t}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = NY$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & -2/\sqrt{5} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{matrix} -\frac{1}{\sqrt{2}} r - \frac{2}{\sqrt{5}} s \\ 1/\sqrt{2}\, r + \frac{1}{\sqrt{5}} s \end{matrix}$$

$$\therefore x = -1/\sqrt{2}\, r - \frac{2}{\sqrt{5}} s = -1/\sqrt{2}\, ce^{3t} - \frac{2}{\sqrt{5}} ke^{2t}$$

$$y = 1/\sqrt{2}\, r + \frac{1}{\sqrt{5}} s = 1/\sqrt{2}\, ce^{3t} + 1/\sqrt{5}\, ke^{2t}$$

$x(0) = 1, \quad y(0) = 0$

$$1 = -1/\sqrt{2}\, c - \frac{2}{\sqrt{5}} k \quad \text{--- (7)}$$

$$0 = 1/\sqrt{2}\, c + 1/\sqrt{5}\, k \quad \text{--- (8)}$$

Solve (7) & (8)

$$\cancel{-1/\sqrt{2}\, c} - 2/\sqrt{5}\, k = 1$$
$$\cancel{1/\sqrt{2}\, c} + 1/\sqrt{5}\, k = 0$$
$$\overline{-1/\sqrt{5}\, k = 1}$$

$$\therefore \boxed{k = -\sqrt{5}}$$

sub $k = -\sqrt{5}$ in (7)

$$1 = -1/\sqrt{2}\, c + 2$$

$$-1 = -1/\sqrt{2}\, c$$

$$\boxed{c = \sqrt{2}}$$

$$\therefore x(t) = -\frac{1}{\cancel{\sqrt{2}}} \cancel{\sqrt{2}} e^{3t} - \frac{2}{\cancel{\sqrt{5}}} (-\cancel{\sqrt{5}}) e^{2t}$$

$$\boxed{x(t) = 2e^{2t} - e^{3t}}$$

$$y(t) = \frac{1}{\cancel{\sqrt{2}}} \cancel{\sqrt{2}} e^{3t} + \frac{1}{\cancel{\sqrt{5}}} (-\cancel{\sqrt{5}}) e^{2t}$$

$$\boxed{y(t) = e^{3t} - e^{2t}}$$

Problem - 2

An elastic membrane in the x_1, x_2 - plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P: (x_1, x_2)$ goes over into the point $Q: (y_1, y_2)$ given by $y_1 = 5x_1 + 3x_2$
$y_2 = 3x_1 + 5x_2$.

Find the principal directions, that is, direction of position vector x of P for which the direction of the position vector y of Q is the same (or) exactly opposite. What shape does the boundary circle take under this deformation?

Soln:-

Given $y_1 = 5x_1 + 3x_2$

$y_2 = 3x_1 + 5x_2$

Let $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

The char. eqn is

$\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 5+5 = 10$

$S_2 = \begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix} = 25-9 = 16$

$\therefore$ The char. eqn is

$\lambda^2 - 10\lambda + 16 = 0$

$(\lambda-8)(\lambda-2) = 0$

$\lambda = 2, 8$

$\therefore$ The eigen values are

2, 8

To find Eigen vector

$(A-\lambda I)X = 0.$

$\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(5-\lambda)x_1 + 3x_2 = 0$ — (1)

$3x_1 + (5-\lambda)x_2 = 0$ — (2)

case(i) Put $\lambda = 2$

$3x_1 + 3x_2 = 0$ — (3)

$3x_1 + 3x_2 = 0$ — (4)

The eqn (3) & (4) are same

$3x_1 = -3x_2$

$x_1 = -x_2$

$\frac{x_1}{-1} = \frac{x_2}{1}.$

$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

case(ii) put $\lambda = 8$ in (1) & (2).

$-3x_1 + 3x_2 = 0$ — (5)

$3x_1 - 3x_2 = 0$ — (6)

The eqn (5) & (6) are same

$-3x_1 + 3x_2 = 0$

$3x_2 = 3x_1$

$\frac{x_2}{1} = \frac{x_1}{1}$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The ~~the~~ Eigen vectors

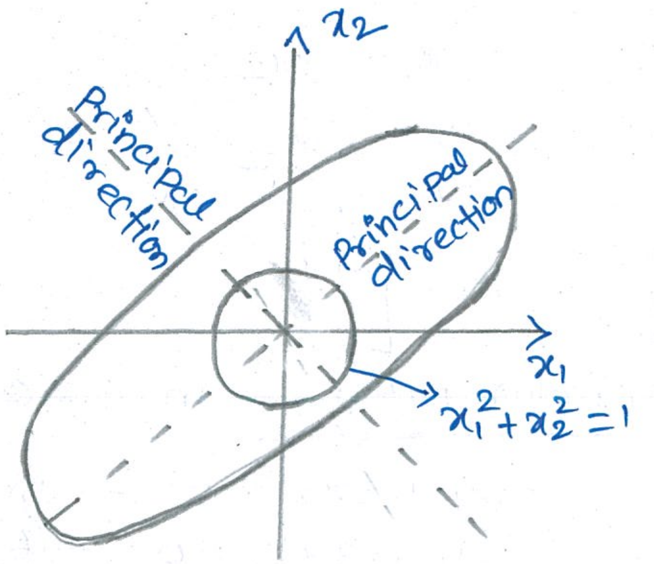
$\lambda = 2 \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 8 \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

These vectors make $45°$ and $135°$ angles with the positive x_1-direction.

They give the principal directions

The eigen values show that in the principal direction the membrane is stretched by factors 8 and 2, respectively; see in Figure



Accordingly, if we choose the principal directions as direction of a new cartesian direction of a new cartesian $u_1 u_2$- coordinate system, say with positive $z_1 = r\cos\theta$ in the first quadrant semi axes in the first quadrant and the positive $z_2 = r\sin\theta$ semi axis in the second quadrant of the $x_1 x_2$-system, then a boundary point of the unstretched circular membrane has coordinates $\cos\theta \sin\theta$

Hence
$$z_1 = 8\cos\theta \quad \text{--- (7)}$$
$$z_2 = 2\sin\theta \quad \text{--- (8)}$$

$(7)^2 + (8)^2$
$$\frac{z_1^2}{8^2} + \frac{z_2^2}{2^2} = \cos^2\theta + \sin^2\theta$$
$$\frac{z_1^2}{64} + \frac{z_2^2}{4} = 1$$

$\therefore$ The deformed boundary is an ellipse.

Problem 3:-

Find the solution of the differential equations

$$u = 2x - 2y + 4z$$
$$v = 3y - 2z$$
$$w = -y + 2z$$

with the initial conditions $x(0) = 2$, $y(0) = 0$, & $z(0) = 2$.

Soln:-

Let $A = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad X' = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

The char. eqn is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 2 + 3 + 2 = 7$

$s_2 = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 0 & 3 \end{vmatrix}$

$= 6 - 2 + 4 - 0 + 6 - 0 = 14$

$s_3 = \begin{vmatrix} 2 & -2 & 4 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{vmatrix}$

$= 2(6-2) + 2(0-0) + 4(0-0)$
$= 8$

$\therefore \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

$\lambda = 1$ is a one of the root

$$\begin{array}{c|cccc} 1 & 1 & -7 & 14 & -8 \\ & 0 & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & \boxed{0} \end{array}$$

$\lambda^2 - 6\lambda + 8 = 0$

$\lambda^2 - 4\lambda - 2\lambda + 8 = 0$

$(\lambda - 4)(\lambda - 2) = 0$

$\lambda = 2, 4.$

$\therefore$ The eigen values are

1, 2, 4

To find Eigen vector

$$\begin{bmatrix} 2-\lambda & -2 & 4 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} (2-\lambda)x - 2y + 4z = 0 \\ 0x + (3-\lambda)y - 2z = 0 \\ 0x - y + (2-\lambda)z = 0 \end{array}\right\} \text{— I}$$

case(i) put $\lambda = 1$ in I

$1x - 2y + 4z = 0$ — 1

$0x + 2y - 2z = 0$ — 2

$0x - y + z = 0$ — 3

solve 1 & 2

$$\frac{x}{\begin{vmatrix} -2 & 4 \\ 2 & -2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & 1 \\ -2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix}}$$

$$\frac{x}{4-8} = \frac{y}{0+4} = \frac{z}{4-0}$$

$$\frac{x}{-4} = \frac{y}{4} = \frac{z}{4}$$

$$\therefore X_1 = \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

case(ii) put $\lambda = 2$ in I

$0x - 2y + 4z = 0$ — 4

$0x + y - 2z = 0$ — 5

$0x - y + 0z = 0$ — 6

solve 4 & 5

$$\frac{x}{\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & 0 \\ -2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x}{4-4} = \frac{y}{0-0} = \frac{z}{0-0}$$

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{0}$$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

case(iii) $\lambda = 4$ in I

$-2x - 2y + 4z = 0$ — 7

$0x - y - 2z = 0$ — 8

$0 - y - 2z = 0$ — 9

solve 7 & 8

$$\frac{x}{\begin{vmatrix} -2 & 4 \\ -1 & -2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & -2 \\ -2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & -2 \\ 0 & -1 \end{vmatrix}}$$

$$\frac{x}{8} = \frac{y}{-4} = \frac{z}{2}$$

$$\therefore X_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

Since the eigen values of the coefficient matrix A are distinct the corresponding eigenvectors are linearly independent.

This shows that A is diagonalizable and Hence

$N = [X_1, X_2, X_3]$ is the diagonalizing matrix.

So we have $N^{-1}AN = D$

ie $N^{-1} AN = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}$$

The relation $X = PY$. then Since N is Normalized matrix of constants, we have

$X' = PY'$

So $X' = AX$ becomes

$NY' = AX$
$= A(NY)$

$Y' = (N^{-1}AN)Y$

ie, $\begin{bmatrix} u'(t) \\ v'(t) \\ w'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}$

$$= \begin{bmatrix} u(t) \\ 2v(t) \\ 4w(t) \end{bmatrix}$$

then

$u(t) = c_1 e^{t}$, $v(t) = c_2 e^{2t}$

and $w(t) = c_3 e^{4t}$.

where c_1, c_2 and c_3 are constants.

$X = NY$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 4 \\ 1 & 0 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{t} \\ c_2 e^{2t} \\ c_3 e^{4t} \end{bmatrix}$$

$x(t) = -2c_1 e^{t} + c_2 e^{2t} + 4c_3 e^{4t}$

$y(t) = c_1 e^{t} + 0c_2 e^{2t} - 2c_3 e^{4t}$

$z(t) = c_1 e^{t} + 0c_2 e^{2t} + c_3 e^{4t}$

Given $x(0) = 2$
$y(0) = 0$
$z(0) = 2$

$x(0) = -2c_1e^0 + c_2e^0 + 4c_3e^0$
$y(0) = c_1e^0 + 0c_2e^0 - 2c_3e^0$
$z(0) = c_1e^0 + 0c_2e^0 + c_3e^0$

$2 = -2c_1 + c_2 + 4c_3$ — (10)
$0 = c_1 - 2c_3$ — (11)
$2 = c_1 + c_3$ — (12)

Solve (11) & (12)

$$\begin{array}{rcl} \cancel{c_1} - 2c_3 & = & 0 \\ \cancel{c_1} + c_3 & = & 2 \\ - \quad - & & - \\ \hline -3c_3 & = & -2 \end{array}$$

$$\boxed{c_3 = \frac{2}{3}}$$

Sub c_3 in (11)

(11) $\Rightarrow c_1 = 2c_3$

$c_1 = 2(2/3)$

$$\boxed{c_1 = \frac{4}{3}}$$

$-2c_1 + c_2 + 4c_3 = 2$

$-2\left(\frac{4}{3}\right) + c_2 + 4(2/3) = 2$

$-8/3 + c_2 + 8/3 = 2$

$$\boxed{c_2 = 2}$$

$x(t) = -\frac{8}{3}e^t + 2e^{2t} + \frac{8}{3}e^{4t}$

$y(t) = \frac{4}{3}e^t - \frac{4}{3}e^{4t}$

$z(t) = 2/3\, e^t + 2/3\, e^{4t}.$

$\therefore$ By cayley Hamilton thro,

$A^2 - 4A - 5I = 0 \longrightarrow (1)$

$$
\begin{array}{r|l}
 & A^2 \\
A^2-4A-5I & A^4 - 4A^3 - 5A^2 + A + 2I \\
 & A^4 - 4A^3 - 5A^2 \\
 & (-) \\
 & A + 2I
\end{array}
$$

$\therefore A^4 - 4A^3 - 5A^2 + A + 2I$

$= (A^2 - 4A - 5I)\, A^2 + (A+2I)$

$= A + 2I$

$= \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$

Problem : 19 :-

Use cayley - Hamilton theorem to find $A^4 - 8A^3 + 12A^2$ when $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$

Soln:-

The char. eqn is $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 5 + 3 = 8$

$S_2 = \begin{vmatrix} 5 & 3 \\ 1 & 3 \end{vmatrix} = 15 - 3 = 12$

$\therefore \lambda^2 - 8\lambda + 12 = 0$

By cayley - Hamilton thm,

$A^2 - 8A + 12I = 0$

$\times$ by A^2 $\quad A^4 - 8A^3 + 12A^2 = 0.$

Problem-20:-

Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$

Soln:-

$A = \begin{bmatrix} \text{co-eff } x^2 & \frac{1}{2}\text{coeff } xy & \frac{1}{2}\text{co-eff } xz \\ \frac{1}{2}\text{co-eff } yx & \text{co-eff } y^2 & \frac{1}{2}\text{co-eff } yz \\ \frac{1}{2}\text{coeff } zx & \frac{1}{2}\text{coeff } zy & \text{co-eff } z^2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$

Problem 21:-

Discuss the nature of the following quadratic form

$3x_1^2 + 3x_2^2 - 5x_3^2 - 2x_1x_2 - 6x_2x_3 - 6x_3x_1.$

Soln:-

$A = \begin{bmatrix} 3 & -1 & -3 \\ -1 & 3 & -3 \\ -3 & -3 & -5 \end{bmatrix}$

$D_1 = |3| = 3$, (+ive)

$D_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8$ (+ive)

$D_3 = \begin{vmatrix} 3 & -1 & -3 \\ -1 & 3 & -3 \\ -3 & -3 & -5 \end{vmatrix}$

$= 3(15-9) + 1(5-9) - 3(3-9)$

$= 3(-24) + 1(-4) - 3(-6)$

$= -112$, (-ive)

$\therefore$ The Q.F is indefinite

Soln:-

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

Given $\lambda_1 = 2$, $\lambda_2 = 3$

$\lambda_1 + \lambda_2 + \lambda_3 =$ sum of the main diagonal elements.

$2 + 3 + \lambda_3 = 2 + 2 + 2$

$5 + \lambda_3 = 6$

$\boxed{\lambda_3 = 1}$

W.K.T $\lambda_1 \lambda_2 \lambda_3 = |A|$

$2 \times 3 \times 1 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{vmatrix}$

$\Rightarrow 6 = 2(4-0) - 0(0) + 1(-2a)$

$6 = 8 - 2a$

$2a = 2 \Rightarrow \boxed{a = 1}$

Problem-16:-

Find the eigen values of a matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ corresponding to the eigen vector $[-4, -2, 4]^T$. [A/M-2017]

Soln:- W.K.T $(A - \lambda I)X = 0$

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(7-\lambda)(2) - 2 + 0 = 0$

$14 - 2\lambda - 2 = 0$

$12 - 2\lambda = 0$

$2\lambda = 12$

$\boxed{\lambda = 6}$

Hence the corresponding eigen value is 6.

Problem-17:-

State Cayley Hamilton theorem

Soln:-

Every square matrix satisfies its own characteristic equation.

Problem-18:-

Use cayley Hamilton theorem to find $A^4 - 4A^3 - 5A^2 + A + 2I$ when $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

Soln:-

Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

The char. eqn of A is

$\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 1 + 3 = 4$

$S_2 = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5$

$\therefore$ The char. eqn of A is

$\lambda^2 - 4\lambda - 5 = 0$

$\lambda_1+\lambda_2+\lambda_3 = \lambda_1+\lambda_2$

$\therefore \boxed{\lambda_3 = 0}$

W.K.T

Product of the Eigen values $= |A|$

$\lambda_1\lambda_2\lambda_3 = \lambda_1\lambda_2(0)$

$= 0$

Problem : 9

Find the sum and product of all the Eigen values of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{bmatrix}$

Soln:-

Sum of the Eigen values

= sum of the main diagonal elements.

$= 1+0-3 = -2$

Product of the Eigen values

$= \begin{vmatrix} 1 & 2 & -2 \\ 1 & 0 & 3 \\ -2 & -1 & -3 \end{vmatrix}$

$= 1(0+3)-2(-3+6)-2(-1+0)$

$= 3-6+2$

$= -1.$

Problem 10:-

If 3 and 6 are two Eigen values of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ write down all the eigen values of A^{-1}.

Soln:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of A.

Given $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = ?$

W.K.T

Sum of the Eigen values = sum of the main diagonal elements.

$\lambda_1+\lambda_2+\lambda_3 = 1+5+1$

$3+6+\lambda_3 = 7$

$\therefore \boxed{\lambda_3 = -2}$

W.K.T

Eigen values of A^{-1} are

$1/\lambda_1, \frac{1}{\lambda_2}, 1/\lambda_3$

$\therefore 1/3, 1/6, -\frac{1}{2}$ are the eigen values of A^{-1}.

Problem 11:-

Find the Eigen values of A^{-1} where $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Soln:-

A is a upper triangular matrix. Hence the Eigen values of A are 3, 2, 5

$\therefore 1/3, 1/2, 1/5$ are the Eigen values of A^{-1}

Problem 12:-

Find the Eigen values of A^3 given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Soln:-

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

clearly given A is a upper triangular matrix.

Hence the Eigen values are 1, 2, 3

W.K.T "If λ is eigen value of A, Then λ^n is an Eigen value of A^n"

$\therefore 1^2, 2^2, 3^2$ are the eigen values of A^3

$\Rightarrow$ 1, 8, 27 are the eigen values of A^3.

Problem-13:-

If 2, −1, −3 are the eigen values of the matrix A then find the Eigen values of the matrix $A^2 - 2I$.

Soln:-

Given the Eigen values of A are 2, −1, −3.

The Eigen values of $A^2 - 2I$ are $2^2 - 2(1), (-1)^2 - 2(1), (-3)^2 - 2(1)$

$\Rightarrow$ 2, −1, 7 are the eigen values of $A^2 - 2I$.

Problem: 14:-

Find the constants a and b such that matrix $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ has 3 and −2 as it eigen values.

Apr-2017

Soln:-

Sum of the eigen values of a matrix = sum of the main diagonal elements.

$3 - 2 = a + b$

$\Rightarrow a + b = 1 \longrightarrow (1)$

Product of the eigen values = $|A|$

$(3) \times (-2) = \begin{vmatrix} a & 4 \\ 1 & b \end{vmatrix}$

$-6 = ab - 4$

$\Rightarrow ab = -2 \quad — (2)$

Solving (1) & (2)

$x^2 - (a+b)x + ab = 0$

$\Rightarrow x^2 - x - 2 = 0$

$(x-1)(x+1) = 0$

i.e., $x - 2 = 0, \quad x + 1 = 0$

$x = 2, \quad x = -1$

$\therefore a = 2, \ b = -1$ (or) $a = -1, \ b = 2$.

Problem-15:-

If 2 and 3 are the eigen values of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 2 \end{bmatrix}$, find the value of 'a'.

PART-A

P-1

If 3 and 15 are two eigen values of the matrix $A=\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then find its third Eigen values and hence find $|A|$.

Sol

Given $\lambda_1=3$, $\lambda_2=15$

W.K.T sum of the Eigen values = sum of the main diagonal elements

$\therefore \lambda_1+\lambda_2+\lambda_3 = 8+7+3$

$3+15+\lambda_3 = 18$

$\boxed{\lambda_3=0}$

Also W.K.T

Product of the Eigen value $= |A|$

$3\times15\times0 = |A|$

$\therefore \boxed{|A|=0}$

Problem 2:-

Show that the Eigen values of a null matrix are zero. [Apr-2018]

Soln:-

If A is a null square matrix of order 3 then its char. eqn is $(-1)^3\lambda^3=0$

$\lambda^3=0$

$\therefore$ The Eigen values are 0, 0, 0.

In general,

The char eqn of a null square matrix of order n is

$(-1)^n\lambda^n=0$

$\therefore \lambda^n=0$

$\therefore$ The Eigen values of A are zero

Problem-3:-

If λ is the Eigen value of the matrix A, then Prove that λ^2 is the eigen value of A^2 (APR-2019)

Soln:-

Given: λ is the eigen value of the matrix A.

$AX=\lambda X$ ——(1)

$A(AX)=A(\lambda X)$

$A^2X=\lambda(AX)$

$=\lambda(\lambda X)$ [using (1)]

$=\lambda^2X$

$\therefore \lambda^2$ is an eigen value of A^2.

Problem 4:-

If the Eigen values of the matrix A of order 3x3 are matrix 2,3 and 1, then find the det of A.

Soln:-

Given Eigen values of A are 2,3,1

$|A|$ = Product of the eigen values.

$= 2\times3\times1$

$\boxed{|A|=6}$

Problem 5:-

Prove that any square matrix A and its transpose have the same Eigen values.

N/D-2019

Soln:-

Let A be a square matrix of order n.

The char eqn of A and A^T are

$|A-\lambda I| = 0 \rightarrow (1)$

& $|A^T-\lambda I| = 0 - (2)$

W.K.T $|A| = |A^T|$

Hence (1) & (2) are identical.

$\therefore$ The Eigen values of A & A^T are same.

Problem-6:-

If $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$ then find $2A^2-8A-10I$, where I is the unit matrix.

Soln:-

Given $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}$

The char eqn of A is $|A-\lambda I|=0$

i.e., $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 4$

$S_2 = \begin{vmatrix}1 & 4\\ 2 & 3\end{vmatrix} = 3-8 = -5$

$\therefore \lambda^2 - 4\lambda - 5 = 0$

By C-H Theorem,

we have $A^2-4A-5I=0 \rightarrow (1)$

$\therefore 2A^2-8A-10I = 2[A^2-4A-5I]$

$= 0$

Problem 7.

The Product of the two Eigen values of the matrix $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$ is 16. Find the third Eigen value.

Soln:-

Let $\lambda_1, \lambda_2, \lambda_3$ are the Eigen values of A.

Given $\lambda_1\lambda_2 = 16$

W.K.T $\lambda_1\lambda_2\lambda_3 = |A|$

$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{vmatrix}$

$16\lambda_3 = 6(9-1)+2(-6+2)+2(2-6)$

$16\lambda_3 = 32$

$\boxed{\lambda_3 = 2}$

Problem 8:-

If the Sum of two Eigen values and trace of a 3X3 matrix A are equal find the value of |A|

Soln:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the Eigen values of the given 3X3 matrix A.

W.K.T

Sum of the Eigen values = Trace of A. - (1)

Trace of A = Sum of the two Eigen value

Problem 21:-

Find the index, signature of the Q.F. $x_1^2+2x_2^2-3x_3^2$.

Soln:-

Here Q.F = C.F

$f(x_1,x_2,x_3) = x_1^2+2x_2^2-3x_3^2$

Index = No. of +ive terms in the canonical form

$= 2$

Signature = No. of positive terms in the canonical form − No. of negative terms in the canonical form

$= 2-1 = 1$

Problem 22:-

Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

Soln

Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

$\therefore$ Q.F is

$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2$

$+2a_{12}x_1x_2 + 2a_{23}x_2x_3$

$+2a_{13}x_1x_3$

$= x_2^2+2x_3^2+10x_1x_2+12x_2x_3-2x_1x_3.$

Problem 23:-

Write down the Q.F corresponding to the matrix $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{bmatrix}$

Soln:-

Let $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{bmatrix}$

Q.F is $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_2x_3 + 2a_{13}x_1x_3$

$= 2x_1^2+2x_2^2-2x_3^2-4x_1x_3+2x_2x_3$

Problem 24:-

Write down nature of a quadratic form whose matrix is $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

Soln:-

Eigen values of the given matrix are $-1, -1, -2$

All the eigen values are negative numbers.

$\therefore$ The nature of the Q.F is negative definite.

Problem 25:-

Can $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be diagonalized? Why

Soln:-

The given matrix A is real symmetric and non-Sigular matrix.

Hence

A can be diagonalised.

Unit - II – DIFFERENTIAL CALCULUS

Representation of functions - Limit of a function - Continuity - Derivatives - Differentiation Rules (sum, product, quotient, chain rules) - Implicit Differentiation - Logarithmic differentiation - Applications : Maxima and minima of functions of one variable.

REPERENTAION OF FUNCTIONS :

A function is a rule that assigns to each element 'x' in a set 'A' to exactly one element called '$f(x)$' in a set 'B'

* Domain : Let $f: A \to B$ then set 'A' is called the domain of the function.

* Co-Domain : Set 'B' is called co-domain of the function.

* Range : The set of all images of all the elements of 'A' under the function 'f' is called the range of 'f' and it is denoted by $f(A)$.

PROBLEMS

1. Find the domain of the function $f(x) = \sqrt{x+2}$

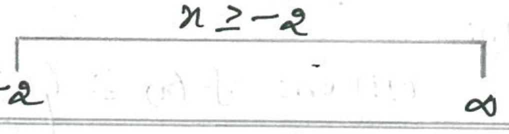
Soln:

Since the square root of a negative number is not defined, the domain of 'f' must be positive.

$\therefore x + 2 \geq 0$

$\Rightarrow x \geq -2$

$\therefore \boxed{\text{Domain is } [-2, \infty)}$



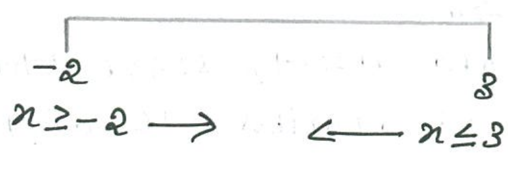
2. Find the domain of the function $f(x) = \sqrt{3-x} - \sqrt{2+x}$

Soln: NOV/DEC 2018

Let, $3 - x \geq 0$ and $2 + x \geq 0$

$\Rightarrow x \leq 3$ and $x \geq -2$

$\therefore \boxed{\text{Domain is } [-2, 3]}$



[Since the square root of a negative number is not defined, the domain of 'f' must be positive]

3. Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$

Soln:

Given: $f(x) = \frac{x+4}{x^2-9}$

The function is not defined at $x=3$ and $x=-3$

$$\boxed{\text{Domain: } \{x \,/\, x \neq 3,\ x \neq -3\}}$$

(or)

$$\boxed{\text{Domain: } (-\infty,-3) \cup (-3,3) \cup (3,\infty)}$$

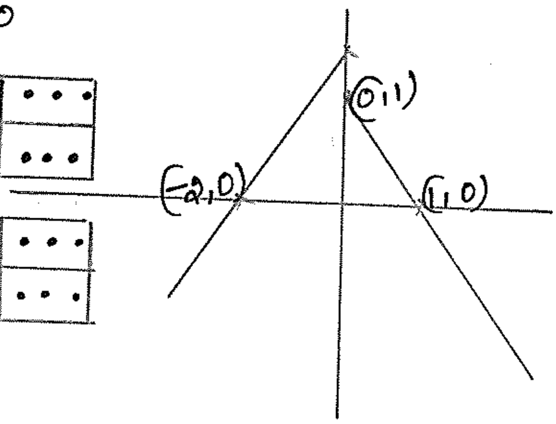
4. Find the domain, range also sketch the graph for the following functions, $f(x) = \begin{cases} x+2, & x<0 \\ 1-x, & x \geq 0 \end{cases}$.

Soln:

Given: $f(x) = \begin{cases} x+2, & x<0 \\ 1-x, & x\geq 0 \end{cases}$

Domain : x	-1	-2	-3	$\cdots$
Range : $y = x+2$	1	0	-1	$\cdots$

Domain : x	0	1	2	$\cdots$
Range : $y = 1-x$	1	0	-1	$\cdots$



$$\boxed{\begin{array}{l}\text{Domain : } (-\infty, \infty) \\ \text{Range : } (-\infty, 1)\end{array}}$$

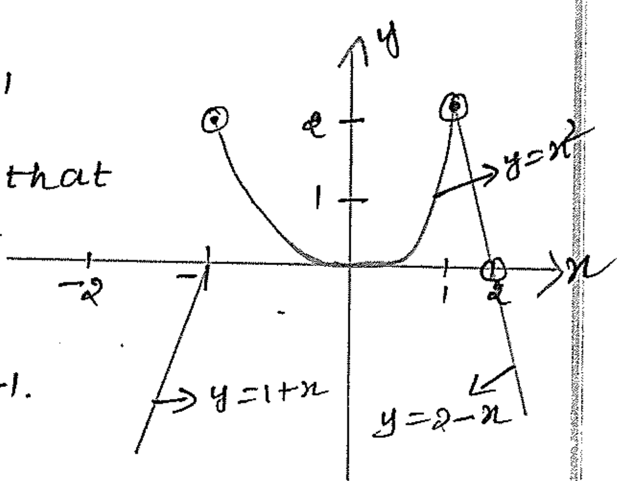
5. Sketch the graph of the function $f(x) = \begin{cases} 1+x, & x<-1 \\ x^2, & -1 \leq x \leq 1 \\ 2-x, & x \geq 1 \end{cases}$ and use it to determine the values of 'a' for which $\lim_{x \to a} f(x)$ exists?

JAN 2018

Soln:

Given: $f(x) = \begin{cases} 1+x, & x<-1 \\ x^2, & -1 \leq x \leq 1 \\ 2-x, & x \geq 1 \end{cases}$

From the graph, it is observed that $\lim_{x\to a} f(x)$ exists for all 'a' except when $a=-1$, since the right and left limits are different at $a=-1$.



LIMIT OF A FUNCTION:

Let $f(x)$ be a function of a real variable 'x'. Let 'a' and 'l' be a fixed numbers. If '$f(x)$' approaches 'l' as 'x' approaches 'a', then we say 'l' is the limit of $f(x)$ as 'x' tends to 'a' & we write $\lim\limits_{x\to a} f(x) = l$.

LEFT HAND LIMIT:

If $f(x)$ approaches the value 'l' as 'x' approaches 'a' from the left, then $\lim\limits_{x\to a^-} f(x) = l$.

RIGHT HAND LIMIT:

If $f(x)$ approaches the value 'l' as 'x' approaches 'a' from the right, then $\lim\limits_{x\to a^+} f(x) = l$.

Result:

$\lim\limits_{x\to a} f(x) = l$ if and only if $\lim\limits_{x\to a^-} f(x) = l = \lim\limits_{x\to a^+} f(x)$.

Problems

1. Evaluate $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$

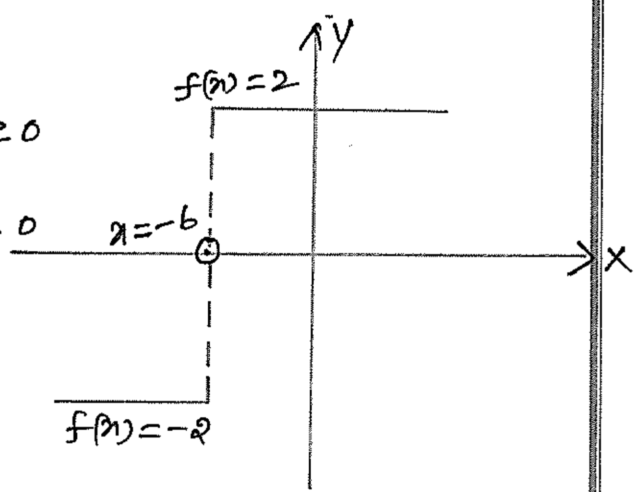
Soln:

We know that $\lim\limits_{x\to a} f(x) = l$ iff $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = l$

Given: $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$

Now, $f(x) = \begin{cases} \dfrac{2(x+6)}{x+6}, & x+6 \geq 0 \\ \dfrac{2(x+6)}{-(x+6)}, & x+6 < 0 \end{cases}$

$= \begin{cases} 2, & x \geq -6 \\ -2 & x < -6 \end{cases}$



$\therefore \lim\limits_{x\to 6^-} \dfrac{2x+12}{|x+6|} = -2$ —— ①

& $\lim\limits_{x\to 6^+} \dfrac{2x+12}{|x+6|} = 2$ —— ②

① ≠ ② $\therefore$ Limit does not exists.

2. Check whether $\lim\limits_{x \to -3} \dfrac{3x+9}{|x+3|}$ exists.

April/May 2019

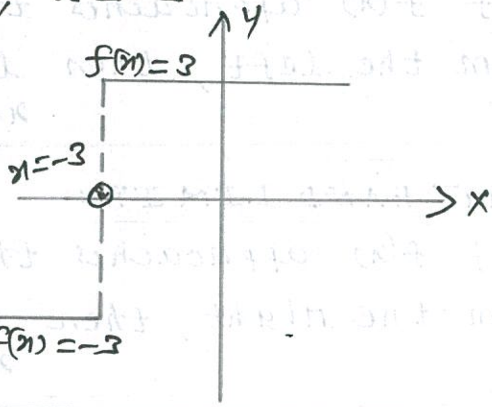
Soln:

Given: $\lim\limits_{x \to -3} \dfrac{3x+9}{|x+3|}$, $f(x) = \begin{cases} \dfrac{3(x+3)}{x+3}, & x+3 \geq 0 \\ \dfrac{3(x+3)}{-(x+3)}, & x+3 < 0 \end{cases}$

$= \begin{cases} 3, & x \geq -3 \\ -3, & x < -3 \end{cases}$

Now, $\lim\limits_{x \to 3^+} \dfrac{3x+9}{|x+3|} = 3$ —— ①

Also, $\lim\limits_{x \to 3^-} \dfrac{3x+9}{|x+3|} = -3$ —— ②



① ≠ ② $\therefore$ limit does not exists.

3. Guess the value of the limit (if it exists) for the function $\lim\limits_{x \to 0} \dfrac{e^{5x}-1}{x}$ by evaluating the function at the given numbers $x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$ [correct to six decimal places]

NOV/DEC 2018

Soln:

Let $f(x) = \dfrac{e^{5x}-1}{x}$

x	-0.5	-0.1	-0.01	-0.001	-0.0001
$f(x)$	1.835830	3.934693	4.877058	4.987521	4.998750

x	0.5	0.1	0.01	0.001	0.0001
$f(x)$	22.364988	6.487213	5.127110	5.012521	5.001250

As 'x' approaches to '0', the function $f(x) = \dfrac{e^{5x}-1}{x}$ approaches to '5'.

$$\therefore \lim_{x \to 0} \frac{e^{5x}-1}{x} = 5.$$

4. Investigate $\lim_{x \to 0} \frac{1}{x^2}$

Soln:

Let $f(x) = \frac{1}{x^2}$

x	-1	-0.5	-0.1	-0.05	-0.01	-0.001	0.001	0.01	0.05	0.1	0.5	1
$f(x)$	1	4	100	400	10000	1000000	1000000	10000	400	100	4	1

As 'x' approaches to '0', the function $f(x) = \frac{1}{x^2}$ becomes very large and does not approaches to a number.

$$\therefore \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

5. Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$

Soln:

Given: $\lim_{x \to 0} \frac{\sin x}{x}$, Here $f(x) = \frac{\sin x}{x}$

x	-1	-0.5	-0.1	-0.05	-0.01	-0.001	0.001	0.01	0.05	0.1	0.5	1
$f(x)$	0.8415	0.9589	0.9983	0.9996	0.9999	0.99999	0.99999	0.9999	0.9996	0.9983	0.9589	0.8415

As 'x' approaches to '0', the function $f(x) = \frac{\sin x}{x}$ approaches to 1.

$$\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

6. Prove that $\lim_{x \to 0} |x| = 0$

Proof:

Let $f(x) = |x|$

$$= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

We know that $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Now, $\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$

Also, $\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$ $\therefore \lim_{x \to 0} |x| = 0$

7. Determine $\lim_{x \to \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi - 2x)^2}$.

AU JAN 2016 R-15 MA7151

Soln:

Let, $\lim_{x \to \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2} = \frac{1+\cos \cancel{2}\frac{\pi}{\cancel{2}}}{\left(\pi - \cancel{2}\frac{\pi}{\cancel{2}}\right)^2} = \frac{1-1}{0} = \frac{0}{0}$ ①

Apply L'Hospital rule,

$$① \Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{-2\sin 2x}{2(\pi - 2x)(-2)} = \lim_{x \to \frac{\pi}{2}} \frac{\sin 2x}{2(\pi - 2x)} \quad ②$$

$$= \frac{\sin 2\frac{\pi}{2}}{2\left(\pi - 2\frac{\pi}{2}\right)} = \frac{0}{0}$$

Apply L'Hospital rule,

$$② \Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{2\cos 2x}{2\ (-2)} = \frac{\cos 2\frac{\pi}{2}}{-2} = \frac{-1}{-2}$$

$$= \frac{1}{2}$$

8. Evaluate $\lim_{x \to \infty} \left[x\sqrt{x^2+1} - x\right]$

Soln:

Let, $\lim_{x \to \infty} \left[x\sqrt{x^2+1} - x\right] = \lim_{x \to \infty} x\sqrt{x^2+1} - x \left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right]$

$$= \lim_{x \to \infty} \frac{x\left[x^2+1-x^2\right]}{\sqrt{x^2+1}+x}$$

$$= \lim_{x \to \infty} \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}+x}$$

$$= \lim_{x \to \infty} \frac{x}{x\sqrt{1+\frac{1}{x^2}}+x}$$

$$= \lim_{x \to \infty} \frac{x}{x\left[\sqrt{1+\frac{1}{x^2}}+1\right]}$$

$$= \frac{1}{\sqrt{1+\frac{1}{\infty}}+1} = \frac{1}{2}$$

9. Evaluate $\lim\limits_{x \to \infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Soln:

Let $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

$$\therefore \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

$$= \lim_{x \to \infty} \frac{x^2\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{x^2\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{3 - \frac{1}{\infty} - \frac{2}{\infty}}{5 + \frac{4}{\infty} + \frac{1}{\infty}} = \frac{3}{5}$$

HORIZONTAL ASYMPTOTE:

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim\limits_{x \to \infty} f(x) = L$ (or) $\lim\limits_{x \to -\infty} f(x) = L$.

PROBLEMS

1. Find the horizontal asymptote of the curve $\dfrac{x^2 - 1}{x^2 + 1}$

Soln:

Given: $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1}$$

$$= \lim_{x \to -\infty} \frac{x^2\left(1 - \frac{1}{x^2}\right)}{x^2\left(1 + \frac{1}{x^2}\right)}$$

$$= \frac{1 - 0}{1 + 0} = 1.$$

and $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{x^2 - 1}{x^2 + 1} = \lim\limits_{x \to \infty} \dfrac{x^2\left(1 - \frac{1}{x^2}\right)}{x^2\left(1 + \frac{1}{x^2}\right)}$

$= \dfrac{1 - 0}{1 + 0} = 1.$

Hence the line $y = 1$ is a Horizontal asymptote of the given curve.

2. Find the Horizontal and vertical asymptotes of the curve $\frac{\sqrt{2x^2+1}}{3x-5}$

Soln:

$$\text{Let } f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{x\left[3-\frac{5}{x}\right]}$$

$$= \frac{\sqrt{2+0}}{3}$$

$$= \frac{\sqrt{2}}{3}.$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

when $x \to \infty$ we have $\sqrt{x^2} = -x$, $x<0$

$$= \lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

$$= \frac{-\sqrt{2}}{3}$$

$\therefore$ Both the line $y = \frac{-\sqrt{2}}{3}$ and $\frac{\sqrt{2}}{3}$ are horizontal asymptotes. The vertical asymptotes occurs when the given function becomes either $-\infty$ or ∞.

For $x = \frac{5}{3}$ the function becomes ∞.

$$\therefore \lim_{x\to\left(\frac{5}{3}\right)^+} f(x) = \lim_{x\to\frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

$$\text{and } \lim_{x\to\frac{5}{3}^-} f(x) = \lim_{x\to\frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty.$$

SQUEEZE THEOREM:

If $f(x) \le g(x) \le h(x)$ when 'x' is near to 'a' [except possible at 'a'] and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$.

PROBLEMS

1. Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

Proof:

Let, $f(x) = \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right), \; x \ne 0.$

If $x = 0$, $f(x)$ is not defined.

If $x \ne 0$, $\frac{1}{x}$ is real.

$\therefore \sin\frac{1}{x}$ is defined

$$-1 \le \sin\frac{1}{x} \le 1$$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\lim_{x \to 0}(-x^2) \le \lim_{x \to 0} x^2 \sin\frac{1}{x} \le \lim_{x \to 0} x^2$$

Since $\lim_{x \to 0}(-x^2) = 0$ and $\lim_{x \to 0}(x^2) = 0$

$\therefore$ By squeeze theorem,

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

2. Show that $\lim_{x \to 0} \sqrt{x^3 + x^2}\, \sin\frac{\pi}{x} = 0$

Proof:

Let, $f(x) = \lim_{x \to 0} \sqrt{x^3 + x^2}\, \sin\left(\frac{\pi}{x}\right)$

If $x = 0$, $f(x)$ is not defined

If $x \ne 0$, $\frac{\pi}{x}$ is real.

$\therefore \sin\frac{\pi}{x}$ is defined.

$$-1 \le \sin\frac{\pi}{x} \le 1$$

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$$-\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} \le \sqrt{x^3+x^2}$$

$$\lim_{x\to 0} -\sqrt{x^3+x^2} \le \lim_{x\to 0}\sqrt{x^3+x^2}\,\sin\frac{\pi}{x} \le \lim_{x\to 0}\sqrt{x^3+x^2}$$

since $\lim\limits_{x\to 0} -\sqrt{x^3+x^2} = 0$ & $\lim\limits_{x\to 0}\sqrt{x^3+x^2} = 0$.

$\therefore$ By squeeze theorem, $\lim\limits_{x\to 0}\sqrt{x^3+x^2}\,\sin\frac{\pi}{x} = 0$.

8. Show that $\lim\limits_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right)$.

Proof:

Let, $-1 \le \cos\frac{1}{x^2} \le 1$

$$-x^2 \le x^2\cos\frac{1}{x^2} \le x^2$$

$$\lim_{x\to 0} -(x^2) \le \lim_{x\to 0} x^2\cos\frac{1}{x^2} \le \lim_{x\to 0} x^2$$

since $\lim\limits_{x\to 0}(-x^2) = 0$ & $\lim\limits_{x\to 0}(x^2) = 0$

By squeeze theorem, $\lim\limits_{x\to 0} x^2\cos\left(\frac{1}{x^2}\right) = 0$.

CONTINUITY

A function 'f' is continuous at the point 'a' if $\lim\limits_{x\to a} f(x) = f(a)$.

* A function 'f' is continuous from right at a point 'a' if $\lim\limits_{x\to a^+} f(x) = f(a)$.

* A function 'f' is continuous from left at a point 'a' if $\lim\limits_{x\to a^-} f(x) = f(a)$.

RESULT: $f(x)$ is continuous if $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = f(a)$.

PROBLEMS

1. Show that the function $f(x) = 1-\sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

Soln:

Let, $f(x) = 1-\sqrt{1-x^2}$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} 1-\sqrt{1-x^2}$$
$$= 1-\sqrt{1-(-1)^2}$$
$$= 1$$

& $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-\sqrt{1-x^2}$$
$$= 1-\sqrt{1-1}$$
$$= 1$$

$$\therefore \lim_{x\to -1^+} f(x) = \lim_{x\to 1^-} f(x) = 1.$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} 1-\sqrt{1-x^2}$$
$$= 1-\sqrt{1-a^2}$$
$$= f(a)$$
$\therefore -1 < a < 1$

f is continuous.

$\therefore$ $f(x)$ is continuous on the interval $[-1, 1]$.

2. For what value of the constant 'c' is the function 'f' continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} cx^2+2x; & x<2 \\ x^3-cx; & x\ge 2\end{cases}$

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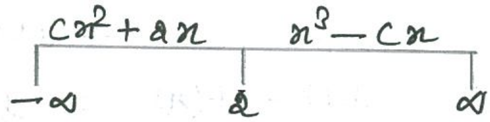
Soln:

The given function $f(x)$ is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now,



$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (cx^2+2x)$$
$$= c(4)+2(2)$$
$$= 4c+4$$

Also, $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^3-cx)$$
$$= 8-2c$$

We know that a function 'f' is continuous at a point 'a' if

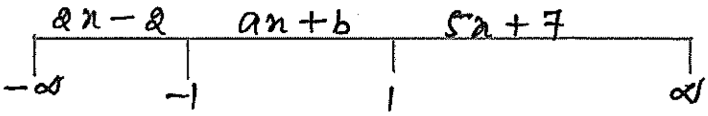
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

$$\therefore 4c+4 = 8-2c$$
$$4c+2c = 4$$
$$c = 4/6$$
$$\boxed{c = 2/3}$$

8. Let $f(x) = \begin{cases} 2x-2 & , x<-1 \\ ax+b & ; -1 \le x \le 1 \\ 5x+7 & , x \ge 1 \end{cases}$ is continuous for all real 'x', Find the values of 'a' & 'b'.

Soln:

Given: $f(x)$ is continuous.



$$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 2x-2$$

$$= -2-2 = -4$$

$$\& \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} ax+b$$

$$= a(-1)+b = -a+b$$

$\because$ f is continuous, $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

$$-4 = -a+b$$

$$\Rightarrow a-b = 4 \longrightarrow (1)$$

Also, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} ax+b$

$$= a(1)+b = a+b$$

$$\& \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5x+7$$

$$= 5+7 = 12$$

$\because$ f is continuous, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$$a+b = 12 \longrightarrow (2)$$

From (1) & (2)

$$\begin{array}{r} a+b = 12 \\ a-b = 4 \\ \hline 2a = 16 \end{array} \quad (+)$$

$$\boxed{a = 8}$$

$$\therefore (1) \Rightarrow 8-b = 4$$

$$\boxed{b = 4}$$

$\therefore a = 8$

$\& \ b = 4.$

4. Find the domain where the function 'f' is continuous. Also find the number at which the function 'f' is discontinuous where $f(x) = \begin{cases} 1+x^2, & x \le 0 \\ 2-x, & 0 < x \le 2 \\ (x-2)^2, & x > 2 \end{cases}$

Soln:

$1+x^2 \quad 2-x \quad (x-2)^2$

$-\infty \quad 0 \quad 2 \quad \infty$

At $x=0$,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (1+x^2)$$
$$= 1 \quad \text{——} \quad (1)$$

& $$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2-x)$$
$$= 2 \quad \text{——} \quad (2)$$

(1) $\neq$ (2) $\therefore$ $f(x)$ is discontinuous at $x=0$.

At $x=2$,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2-x$$
$$= 0 \quad \text{——} \quad (3)$$

& $$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-2)^2$$
$$= 0 \quad \text{——} \quad (4)$$

(3) = (4) $\therefore$ $f(x)$ is continuous at $x=2$.

$\therefore$ Domain: $(-\infty, 0) \cup (0, \infty)$.

5. Find the values of 'a' and 'b' that make 'f' continuous on $(-\infty, \infty)$, $f(x) = \begin{cases} \dfrac{x^2-8}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x \le 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$

Soln: NOV/DEC 2018

$\dfrac{x^2-8}{x-2} \quad ax^2-bx+3 \quad 2x-a+b$

$-\infty \quad 2 \quad 3 \quad \infty$

$x<2 \quad 2 \le x \le 3 \quad x \ge 3$

At $x=2$,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-8}{x-2} = \frac{2^2-8}{2-2} = \frac{-4}{0} = \infty$$

Apply L' Hospital's Rule,

$$= \lim_{x \to 2^-} \frac{2x}{1} = 2(2) = 4$$

Also, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3$

$$= 4a - 2b + 3$$

Given: f is continuous.

$$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\Rightarrow 4a - 2b + 3 = 4$$

$$4a - 2b = 1 \quad \text{——— (1)}$$

At $x = 3$,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2 - bx + 3$$

$$= 9a - 3b + 3 \quad \text{——— (2)}$$

Also,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b$$

$$= 6 - a + b \quad \text{——— (3)}$$

Given: f is continuous,

$$\therefore \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$

$$9a - 3b + 3 = 6 - a + b \quad \text{By (2) \& (3)}$$

$$10a - 4b = 3 \quad \text{——— (4)}$$

From (1) & (4)

$$10a - 4b = 3$$

$$(-) \quad 8a - 4b = 2 \quad [\text{(1)} \times 2]$$

$$2a = 1$$

$$\boxed{a = 1/2}$$

$$\therefore \text{(1)} \Rightarrow 4\frac{1}{2} - 2b = 1$$

$$2 - 1 = 2b$$

$$\boxed{b = 1/2}$$

DERIVATIVE:

The derivative of a function 'f' at a number 'a', denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ if this limit exists.

PROBLEMS

1. If $f(x) = \sqrt{x}$, find the derivative of $f(x)$.

Soln:

By the definition of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

2. If $f(x) = \sin x$, find the derivative of $f(x)$.

Soln:

Let, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$\because \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\sin\frac{h}{2}}{h/2} \cos\left(x + \frac{h}{2}\right)$$

$$= \lim_{h \to 0} \frac{\sin h/2}{h/2} \times \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right)$$

$$= 1 \times \cos x$$ $\because \lim_{x \to 0} \frac{\sin x}{x} = 1.$

$$= \cos x$$

3. Find the derivative of the function $f(x) = \frac{1}{\sqrt{x}}$.

Soln:

Let, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}\cdot h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}\cdot h} \times \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}\, h\left[\sqrt{x} + \sqrt{x+h}\right]}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left[\sqrt{x} + \sqrt{x+h}\right]}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = \frac{-1}{2x\sqrt{x}}$$

RULES OF DIFFERENTIATION:

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(c \cdot u) = c\frac{du}{dx}$, [u is a function of x & c is a constant]

3. Product rule, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

4. Quotient rule, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

5. Chain rule,

(i) If 'y' is a function of 'u' and 'u' itself is a function of x, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

(ii) If 'y' is a function of 'u', 'u' is a function of 'v', 'v' is a function of 'w' and 'w' is a function of 'x', then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$

PROBLEMS

1. If $f(x) = e^x(x + x\sqrt{x})$, find the derivative of $f(x)$.

Soln:

Given: $f(x) = e^x(x + x\sqrt{x})$

$= e^x(x + x^{3/2})$

$f'(x) = \frac{d}{dx}\left[e^x(x + e^{3/2})\right]$

$= e^x \frac{d}{dx}(x + x^{3/2}) + (x + x^{3/2})\frac{d}{dx}e^x$

$= e^x\left(1 + \frac{3}{2}x^{\frac{3}{2}-1}\right) + (x + x^{3/2})e^x$

$= e^x\left[1 + \frac{3}{2}x^{1/2} + x + x^{3/2}\right]$

$= e^x\left[1 + x + \frac{3}{2}\sqrt{x} + x\sqrt{x}\right]$

2. Find $\frac{dy}{dx}$ if $y = x^2 e^{2x}(x^2+1)^4$. APRIL/MAY 2019

Soln:

Given: $y = x^2 e^{2x}(x^2+1)^4$

$\frac{dy}{dx} = \frac{d}{dx}\left[x^2 e^{2x}(x^2+1)^4\right]$

$= x^2 e^{2x}\frac{d}{dx}(x^2+1)^4 + x^2(x^2+1)^4\frac{d}{dx}e^{2x} + e^{2x}(x^2+1)^4\frac{d}{dx}(x^2)$

$= x^2 e^{2x}\,4(x^2+1)^3(2x) + x^2(x^2+1)^4(2e^{2x}) + e^{2x}(x^2+1)^4(2x)$

$= 8x^3 e^{2x}(x^2+1)^3 + 2x^2 e^{2x}(x^2+1)^4 + 2xe^{2x}(x^2+1)^4$

$= 2xe^{2x}(x^2+1)^3\left[4x^2 + x(x^2+1) + x^2+1\right]$

$= 2xe^{2x}(x^2+1)^3\left[4x^2 + x^3 + x + x^2 + 1\right]$

$= 2xe^{2x}(x^2+1)^3(x^3 + 5x^2 + x + 1)$

3. If $f(x) = \frac{1 - xe^x}{x + e^x}$, find the derivative of $f(x)$.

Soln:

Let, $f'(x) = \frac{d}{dx}\left[\frac{1 - xe^x}{x + e^x}\right]$

$= \frac{(x+e^x)\frac{d}{dx}(1 - xe^x) - (1 - xe^x)\frac{d}{dx}(x + e^x)}{(x+e^x)^2}$

$$= \frac{(x+e^x)(-e^x - xe^x) - (1-xe^x)(1+e^x)}{(x+e^x)^2}$$

$$= \frac{-xe^x - x^2e^x - e^{2x} - xe^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x+e^x)^2}$$

$$= \frac{-x^2e^x - e^{2x} - 1 - e^x}{(x+e^x)^2}$$

$$= \frac{-(x^2e^x + e^{2x} + e^x + 1)}{(x+e^x)^2}$$

4. If $y = (1-x^2)^{10}$, find the derivative of y.

Soln:

Given: $y = (1-x^2)^{10}$

Let $u = 1-x^2 \quad \therefore \quad y = u^{10}$

$$\frac{du}{dx} = -2x \qquad \frac{dy}{du} = 10u^9$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 10u^9(-2x) = -20x(1-x^2)^9.$$

5. If $y = \tan(\sin x)$, find the derivative of y.

Soln:

Let $u = \sin x \quad \therefore \quad y = \tan u$

$$\frac{dy}{du} = \sec^2 u = \sec^2(\sin x)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \sec^2(\sin x)\cos x$$

6. If $y = \log\left(x + \sqrt{x^2-1}\right)$, find the derivative of 'y'.

Soln:

Let $y = \log u$ where $u = x + \sqrt{x^2-1}$

$$\frac{dy}{du} = \frac{1}{u} \qquad \frac{du}{dx} = 1 + \frac{1}{\cancel{2}\sqrt{x^2-1}}(\cancel{2}x)$$

$$= 1 + \frac{x}{\sqrt{x^2-1}}$$

(19)

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

7. If $y = \sin(\sin(\sin x))$, find the derivative of 'y'.

Soln:

Let $y = \sin u$, $u = \sin v$, $v = \sin x$

$$\frac{dy}{du} = \cos u, \quad \frac{du}{dv} = \cos v, \quad \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \cos u \cos v \cos x$$

$$= \cos(\sin v) \cos(\sin x) \cos x$$

$$= \cos(\sin(\sin x)) \cos(\sin x) \cos x.$$

8. If $y = \cos\sqrt{\sin(\tan \pi x)}$, find the derivative of 'y'.

Soln:

Let $y = \cos\sqrt{u}$, $u = \sin v$, $v = \tan \pi x$

$$\frac{dy}{du} = -\sin\sqrt{u}\left(\frac{1}{2\sqrt{u}}\right), \quad \frac{du}{dv} = \cos v, \quad \frac{dv}{dx} = \sec^2 \pi x(\pi)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= \frac{-1}{2\sqrt{u}} \sin\sqrt{u} \times \cos v \times \pi \sec^2 \pi x$$

$$= \frac{-1}{2\sqrt{\sin v}} \sin\sqrt{\sin v} \times \cos(\tan \pi x)\, \pi \sec^2 \pi x$$

$$= -\frac{\pi}{2} \frac{\sin\sqrt{\sin(\tan \pi x)}}{\sqrt{\sin(\tan \pi x)}} \times \cos(\tan \pi x) \sec^2 \pi x.$$

II IMPLICIT DIFFERENTIATION:

PROBLEMS

1. If $\sqrt{x}+\sqrt{y}=1$ then find $\frac{dy}{dx}$

Soln:

Given: $\sqrt{x}+\sqrt{y}=1$

Differentiate with respect to 'x'

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2\sqrt{y}}{2\sqrt{x}}=-\sqrt{\frac{y}{x}}$$

2. Find y'' if $x^4+y^4=16$

JAN 2018

Soln:

Given $x^4+y^4=16$

Differentiate w.r.to 'x'.

$$4x^3+4y^3\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-x^3}{y^3}=y'$$

$$\therefore y''=\frac{d^2y}{dx^2}=-\frac{y^3(3x^2)-(x^3)(3y^2)\frac{dy}{dx}}{y^6}$$

$$=-\frac{3x^2y^3-3x^3y^2\left(\frac{-x^3}{y^3}\right)}{y^6}$$

$$=-\frac{3x^2y^3+\frac{3x^6}{y}}{y^6}$$

$$=-\frac{3x^2y^4+3x^6}{y^7}$$

$$=-\frac{3x^2}{y^7}\left[y^4+x^4\right]=\frac{-3x^2(16)}{y^7}$$

$$=\frac{-48x^2}{y^7}.$$

3. Find y' if $\cos(xy) = 1+\sin y$

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Soln:

Given $\cos xy = 1+\sin y$

Diff., w.r. to 'x'.

$$-\sin xy\left[x\frac{dy}{dx} + y(1)\right] = \cos y\frac{dy}{dx}$$

$$-\sin xy\, x\frac{dy}{dx} - y\sin xy - \cos y\frac{dy}{dx} = 0$$

$$-\frac{dy}{dx}\left[x\sin xy + \cos y\right] = y\sin xy$$

$$\therefore \frac{dy}{dx} = \frac{-y\sin xy}{x\sin xy + \cos y}$$

4. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.

Soln:

Let $x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring on both sides,

$$x^2(1+y) = y^2(1+x)$$

$$x^2 + x^2y = y^2 + y^2x$$

$$(x^2-y^2) + x^2y - y^2x = 0$$

$$(x+y)(x-y) + xy(x-y) = 0$$

$$x-y\,[x+y+xy] = 0$$

If $x-y=0$, then $1-\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 1.$$

III^{ly} If $x+y+xy = 0$

$$y+xy = -x$$

$$y(1+x) = -x$$

$$y = \frac{-x}{1+x}$$

$$\frac{dy}{dx} = -\frac{(1+x)(1) - x(1)}{(1+x)^2}$$

$$= -\frac{1+x-x}{(1+x)^2} = \frac{-1}{(1+x)^2}.$$

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS:

FORMULA'S

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\frac{d}{dx}(\sec x) = \sec x \tan x$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

1. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
2. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
3. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
4. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
5. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
6. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

---

1. If $f(x) = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$, find the derivative of $f(x)$. NOV/DEC 2018

Soln:

Let $u = \frac{b+a\cos x}{a+b\cos x}$

$\therefore$ Let $y = f(x) = \cos^{-1}(u)$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-u^2}}\left[\frac{(a+b\cos x)(-a\sin x) - (b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}\right]$$

$$= \frac{-1}{\sqrt{1-\frac{(b+a\cos x)^2}{(a+b\cos x)^2}}}\left[\frac{-a^2\sin x - \cancel{ab\sin x\cos x} + b^2\sin x + \cancel{ab\sin x\cos x}}{(a+b\cos x)^2}\right]$$

$$= \frac{-(a+b\cos x)}{\sqrt{\cancel{a^2} + b^2\cos^2 x + \cancel{2ab\cos x} - b^2 - \cancel{a^2}\cos^2 x - \cancel{2ab\cos x}}} \cdot \frac{-(a^2-b^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{(a^2-b^2)\sin x}{\sqrt{a^2(1-\cos^2 x)-b^2(1-\cos^2 x)}\,(a+b\cos x)}$$

$$= \frac{(a^2-b^2)\sin x}{\sqrt{a^2\sin^2 x-b^2\sin^2 x}\,(a+b\cos x)}$$

$$= \frac{(a^2-b^2)\cancel{\sin x}}{\cancel{\sin x}\sqrt{a^2-b^2}\,(a+b\cos x)}$$

$$= \frac{\sqrt{a^2-b^2}\,\cancel{\sin x}}{a+b\cos x}$$

2. Find $\frac{dy}{dx}$ if $y=\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Soln:

Let $x=\tan\theta$ ——— (1)

$$\therefore\ y=\tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{1/\cos\theta-1}{\sin\theta/\cos\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{\cancel{2}\sin^{\cancel{2}}(\theta/2)}{\cancel{2}\cancel{\sin(\theta/2)}\cos(\theta/2)}\right)$$

$$=\tan^{-1}\left(\tan(\theta/2)\right)$$

$$y=\theta/2$$

$$y=\frac{\tan^{-1}x}{2} \quad \text{By (1)}$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

DERIVATIVE OF LOGARITHMIC FUNCTIONS:-

PROBLEMS

1. Differentiate $y = x^x$

Soln:

Given $y = x^x$

Taking 'log' on both sides,

$$\log y = \log x^x$$

$$\log y = x \log x$$

Differentiate with respect to 'x'.

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x \,(1)$$

$$= 1 + \log x$$

$$\frac{dy}{dx} = y(1 + \log x)$$

$$= x^x(1 + \log x)$$

2. Find y' if $x^y = y^x$

Soln:

Given $y^x = x^y$

Taking 'log' on both sides,

$$\log x^y = \log y^x$$

$$y \log x = x \log y$$

Differentiate with respect to 'x'.

$$y\frac{1}{x} + \log x\frac{dy}{dx} = x\frac{1}{y}\frac{dy}{dx} + \cancel{\log y}\log y\,(1)$$

$$\frac{dy}{dx}\left(\log x - \frac{x}{y}\right) = \log y - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x\log y - y}{x} \times \frac{y}{y\log x - x}$$

$$\frac{dy}{dx} = \frac{y(x\log y - y)}{x(y\log x - x)}.$$

3. Differentiate $y = x^{\cos x}$

Soln:

Given: $y = x^{\cos x}$

Taking 'log' on both sides

$$\log y = \log\left(x^{\cos x}\right)$$

$$\log y = \cos x \log x$$

Differentiate w.r.to 'x'

$$\frac{1}{y}\frac{dy}{dx} = \cos x\left(\frac{1}{x}\right) + \log x(-\sin x)$$

$$\frac{dy}{dx} = y\left(\frac{\cos x - x\sin x \log x}{x}\right)$$

$$= x^{\cos x}\left(\frac{\cos x - x\sin x \log x}{x}\right)$$

TANGENT & NORMAL:

* The equation of tangent at a given point (x_1, y_1) is given by $y - y_1 = m(x - x_1)$.
* The equation of normal at a given point (x_1, y_1) is given by $y - y_1 = \frac{-1}{m}(x - x_1)$.

PROBLEMS

1. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal?

Soln:

Given: $y = x^4 - 6x^2 + 4$

Differentiate w.r.to 'x',

$$\frac{dy}{dx} = 4x^3 - 12x$$

The tangent line is horizontal if $y' = 0$

$$\therefore y' = 4x^3 - 12x = 0$$

$$\Rightarrow 4x(x^2 - 3) = 0$$

$$4x = 0, \quad x^2 - 3 = 0$$

$$\boxed{x = 0}, \quad \boxed{x = \pm\sqrt{3}}$$

$\therefore$ So the given curve has horizontal tangents when $x = 0, \sqrt{3}$ & $-\sqrt{3}$.

when $x = 0 \Rightarrow y = 0 - 6(0) + 4 \Rightarrow \boxed{y = 4}$

$\Rightarrow (0, 4)$

when $x = \sqrt{3} \Rightarrow (\sqrt{3})^4 - 6(\sqrt{3})^2 + 4 = y$

$\Rightarrow \boxed{y = -5} \Rightarrow (\sqrt{3}, -5)$

When $x = -\sqrt{3} \Rightarrow y = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 4$

$\boxed{y = -5} \Rightarrow (-\sqrt{3}, -5)$.

$\therefore$ The corresponding points are $(0,4)$, $(\sqrt{3}, -5)$ & $(-\sqrt{3}, -5)$.

2. Find the equation of the tangent line to the curve $x^3 + y^3 = 6xy$ at the point $(3,3)$ and what point the tangent line horizontal in the first quadrant.

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Soln:

Given: $x^3 + y^3 = 6xy$, Diff. w.r.to 'x',

$$3x^2 + 3y^2 \frac{dy}{dx} = 6\left[x\frac{dy}{dx} + y(1)\right]$$

$$x^2 + y^2 \frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}, \quad \text{at } (3,3)\ \frac{dy}{dx} = \frac{6-9}{9-6} = -1. \quad \text{——} (1)$$

$\therefore \boxed{m = -1}$

The equation of the tangent at the point $(3,3)$ is

$$y - 3 = -1(x-3) \Rightarrow y - 3 = -x + 3$$

$$\boxed{x + y = 6}$$

The tangent line is horizontal if $y' = 0$

$$(1) \Rightarrow \frac{2y - x^2}{y^2 - 2x} = 0 \Rightarrow 2y - x^2 = 0$$

$$\boxed{y = x^2/2} \quad \text{——} (2)$$

Let $x^3 + y^3 = 6xy$

$$x^3 + \left(\frac{x^2}{2}\right)^3 = 6x\frac{x^2}{2} \quad \text{By } (2)$$

$$x^3 + \frac{x^6}{8} = \frac{6x^3}{2} \Rightarrow x^3 + \frac{x^6}{8} = 3x^3$$

$$\Rightarrow 2x^3 = \frac{x^6}{8}$$

$$x^6 = 16x^3$$

$$\boxed{x^3 = 16}$$

$$\therefore x = 16^{1/3}$$

$$\Rightarrow \boxed{x = 2^{4/3}}$$

$$\therefore (2) \Rightarrow y = \frac{(2^{4/3})^2}{2} = \frac{2^{8/3}}{2} = 2^{5/3}$$

$\therefore$ The tangent is horizontal at $(2^{4/3}, 2^{5/3})$.

Applications: Maxima and minima of functions of one variable.

Let 'c' be a point in a domain 'D' of the function 'f'.
Then $f(c)$ is the
* absolute maximum value of 'f' on 'D' if $f(c) \geq f(x)$, for all 'x' in 'D'.
* absolute minimum value of 'f' on 'D' if $f(c) \leq f(x)$ for all 'x' in 'D'.

Absolute Maximum and Minimum of $f(x)$:

Step(i): Find the critical numbers of $f(x)$.

i.e $\boxed{f'(x) = 0}$

Step(ii): Substitute the critical points at $f(x)$.

Step(iii): Then the maximum value of $f(x)$ is absolute maximum and the minimum value of $f(x)$ is absolute minimum.

1. Find the absolute maximum and absolute minimum value of $f(x) = x - 2\sin x$ on $[0, 2\pi]$.

Soln:

Given: $f(x) = x - 2\sin x$

$\therefore f'(x) = 1 - 2\cos x$

Critical numbers: $f'(x) = 0$

$$\Rightarrow 1 - 2\cos x = 0$$

$$\cos x = 1/2$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$\therefore f(\pi/3) = \frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = -0.68485$ [Radian mode]

$f(5\pi/3) = \frac{5\pi}{3} - 2\sin\left(\frac{5\pi}{3}\right) = 6.968039$

At end points, $f(0) = 0 - 2\sin 0 = 0$

$f(2\pi) = 2\pi - 2\sin 2\pi = 2\pi = 6.28$.

$\therefore$ The absolute minimum value is $f\left(\frac{\pi}{3}\right) = -0.68485$

The absolute maximum value is $f\left(\frac{5\pi}{3}\right) = 6.9680$.

2. Find the absolute maximum and absolute minimum value of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on $[-2, 3]$.

Soln:

Given: $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

$f'(x) = 12x^3 - 12x^2 - 24x$

The critical numbers of $f(x)$ are occured at

$f'(x) = 0 \Rightarrow 12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$ $\quad -2 \quad -1$; $\quad -2\times 1 \quad -2+1$

$x = 0,\ x = 2,\ x = -1$

The values of $f(x)$ at these critical numbers are

$f(0) = 0 + 1 = 1$

$f(2) = 3(16) - 4(8) - 12(4) + 1 = -4$

$f(-1) = 3(1) - 4(-1) - 12(1) + 1 = -31$

At end points,

$f(-2) = 33$

$f(3) = 28$

$\therefore$ The absolute minimum value is $f(2) = -31$

The absolute maximum value is $f(-2) = 33$.

FIRST AND SECOND DERIVATIVE TEST:

INCREASING / DECREASING / CONCAVITY / INFLECTION POINTS:-

INCREASING & DECREASING FUNCTIONS:-

* If $f'(x) > 0$ in an interval (a, b), then 'f' is increasing ↑.
* If $f'(x) < 0$ in an interval (a, b), then 'f' is decreasing ↓.

CRITICAL NUMBER:

A critical number of a function 'f' is a number 'c' in the domain of 'f' such that $f'(c) = 0$ (or) $f'(c)$ does not exists.

FIRST DERIVATIVE TEST:

Suppose that 'c' is a critical number of a continuous function 'f' then

* If $f'(x)$ changes from positive to negative at 'c', then $f(x)$ has local maximum at 'c'.
* If $f'(x)$ changes from negative to positive at 'c', then $f(x)$ has local minimum at 'c'.

* If $f'(x)$ does not changes sign at 'c', then $f(x)$ has no local maximum or minimum at 'c'.

Note: The first derivative test is a consequence of the increasing and decreasing test.

CONCAVITY: CONCAVE UPWARDS / CONCAVE DOWNWARDS

* If $f''(x) > 0$ in any interval, then $f(x)$ is concave upwards [convex downwards].
* If $f''(x) < 0$ in any interval, then $f(x)$ is convex upwards [concave downwards].

INFLECTION POINTS:

A point 'P' on a curve $y = f(x)$ is called an inflection point if $f(x)$ is continuous and the curve changes from concave upwards to concave downwards (or) from concave downward to concave upward at 'P'.

SECOND DERIVATIVE TEST:

Suppose $f''(x)$ is continuous near 'c',

* If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at 'c'.
* If $f'(c) = 0$ and $f''(c) < 0$, then $f(x)$ has a local maximum at 'c'.

PROBLEMS

1. The profit function of acrosonic company is given by $P(x) = -0.02x^2 + 300x - 200{,}000$, dollars, where the function 'P' is increasing and where it is decreasing.

Soln:

The derivative p' of the function p is

$$p'(x) = -0.04x + 300$$

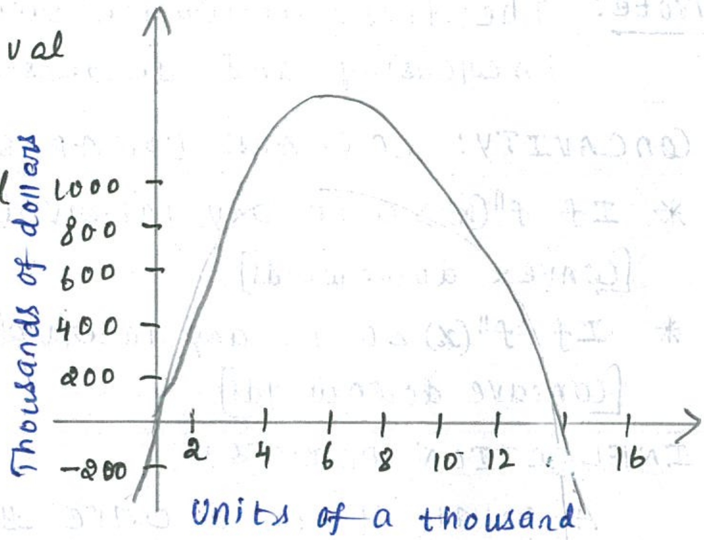
$$p'(x) = -0.04(x - 7500)$$

To find critical points: $p'(x) = 0$

$$-0.04(x - 7500) = 0$$

$$\boxed{x = 7500}$$

0 7500 ∞



* $p'(x) > 0$ for 'x' in the interval $(0, 7500)$.
* $p'(x) < 0$ for 'x' in the interval $(7500, \infty)$.

This means that the profit function 'P' is increasing on $(0, 7500)$ and decreasing on $(7500, \infty)$.

2. For the function $f(x) = 2 + 2x^2 - x^4$, find the intervals of increasing (or) decreasing, local maximum (or) minimum values and the intervals of concavity also the inflection points.

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Soln:

Given: $f(x) = 2 + 2x^2 - x^4$

$f'(x) = 4x - 4x^3$

To find the critical points:-

$f'(x) = 0 \Rightarrow 4x - 4x^3 = 0$

$4x(1 - x^2) = 0$

$x = 0, \quad 1 = x^2$

$\boxed{x = 0}, \boxed{x = \pm 1}$

$-\infty \quad -1 \quad 0 \quad 1 \quad \infty$

$\therefore$ The critical points are $-1, 0, 1$.

Interval	Sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty, -1)$	$f'(-2) = 4(-2) - 4(-2)^3 = 24$ (+ive)	Increasing.
$(-1, 0)$	$f'(-1/2) = 4(-1/2) - 4(-1/2)^3 = -3/2$ (-ive)	Decreasing
$(0, 1)$	$f'(1/2) = 4(1/2) - 4(1/2)^3 = 3/2$, (+ive)	Increasing
$(1, \infty)$	$f'(2) = 4(2) - 4(2)^3 = -24$ (-ive)	Decreasing

Local maximum (or) minimum using First derivative Test:

From the above table,

* $f'(x)$ changes from +ive to −ive at $x=-1$.

∴ $f(x)$ has a local maximum at $x=-1$.

∴ Local maximum value at $x=-1$, $f(-1) = 2+2(-1)^2-(-1)^4$
$= 3$.

* $f'(x)$ changes from −ive to +ive at $x=0$.

∴ $f(x)$ has a local minimum at $x=0$.

∴ Local minimum value at $x=0$, $f(0) = 2+2(0)-0$
$=2$.

* $f'(x)$ changes from +ive to −ive at $x=1$.

∴ $f(x)$ has a local maximum at $x=1$.

∴ Local maximum value at $x=1$, $f(1) = 2+2(1)-1$
$=3$.

Concavity:

$$f''(x) = 4-12x^2$$

$$f''(x) = 0 \Rightarrow 4-12x^2 = 0$$

$$4 = 12x^2$$

$$x^2 = 1/3 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$= \pm 0.58$$

$-\infty$ -0.58 0.58 ∞

Interval	Sign of $f''(x)$	Behaviour of $f(x)$
$(-\infty,-0.58)$	$f(-1) = 4-12(-1)^2 = -8$, (−ive)	concave downwards
$(-0.58, 0.58)$	$f(0) = 4-12(0) = 4$, (+ive)	concave upwards
$(0.58, \infty)$	$f(1) = 4-12(1)^2 = -8$, (−ive)	concave downwards

Local maximum (or) minimum using second derivative Test:

* $f'(-1) = 0$ and $f''(-1) = 4-12(-1)^2 = -8 < 0$ (From the above table)

∴ $f(x)$ is Local maximum at $x=-1$.

* $f'(0) = 0$ and $f''(0) = 4-12(0) = 4 > 0$

∴ $f(x)$ is Local minimum at $x=0$.

* $f'(1) = 0$, $f''(1) = 4 - 12(1)^2 = -8 < 0$

$\therefore$ $f(x)$ is local maximum at $x = 1$.

<u>Inflection points :</u>

* Curve changes from concave downward to concave upward at $x = -0.58$.

$$\therefore f(-0.58) = 2 + 2(-0.58) - (-0.58)^4$$
$$= 2.56$$

$\therefore$ Inflection pts are $(-0.58, 2.56)$.

* Also, curve changes from concave upward to concave downward at $x = 0.58$.

$$\therefore f(0.58) = 2 + (0.58) - (0.58)^4$$
$$= 2.56.$$

$\therefore$ Inflection pts are $(0.58, 2.56)$.

2. Find the local maximum (or) minimum values of $f(x) = \sqrt{x} - \sqrt[4]{x}$ using both first and second derivative tests.

Soln:

Given: $f(x) = \sqrt{x} - \sqrt[4]{x}$

$$= x^{1/2} - x^{1/4}$$

$$f'(x) = \frac{1}{2} x^{-1/2} - \frac{1}{4} x^{-3/4}$$

$$= \frac{x^{-3/4}}{4} (2x^{1/4} - 1)$$

$$= \frac{2x^{1/4} - 1}{4x^{3/4}}$$

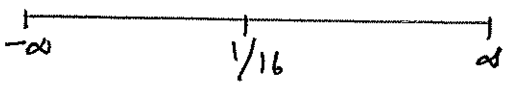
To find critical points :

$$f'(x) = 0 \Rightarrow \frac{2x^{1/4} - 1}{4x^{3/4}} = 0$$

$$\Rightarrow 2x^{1/4} - 1 = 0$$

$$\Rightarrow 2x^{1/4} = 1$$

$$x^{1/4} = 1/2$$



$$\Rightarrow x = (1/2)^4 = 1/16$$

Increasing & Decreasing:

Interval	sign of $f'(x)$	Behaviour of $f(x)$
$(-\infty, 1/16)$	$f'(1/17) = f'(0.06) = \dfrac{2\sqrt[4]{0.06}-1}{4\sqrt[4]{(0.06)^3}} < 0$	Decreasing.
$(1/16, \infty)$	$f'(1) = \dfrac{2-1}{4} = 1/4$, +ive > 0	Increasing.

First Derivative Test (Local maximum (or) minimum):

* $f''(x)$ changes sign from negative to positive at $x = 1/16$.
  $\therefore$ $f(x)$ has local minimum at $x = 1/16$.

$$f(1/16) = \sqrt{\frac{1}{16}} - \sqrt[4]{1/16} = -0.25$$

Second Derivative Test:

$$f''(x) = \frac{-1}{4}x^{-3/2} + \frac{3}{16}x^{-7/4}$$

$$= \frac{-1}{4\sqrt{x^3}} + \frac{3}{16\sqrt[4]{x^7}}$$

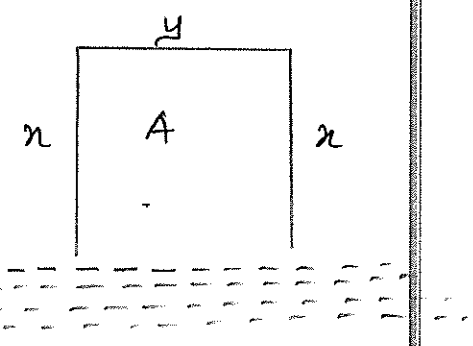
$$\therefore f''(1/16) = \frac{-1}{4\sqrt{(1/16)^3}} + \frac{3}{16\sqrt[4]{(1/16)^7}}$$

$$= -16 + 24$$

$$= 8 > 0.$$

$\therefore$ $f(x)$ has a local minimum at $x = 1/16$.

3. A farmer has 2400 ft. of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fencing along the river. What are the dimensions of the field that has the largest area?



Soln:

Given: Surface area $= 2400$ ft.

$$\therefore S = 2x + y$$

$$S = 2x + y = 2400$$

$$\boxed{y = 2400 - 2x}$$

Area $= xy$

$A = xy = x(2400 - 2x)$

$$\boxed{A = 2400x - 2x^2}$$

To find critical numbers:

$A'(x) = 0$

$\Rightarrow 2400 - 4x = 0$

$$\boxed{x = 600}$$

To find the dimensions of field,

$y = 2400 - 2(600)$

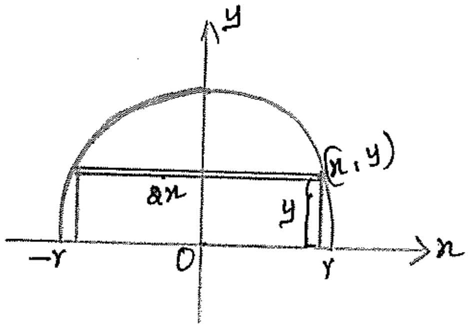
$= 1200$ ft.

$\therefore$ The dimensions of the field is 600×1200 ft.

4. Find the area of the largest rectangle that can be inscribed in a semi circle of radius 'r'.

Soln:

The constraint is:



Area of circle : $x^2 + y^2 = r^2$ ——— (1)

& Area of rectangle : $A = 2xy$. ——— (2)

(1) $\Rightarrow y^2 = r^2 - x^2$

$y = \pm\sqrt{r^2 - x^2}$

$\therefore A = 2x\sqrt{r^2 - x^2}$

To find critical numbers:

$A'(x) = 2x \dfrac{1}{2\sqrt{r^2 - x^2}}(-2x) + \sqrt{r^2 - x^2}\,(2) = 0$

$\Rightarrow \dfrac{-4x^2}{2\sqrt{r^2 - x^2}} + 2\sqrt{r^2 - x^2} = 0$

$\dfrac{-4x^2 + 4(r^2 - x^2)}{2\sqrt{r^2 - x^2}} = 0$

Let $-4x^2 + 4(r^2 - x^2) = 0$ ——— (1)

$-x^2 + r^2 - x^2 = 0 \Rightarrow 2x^2 = r^2$

$$\Rightarrow \boxed{x = \pm\frac{r}{\sqrt{2}}}$$

Test critical numbers:

$$① \Rightarrow A'(x) = -4x^2 + 4r^2 - 4x^2$$

$$A''(x) = -16x$$

$$\therefore A''\left(r/\sqrt{2}\right) = -16\left(\frac{r}{\sqrt{2}}\right) < 0, \text{ since } r > 0$$

$\therefore$ A is local maximum at $x = \frac{r}{\sqrt{2}}$.

To find the area of the largest rectangle,

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\left(\frac{r}{\sqrt{2}}\right)\sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2}$$

$$= \frac{2r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}}$$

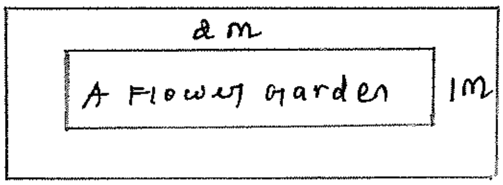
$$= \frac{2r}{\sqrt{2}}\,\frac{r}{\sqrt{2}} = r^2$$

5. A rectangular flower garden with an area of $30m^2$ is surrounded by a grass border 1m wide on two sides and 2m wide on the other two sides. What dimensions of the garden minimize the combined area of the garden and borders?

Soln:

Let $A = xy = 30$

$C = (x+4)(y+2)$



$\therefore xy = 30$

$y = 30/x$

$$\therefore C = (x+4)\left(\frac{30}{x} + 2\right)$$

$$= 30 + 2x + \frac{120}{x} + 8$$

$$\therefore C(x) = 2x + \frac{120}{x} + 38$$

$$\text{Let } C'(x) = 2 - \frac{120}{x^2} = 0 \quad \text{———} \quad ①$$

$$\Rightarrow 2x^2 - 120 = 0$$

$$x^2 = 60 \Rightarrow \boxed{x = \pm\sqrt{60}}$$

Second Derivative Test:

(1) $\Rightarrow c''(x) = \frac{240}{x^3}$

$\Rightarrow c''\sqrt{60} > 0.$

Hence 'c' is a maximum at $x = \sqrt{60}$

$\therefore y = \frac{30}{x} = \frac{30}{\sqrt{60}}$ m.

6. If $f(x) = 2x^3 + 3x^2 - 36x$, find the intervals on which it is increasing (or) decreasing, the local maximum (or) minimum values of $f(x)$.

Soln:

Given $f(x) = 2x^3 + 3x^2 - 36x$ — April/May 2019

$f'(x) = 6x^2 + 6x - 36$

Let $f'(x) = 0 \Rightarrow x^2 + x - 6 = 0$

$-6 \quad 1$

$-2\times 3 \quad -2+3$

$\boxed{x = 2, -3}$

Increasing / Decreasing:

Interval	Sign of $f'(x)$	Behaviour of
$(-\infty, -3)$	$f'(-4) = 6(-4)^2 + 6(-4) - 36 = 36$ (+ive)	Increasing
$(-3, 2)$	$f'(0) = 6(0^2) + 6(0) - 36 = -36$ (-ive)	Decreasing
$(2, \infty)$	$f'(3) = 6(3^2) + 6(3) - 36 = 36 > 0$	Increasing

First Derivative Test: [Local maximum / minimum]

* $f'(x)$ changes positive to negative at $x = -3$

$\therefore f(x)$ has a local maximum at $x = -3$.

$\therefore \boxed{f(-3) = 81}$

* $f'(x)$ changes negative to positive at $x = 2$

$\therefore f(x)$ has a local minimum at $x = 2$.

& $\boxed{f(2) = -44}$.

Second Derivative Test: $f''(x) = 12x + 6$

* $f'(-3) = 0$ and $f''(-3) = -30 < 0$

$\therefore f(x)$ is local maximum at $x = -3$.

* $f'(2) = 0$ & $f''(2) = 30 > 0$ $\therefore f(x)$ is local minimum at $x = 2$.

UNIT II / FUNCTIONS OF SEVERAL VARIABLES

Partial differentiation – Homogeneous functions and Euler's theorem – Total derivative – Change of variables – Jacobians – Partial differentiation of implicit functions – Taylor's series for functions of two variables – Maxima and minima for functions of two variables – Lagrange's method of undetermined multipliers.

PARTIAL DIFFERENTIATION :-

If $z = f(x,y)$ be a function of two variables x & y and if we keep y as constant and vary x alone, then z is a function of x only.

The derivative of z w.r.t x, treating y as constant is called the partial derivatives of z w.r. to x and it is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$ or f_x.

Note:- ① $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_{xx} = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$,

$f_{yx} = \frac{\partial^2 f}{\partial y \partial x}$, $f_{yy} = \frac{\partial^2 f}{\partial y^2}$.

② $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

PROBLEMS:-

① If $u = (x-y)(y-z)(z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Soln

Given:- $u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$$
$$= -(y-z)(x-y) + (z-x)(y-z).$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$
$$= (z-x)(x-y) - (z-x)(y-z).$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(1) + (z-x)(-1)\right]$$
$$= (x-y)(y-z) - (x-y)(z-x)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

② If $u = x^y$, then show that (i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$

(ii) $u_{xxy} = u_{xyx}$

Soln:- Given:- $u = x^y = e^{\log x^y} = e^{y \log x}$

$$\frac{\partial u}{\partial y} = e^{y\log x} \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = e^{y \log x}\left(\frac{1}{x}\right) + \log x \; e^{y\log x} \; y\left(\frac{1}{x}\right)$$

$$= \frac{x^y}{x} + \log x \; \frac{x^y}{x} \cdot y$$

$$= x^{y-1} + x^{y-1} y \log x$$

$$= x^{y-1}\left[1 + y\log x\right] \longrightarrow ①$$

Now $\frac{\partial u}{\partial x} = e^{y\log x} \; y\left(\frac{1}{x}\right) = e^{y\log x} \cdot \frac{y}{x}$

$$\frac{\partial^2 u}{\partial y \partial x} = e^{y \log x}\left(\frac{1}{x}\right) + \frac{y}{x} e^{y\log x} \log x$$

$$= \frac{x^y}{x} + \frac{x^y}{x} y \log x$$

$$= x^{y-1} + x^{y-1} y \log x$$

$$= x^{y-1}\left[1 + y\log x\right] \longrightarrow ②$$

From ① & ② $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$.

Diff. partially w. r. to x on both sides,

$$u_{xxy} = u_{xyx}$$

③ If $f(x,y) = \log \sqrt{x^2+y^2}$, show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Soln:-

Given:- $f = \log \sqrt{x^2+y^2}$

$$= \frac{1}{2} \log (x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0.$$

EULER'S THEOREM FOR HOMOGENEOUS FUNCTIONS:-

A function $f(x,y)$ is said to be a homogeneous function of degree n in x and y if $f(tx,ty) = t^n f(x,y)$ for any positive t.

Euler's theorem:-

If u is a homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$.

PROBLEMS:-

1) If $u = \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$.

[AU M/J 2012, N/D 2014 R-13]

Soln:-

Let $u(x,y,z) = \frac{x}{y}+\frac{y}{z}+\frac{z}{x}$

$$u(tx,ty,tz) = \frac{tx}{ty}+\frac{ty}{tz}+\frac{tz}{tx}$$

$$= t^0 u(x,y,z)$$

$\therefore$ u is a homogeneous function of degree 0.

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu = 0.u = 0.$$

② If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Soln:-

[AU JAN 14 , A/M 17 R-08]

Given:- $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}$$

Let $f(x,y) = \tan u = \frac{x^3+y^3}{x-y}$

$$f(tx,ty) = \frac{t^3x^3 + t^3y^3}{tx - ty}$$

$$= \frac{t^3}{t}\left(\frac{x^3+y^3}{x-y}\right)$$

$$= t^2 f(x,y).$$

∴ $\tan u$ is a homogeneous function of degree 2.

∴ By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}\tan u = 2\tan u$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= 2\frac{\sin u}{\cos u} \times \cos^2 u$$

$$= \sin 2u.$$

③ If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.

Soln:-

[AU N/D 2003, A/M 2017 R-13]

Let $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $f(x,y) = \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t}{t^{1/2}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$= t^{1-1/2} f(x,y)$$

$$= t^{1/2} f(x,y)$$

$\therefore$ f is a homogeneous function of degree $1/2$.

By Euler's theorem, $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$.

$$x\frac{\partial}{\partial x}\cos u + y\frac{\partial}{\partial y}\cos u = \frac{1}{2}\cos u.$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\frac{\cos u}{\sin u} = -\frac{1}{2}\cot u.$$

④ If u is a homogeneous function of degree n in x & y, then show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$. [AU M/J 2010]

Soln:

By Euler's theorem, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu \longrightarrow ①$

Diff ① p.w.r. to 'x'

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$= (n-1)\frac{\partial u}{\partial x} \longrightarrow ②$$

Diff ① p.w.r. to 'y',

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$= (n-1)\frac{\partial u}{\partial y} \longrightarrow ③$$

$(2)\times x + (3)\times y \Rightarrow x^2\dfrac{\partial^2 u}{\partial x^2} + xy\dfrac{\partial^2 u}{\partial x\partial y} + xy\dfrac{\partial^2 u}{\partial y\partial x} + y^2\dfrac{\partial^2 u}{\partial y^2}$

$$= (n-1)x\frac{\partial u}{\partial x} + (n-1)y\frac{\partial u}{\partial y}$$

$$= (n-1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= (n-1)(nu) \qquad [\because \text{by } ①]$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

⑤ If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. [AU Nov 13, UD]

Soln:-

Given:- $u(x,y) = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{y}{x}$

$$u(tx, ty) = t^2x^2\tan^{-1}\frac{ty}{tx} - t^2y^2\tan^{-1}\frac{ty}{tx}$$

$$= t^2\left[x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{y}{x}\right]$$

$$= t^2 u(x,y).$$

$\therefore u$ is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$= 2(2-1)u$$

$$= 2u.$$

⑥ If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then prove that [AU A/M 2014 R-08]

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$.

Soln:-

Given:- $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t}{t^{1/2}}f(x,y)$$

$$= t^{1/2} f(x,y)$$

$\therefore f = \sin u$ is a homogeneous function of degree $1/2$.

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u. \longrightarrow (1)$$

ii) Diff (1) p. w. r. to x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$= \left[\frac{1}{2}\sec^2 u - 1\right]\frac{\partial u}{\partial x} \longrightarrow (2)$$

Diff (1) p. w. r. to 'y',

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}$$

$$= \left[\frac{1}{2}\sec^2 u - 1\right]\frac{\partial u}{\partial y} \longrightarrow (3)$$

Multiplying (2) by x, (3) by y & adding,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \left[\frac{1}{2}\sec^2 u - 1\right]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$= \left[\frac{1}{2\cos^2 u} - 1\right]\frac{1}{2}\tan u$$

$$= -\left[\frac{2\cos^2 u - 1}{2\cos^2 u}\right]\frac{1}{2}\frac{\sin u}{\cos u}$$

$$= -\frac{\sin u\cos 2u}{4\cos^3 u} \qquad [\because 2\cos^2 u - 1 = \cos 2u]$$

TOTAL DERIVATIVES – CHANGE OF VARIABLES – PARTIAL DIFFERENTIATION OF IMPLICIT FUNCTIONS :-

① Find $\frac{dy}{dx}$ when $x^3+y^3=3axy$

Soln:-

Let $f(x,y) = x^3+y^3-3axy$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2-3ay}{3y^2-3ax} = -\frac{x^2-ay}{y^2-ax}.$$

② If $u = x\log(xy)$ where $x^3+y^3+3xy=1$, then find $\frac{du}{dx}$.

Soln:-

Given:- $u = x\log(xy) = x[\log x + \log y]$

$$\frac{\partial u}{\partial x} = x\left[\frac{1}{x}+0\right] + [\log x + \log y](1)$$

$$= 1+\log x+\log y$$

$$\frac{\partial u}{\partial y} = x(0+1/y) + (\log x+\log y)(0) = \frac{x}{y}.$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 1+\log x+\log y + \frac{x}{y}\frac{dy}{dx} \longrightarrow ①$$

Given:- $x^3+y^3+3xy=1$

Diff w.r.to 'x'

$$3x^2+3y^2\frac{dy}{dx}+3\left[y(1)+x\frac{dy}{dx}\right]=0$$

$$x^2+y^2\frac{dy}{dx}+y+x\frac{dy}{dx}=0$$

$$[y^2+x]\frac{dy}{dx} = -(x^2+y)$$

$$\frac{dy}{dx} = -\frac{x^2+y}{y^2+x}$$

$$\therefore ① \Rightarrow \frac{du}{dx} = 1+\log x+\log y + \frac{x}{y}\left[-\left(\frac{x^2+y}{y^2+x}\right)\right]$$

$$= 1+\log x+\log y - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)$$

③ If $g(x,y) = \psi(u,v)$ where $u = x^2 - y^2$ and $v = 2xy$, then prove that $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right]$.

Soln:-

Given:- $g(x,y) = \psi(u,v)$

$u = x^2 - y^2$	$v = 2xy$
$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = -2y$	$\frac{\partial v}{\partial x} = 2y$ $\frac{\partial v}{\partial y} = 2x$
$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial \psi}{\partial u}(2x) + \frac{\partial \psi}{\partial v}(+2y)$ $= 2x\frac{\partial \psi}{\partial u} + 2y\frac{\partial \psi}{\partial v}$	$\frac{\partial g}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}$ $= \frac{\partial \psi}{\partial u}(-2y) + \frac{\partial \psi}{\partial v}(2x)$ $= -2y\frac{\partial \psi}{\partial u} + 2x\frac{\partial \psi}{\partial v}$
$\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}$	$\frac{\partial}{\partial y} = -2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial \psi}{\partial u} + 2y\frac{\partial \psi}{\partial v}\right)$$

$$= 4x^2\frac{\partial^2 \psi}{\partial u^2} + 4xy\frac{\partial^2 \psi}{\partial u\partial v} + 4xy\frac{\partial^2 \psi}{\partial v\partial u} + 4y^2\frac{\partial^2 \psi}{\partial v^2} \rightarrow ①$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \left(-2y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right)\left(-2y\frac{\partial \psi}{\partial u} + 2x\frac{\partial \psi}{\partial v}\right)$$

$$= 4y^2\frac{\partial^2 \psi}{\partial u^2} - 4xy\frac{\partial^2 \psi}{\partial u\partial v} - 4xy\frac{\partial^2 \psi}{\partial v\partial u} + 4x^2\frac{\partial^2 \psi}{\partial v^2} \rightarrow ②$$

① + ②

$$\Rightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4x^2\frac{\partial^2 \psi}{\partial u^2} + \cancel{4xy\frac{\partial^2 \psi}{\partial u\partial v}} + \cancel{4xy\frac{\partial^2 \psi}{\partial v\partial u}} + 4y^2\frac{\partial^2 \psi}{\partial v^2}$$

$$+ 4y^2\frac{\partial^2 \psi}{\partial u^2} - \cancel{4xy\frac{\partial^2 \psi}{\partial u\partial v}} - \cancel{4xy\frac{\partial^2 \psi}{\partial v\partial u}} + 4x^2\frac{\partial^2 \psi}{\partial v^2}$$

$$= 4(x^2+y^2)\frac{\partial^2 \psi}{\partial u^2} + 4(x^2+y^2)\frac{\partial^2 \psi}{\partial v^2}$$

$$= 4(x^2+y^2)\left[\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2}\right].$$

④ Given the transformations $u = e^x \cos y$ and $v = e^x \sin y$ and that ϕ is a function of u and v and also of x & y, prove that $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right)$.

Soln:-

Given:- $u = e^x \cos y$	$v = e^x \sin y$
$\frac{\partial u}{\partial x} = e^x \cos y = u$ $\frac{\partial u}{\partial y} = -e^x \sin y = -v$	$\frac{\partial v}{\partial x} = e^x \sin y = v$ $\frac{\partial v}{\partial y} = e^x \cos y = u$
$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial \phi}{\partial u} u + \frac{\partial \phi}{\partial v} v$ $= u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}$	$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}$ $= \frac{\partial \phi}{\partial u}(-v) + \frac{\partial \phi}{\partial v} u$ $= -v\frac{\partial \phi}{\partial u} + u\frac{\partial \phi}{\partial v}$
$\frac{\partial}{\partial x} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$	$\frac{\partial}{\partial y} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left[u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right]$$

$$= u^2\frac{\partial^2\phi}{\partial u^2} + uv\frac{\partial^2\phi}{\partial u\partial v} + vu\frac{\partial^2\phi}{\partial v\partial u} + v^2\frac{\partial^2\phi}{\partial v^2} \longrightarrow (1)$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \left[-v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right]\left[-v\frac{\partial \phi}{\partial u} + u\frac{\partial \phi}{\partial v}\right]$$

$$= v^2\frac{\partial^2\phi}{\partial u^2} - vu\frac{\partial^2\phi}{\partial u\partial v} - uv\frac{\partial^2\phi}{\partial v\partial u} + u^2\frac{\partial^2\phi}{\partial v^2} \longrightarrow (2)$$

(1) + (2)

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = u^2\frac{\partial^2\phi}{\partial u^2} + v^2\frac{\partial^2\phi}{\partial v^2} + v^2\frac{\partial^2\phi}{\partial u^2} + u^2\frac{\partial^2\phi}{\partial v^2}$$

$$= (u^2+v^2)\frac{\partial^2\phi}{\partial u^2} + (u^2+v^2)\frac{\partial^2\phi}{\partial v^2}.$$

$$= (u^2+v^2)\left[\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}\right].$$

(5) If $Z = f(y-z, z-x, x-y)$, Show that $\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0$

Soln:- [AU Jan 2013, Jan 2014 AU D15/J16 R-08, AU N/D 2016 R-08]

Let $u = y-z$, $v = z-x$, $w = x-y$

$$Z = f(u,v,w)$$

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}.$$

$$\frac{\partial Z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{\partial f}{\partial u}(1) + \frac{\partial f}{\partial v}(0) + \frac{\partial f}{\partial w}(-1) = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}.$$

$$\frac{\partial Z}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}$$

$$= \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}(1) + \frac{\partial f}{\partial w}(0) = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}.$$

$$\therefore \frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w} + \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w} - \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} = 0$$

(6) If z is a function of x and y where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial Z}{\partial x} - y\frac{\partial z}{\partial y}$.

Soln:- (AU A/M 2014 U.D)

Given:- $z = f(x,y)$

$x = e^u + e^{-v}$	$y = e^{-u} - e^v$
$\frac{\partial x}{\partial u} = e^u$ $\frac{\partial x}{\partial v} = -e^{-v}$	$\frac{\partial y}{\partial u} = -e^{-u}$ $\frac{\partial y}{\partial v} = -e^v$
$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$ $= \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u})$ $= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$	$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$ $= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v)$ $= -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^{u}\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^{v}\frac{\partial z}{\partial y}$$

$$= (e^{u}+e^{-v})\frac{\partial z}{\partial x} - (e^{-u}-e^{v})\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

(7) Transform the equation $z_{xx} + 2z_{xy} + z_{yy} = 0$ by changing the independent variables using $u = x-y$ and $v = x+y$.

Soln:-

(AU June 2012)

$u = x-y$	$v = x+y$
$\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = -1$	$\frac{\partial v}{\partial x} = 1$ $\frac{\partial v}{\partial y} = 1$
$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1)$ $= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$	$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$ $= \frac{\partial z}{\partial u}(-1) + \frac{\partial z}{\partial v}(1)$ $= -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$
$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$	$\frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$
$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)$ $= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$ $= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$	$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial y}\right)$ $= \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$ $= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}.$$

$z_{xx} + 2z_{xy} + z_{yy} = 0$

$$\Rightarrow \cancel{\frac{\partial^2 z}{\partial u^2}} + \cancel{\frac{\partial^2 z}{\partial u \partial v}} + \cancel{\frac{\partial^2 z}{\partial v \partial u}} + \frac{\partial^2 z}{\partial v^2} - 2\cancel{\frac{\partial^2 z}{\partial u^2}} + 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u} + 2\frac{\partial^2 z}{\partial v^2}$$

$$+ \cancel{\frac{\partial^2 z}{\partial u^2}} - \cancel{\frac{\partial^2 z}{\partial u \partial v}} - \cancel{\frac{\partial^2 z}{\partial v \partial u}} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$4\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{\partial^2 z}{\partial v \partial u} = 0 \Rightarrow 4Z_{vv} + 2Z_{uv} - Z_{vu} = 0$$

$$\Rightarrow 2Z_{vv} + Z_{uv} - Z_{vu} = 0$$

JACOBIANS:-

If u_1, u_2, u_3 are functions of three variables x_1, x_2, x_3 then

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

① If $x = r\cos\theta$, $y = r\sin\theta$, find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$ (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

[AU N/D 2014 R-08, 13, AU M/J 14, R-08, AU D15/J16 R-13, AU M/J 2016 R-13] AU N/D 2016 R-13]

Soln:-

Given:- $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

(i) $$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r\left[\cos^2\theta + \sin^2\theta\right] = r.$$

(ii) $$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

$$\Rightarrow r \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

$$\Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}.$$

② If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$; find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln:-

$$\frac{\partial u}{\partial x} = \frac{y^2}{2}\left(-\frac{1}{x^2}\right) = -\frac{y^2}{2x^2}.$$

$$\frac{\partial u}{\partial y} = \frac{2y}{2x} = \frac{y}{x}$$

$$v = \frac{x^2+y^2}{2x} = \frac{x^2}{2x} + \frac{y^2}{2x} = \frac{x}{2} + \frac{y^2}{2x}.$$

$$\frac{\partial v}{\partial x} = \frac{1}{2} + \frac{y^2}{2}\left(-\frac{1}{x^2}\right) = \frac{1}{2} - \frac{y^2}{2x^2} = \frac{x^2-y^2}{2x^2}.$$

$$\frac{\partial v}{\partial y} = \frac{2y}{2x} = \frac{y}{x}.$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} -y^2/2x^2 & y/x \\ \dfrac{x^2-y^2}{2x^2} & y/x \end{vmatrix}$$

$$= -\frac{y^3}{2x^3} - \frac{y(x^2-y^2)}{2x^3} = \frac{-y^3-yx^2+y^3}{2x^3}$$

$$= \frac{-yx^2}{2x^3} = -\frac{y}{2x}.$$

③ If $u=2xy$, $v=x^2-y^2$ and $x=r\cos\theta$, $y=r\sin\theta$, Evaluate $\dfrac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.

Soln:-

Given:- $u=2xy$	$v=x^2-y^2$	$x=r\cos\theta$	$y=r\sin\theta$
$\frac{\partial u}{\partial x}=2y$	$\frac{\partial v}{\partial x}=2x$	$\frac{\partial x}{\partial r}=\cos\theta$	$\frac{\partial y}{\partial r}=\sin\theta$
$\frac{\partial u}{\partial y}=2x$	$\frac{\partial v}{\partial y}=-2y$	$\frac{\partial x}{\partial \theta}=-r\sin\theta$	$\frac{\partial y}{\partial \theta}=r\cos\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - 4x^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= -4(x^2+y^2)\, r(\cos^2\theta + \sin^2\theta)$$

$$= -4(x^2+y^2)\, r$$

$$= -4r(r^2\cos^2\theta + r^2\sin^2\theta)$$

$$= -4r.r^2(\cos^2\theta + \sin^2\theta)$$

$$= -4r^3.$$

(4) Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3

$y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$. [AU N/D 2016 - R-13 M/J 14, A/M-15, AM/17]

Soln:-

Given:- $y_1 = \frac{x_2 x_3}{x_1}$	$y_2 = \frac{x_3 x_1}{x_2}$	$y_3 = \frac{x_1 x_2}{x_3}$
$\frac{\partial y_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}$	$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$	$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$
$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$	$\frac{\partial y_2}{\partial x_2} = \frac{-x_3 x_1}{x_2^2}$	$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$
$\frac{\partial y_2}{\partial x_3} = \frac{x_2}{x_1}$	$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$	$\frac{\partial y_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{-x_2x_3}{x_1^2}\left[\frac{x_1^2x_3x_2}{x_2^2x_3^2} - \frac{x_1^2}{x_2x_3}\right] - \frac{x_3}{x_1}\left[\frac{-x_1x_2x_3}{x_2x_3^2} - \frac{x_1x_2}{x_2x_3}\right]$$

$$+ \frac{x_2}{x_1}\left[\frac{x_1x_3}{x_2x_3} + \frac{x_3x_1x_2}{x_3x_2^2}\right]$$

$$= -\frac{x_2x_3x_1^2x_3x_2}{x_1^2x_2^2x_3^2} + \frac{x_2x_3x_1^2}{x_1^2x_2x_3} + \frac{x_1x_2x_3^2}{x_1x_2x_3^2} + \frac{x_1x_2x_3}{x_1x_2x_3}$$

$$+ \frac{x_2x_1x_3}{x_1x_2x_3} + \frac{x_1x_3x_2^2}{x_1x_3x_2^2}$$

$$= \cancel{-1} + \cancel{1} + 1 + 1 + 1 + 1 = 4.$$

(5) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

[AU JAN 14]
[AU D15/D16 R-08]

Soln:-

Given:- $u = \frac{yz}{x}$	$v = \frac{zx}{y}$	$w = \frac{xy}{z}$
$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$	$\frac{\partial v}{\partial x} = \frac{z}{y}$	$\frac{\partial w}{\partial x} = \frac{y}{z}$
$\frac{\partial u}{\partial y} = \frac{z}{x}$	$\frac{\partial v}{\partial y} = \frac{-zx}{y^2}$	$\frac{\partial w}{\partial y} = \frac{x}{z}$
$\frac{\partial u}{\partial z} = \frac{y}{x}$	$\frac{\partial v}{\partial z} = \frac{x}{y}$	$\frac{\partial w}{\partial z} = -\frac{xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left[\frac{zx^2y}{z^2y^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[\frac{-xyz}{yz^2} - \frac{xy}{zy}\right] + \frac{y}{x}\left[\frac{xz}{yz} + \frac{zxy}{y^2z}\right]$$

$$= \frac{-x^2y^2z^2}{x^2y^2z^2} + \frac{x^2yz}{x^2yz} + \frac{xyz^2}{xyz^2} + \frac{xyz}{xyz} + \frac{xyz}{xyz} + \frac{xy^2z}{xy^2z}$$

$$= \cancel{-1} + \cancel{1} + 1 + 1 + 1 + 1 = 4$$

(6) If $x+y+z=u$, $y+z=uv$, $z=uvw$, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.

Soln:-

Given:-

$u = x+y+z$	$y+z = uv$	$z = uvw$
$u = x+uv$	$y = uv - z$	
$\Rightarrow x = u - uv$	$y = uv - uvw$	

$x = u-uv$	$y = uv - uvw$	$z = uvw$
$\frac{\partial x}{\partial u} = 1-v$	$\frac{\partial y}{\partial u} = v - vw$	$\frac{\partial z}{\partial u} = vw$
$\frac{\partial x}{\partial v} = -u$	$\frac{\partial y}{\partial v} = u - uw$	$\frac{\partial z}{\partial v} = uw$
$\frac{\partial x}{\partial w} = 0$	$\frac{\partial y}{\partial w} = -uv$	$\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left[(u-uw)uv + (uv)(uw)\right] + u\left[(v-vw)uv + (uv)(vw)\right]$$

$$= (1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$= u^2v.$$

(7) Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$ of the transformation $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$.

[AU M/J 2011,
AU D15/J16 R-13
AU M/J 2016 R-13]

Soln:-

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$x = r\sin\theta\cos\phi$	$y = r\sin\theta\sin\phi$	$z = r\cos\theta$
$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$ $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$	$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$ $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$	$\frac{\partial z}{\partial r} = \cos\theta$ $\frac{\partial z}{\partial \theta} = -r\sin\theta$ $\frac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Expand using third row,

$$= \cos\theta\left[r^2\cos\theta\sin\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right]$$
$$+ r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$

$$= \cos\theta\left[r^2\cos\theta\sin\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$
$$+ r\sin\theta\left[r\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right)\right]$$

$$= r^2\cos^2\theta\sin\theta + r^2\sin^3\theta$$

$$= r^2\sin\theta\left(\cos^2\theta + \sin^2\theta\right) = r^2\sin\theta.$$

TAYLOR'S SERIES FOR FUNCTION OF TWO VARIABLES :-

FORMULA

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h\, f_x(a,b) + k\, f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k\, f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$

$$+ \dots \quad \text{where } h = x-a, \; k = y-b.$$

① Expand $e^x \cos y$ about $(0, \pi/2)$ up to third degree terms using Taylor's series. [[redacted]]

Soln:-

Function	value at $(0, \pi/2)$
$f(x,y) = e^x \cos y$	$f(0,\pi/2) = e^0 \cos \pi/2 = 0$
$f_x = e^x \cos y$ $f_y = -e^x \sin y$	$f_x(0,\pi/2) = e^0 \cos \pi/2 = 0$ $-e^0 \sin \pi/2 = -1$
$f_{xx} = e^x \cos y$ $f_{xy} = -e^x \sin y$ $f_{yy} = -e^x \cos y$	$e^0 \cos \pi/2 = 0$ $-e^0 \sin \pi/2 = -1$ $-e^0 \cos \pi/2 = 0$
$f_{xxx} = e^x \cos y$ $f_{xxy} = -e^x \sin y$ $f_{xyy} = -e^x \cos y$ $f_{yyy} = e^x \sin y$	$f_{xxx}(0,\pi/2) = e^x \cos \pi/2 = 0$ $f_{xxy} = -e^0 \sin \pi/2 = -1$ $-e^0 \cos \pi/2 = 0$ $e^0 \sin \pi/2 = 1$

Substitute all values in Taylor series

$$f(x,y) = 0 + \left[x(0) + (y-\pi/2)(-1)\right] + \frac{1}{2!}\left[x^2(0) + 2x(y-\pi/2)(-1) + (y-\pi/2)^2(0)\right]$$

$$+ \frac{1}{3!}\left[x^3(0) + 3x^2(y-\pi/2)(-1) + 3x(y-\pi/2)^2(0) + (y-\pi/2)^3(+1)\right]$$

$$= -y + \pi/2 - \frac{1}{2}\left(2x(y-\pi/2)\right) + \frac{1}{6}\left[-3x^2(y-\pi/2) + (y-\pi/2)^3\right]$$

② Expand $\sin xy$ in powers of $x-1$ and $y-\pi/2$ up to second degree terms by using Taylor's series. [N/D 17 R-13
AU N/D 2014, 2015 R-13]

Soln:-

Function	Value at $(1, \pi/2)$
$f(x,y) = \sin(xy)$	$f(1,\pi/2) = \sin\pi/2 = 1$
$f_x = y\cos(xy)$ $f_y = x\cos(xy)$	$\frac{\pi}{2}\cos\pi/2 = 0$ $\cos\pi/2 = 0$
$f_{xx} = -y^2\sin(xy)$ $f_{xy} = y(-x\sin(xy)) + \cos(xy)$ $f_{yy} = -x^2\sin(xy)$	$-\frac{\pi^2}{4}\sin\pi/2 = -\pi^2/4$ $-\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = -\pi/2$ $-\sin\pi/2 = -1$

Taylor series : $f(x,y) = f(a,b) + \frac{1}{1!}[h f_x(a,b) + k f_y(a,b)]$
$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$.

Here $a=1$; $b=\pi/2$
$h = x-a = x-1$; $k = y-b = y-\pi/2$.

$$f(x,y) = 1 + (x-1)(0) + (y-\pi/2)(0) + \frac{1}{2!}\left[(x-1)^2\left(-\frac{\pi^2}{4}\right) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)\right]$$

$$= 1 + \frac{1}{2}\left[-\frac{\pi^2}{4}(x-1)^2 - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\right].$$

③ Expand $e^x \log(1+y)$ in powers of x and y up to third degree terms using Taylor's series. [[redacted]]

Soln:-

$f(x,y) = f(a,b) + \frac{1}{1!}[h f_x(a,b) + k f_y(a,b)]$
$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$
$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$

Function	Value at (0,0)
$f(x,y) = e^x \log(1+y)$	$f(0,0) = e^0 \log 1 = 0$
$f_x = e^x \log(1+y)$ $f_y = e^x\left(\frac{1}{1+y}\right)$	$f_x = e^0 \log 1 = 0$ $f_y = e^0 (1) = 1$
$f_{xx} = e^x \log(1+y)$ $f_{xy} = e^x\left(\frac{1}{1+y}\right)$ $f_{yy} = -e^x \frac{1}{(1+y)^2}$	$f_{xx} = e^0 \log 1 = 0$ $f_{xy} = e^0(1) = 1$ $f_{yy} = -e^0 = -1$
$f_{xxx} = e^x \log(1+y)$ $f_{xxy} = e^x\left(\frac{1}{1+y}\right)$ $f_{xyy} = -e^x \frac{1}{(1+y)^2}$ $f_{yyy} = 2e^x \frac{1}{(1+y)^3}$	$f_{xxx} = e^0 \log 1 = 0$ $f_{xxy} = e^0 = 1$ $f_{xyy} = -e^0 = -1$ $f_{yyy} = e^0 = 2$

Here $a=0$ & $b=0$

$h = x-a = x$

$k = y-b = y$

$$\therefore f(x,y) = 0 + x(0) + y(1) + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(-1)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\right]$$
$$= y + \frac{1}{2}(2xy - y^2) + \frac{1}{6}\left[3x^2y - 3xy^2 + 2y^3\right].$$

4. Expand $e^x \sin y$ in powers of x & y up to third degree terms using Taylor's series. [JAN 16 R-13]

Soln:-

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$

Function	Value at (0,0)
$f(x,y) = e^x \sin y$	$f(0,0) = e^0 \sin 0 = 0$
$f_x = e^x \sin y$ $f_y = e^x \cos y$	$f_x = 0$ $f_y = 1$
$f_{xx} = e^x \sin y$ $f_{xy} = e^x \cos y$ $f_{yy} = -e^x \sin y$	$f_{xx} = 0$ $f_{xy} = 1$ $f_{yy} = 0$
$f_{xxx} = e^x \sin y$ $f_{xxy} = e^x \cos y$ $f_{xyy} = -e^x \sin y$ $f_{yyy} = -e^x \cos y$	$f_{xxx} = 0$ $f_{xxy} = 1$ $f_{xyy} = 0$ $f_{yyy} = -1$

Here $a=0$; $b=0$

$h = x-a = x$

$k = y-b = y$

$$f(x,y) = 0 + \frac{1}{1!}\left[x(0) + y(1)\right] + \frac{1}{2!}\left[x^2(0) + 2xy(1) + y^2(0)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)\right]$$
$$= y + \frac{1}{2}(2xy) + \frac{1}{6}(3x^2y) + \frac{1}{6}(-y^3)$$
$$= y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3.$$

5. Obtain the Taylor's series expansion of $x^3+y^3+xy^2$ interms of powers of $x-1$ & $y-2$ up to third degree terms.
[JAN 18 R-13, A/M 15 R-13, A/M 17 R-13]

Soln:-

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right]$$

Function	Value at (0,0)
$f(x,y) = x^3+y^3+xy^2$	$f(1,2) = 1+8+4 = 13$
$f_x = 3x^2+y^2$ $f_y = 3y^2+2xy$	$3+4=7$ $12+4=16$
$f_{xx} = 6x$ $f_{xy} = 2y$ $f_{yy} = 6y+2x$	6 4 $12+2=14$
$f_{xxx} = 6$ $f_{xxy} = 0$ $f_{xyy} = 2$ $f_{yyy} = 6$	6 0 2 6

$$f(x,y) = 13 + (x-1)(7) + (y-2)(16) + \frac{1}{2!}\left[(x-1)^2(6) + 2(x-1)(y-2)(4) + (y-2)^2(14)\right] + \frac{1}{3!}\left[(x-1)^3(6) + 3(x-1)^2(y-2)(0) + 3(x-1)(y-2)^2(2) + (y-2)^3(6)\right]$$

$$= 13 + 7(x-1) + 16(y-2) + \frac{1}{2}\left[6(x-1)^2 + 8(x-1)(y-2) + 14(y-2)^2\right] + \frac{1}{6}\left[6(x-1)^3 + 6(x-1)(y-2)^2 + 6(y-2)^3\right]$$

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES :-

Necessary conditions for a maximum or minimum

$f_x(a,b) = 0$ and $f_y(a,b) = 0$

Notations :- $f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$, $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, $f_{yy} = \dfrac{\partial^2 f}{\partial y^2}$, $f_{xy} = \dfrac{\partial^2 f}{\partial x \partial y}$

Sufficient conditions :- If $f_x(a,b) = 0$, $f_y(a,b) = 0$ and $f_{xx}(a,b) = A$, $f_{xy}(a,b) = B$, $f_{yy}(a,b) = C$ then

i) $f(a,b)$ is maximum value if $AC-B^2>0$ and $A<0$ (or $B<0$).

ii) $f(a,b)$ is minimum value if $AC-B^2>0$ and $A>0$ (or $B>0$).

iii) $f(a,b)$ is not extremum (saddle) if $AC-B^2<0$

iv) If $AC-B^2=0$, then the test is inconclusive.

Stationary value :-

A function $f(x,y)$ is said to be stationary at (a,b) or $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b)=0$ and $f_y(a,b)=0$.

PROBLEMS :-

① Find the extreme values of the function $f(x,y)=x^3+y^3-3x-12y+20$ [AU N/D 14, R-13]

Soln:-

Given :- $f(x,y) = x^3+y^3-3x-12y+20$

$f_x = 3x^2-3$

$f_y = 3y^2-12$

$A = f_{xx} = 6x$

$B = f_{xy} = 0$

$C = f_{yy} = 6y$

$AC-B^2 = (6x)(6y)-(0) = 36xy$

To find stationary points :

$f_x=0$	$f_y=0$
$3x^2-3=0$	$3y^2-12=0$
$3(x^2-1)=0$	$3(y^2-4)=0$
$x^2-1=0$	$y^2-4=0$
$x^2=1$	$\boxed{y=\pm 2}$
$\boxed{x=\pm 1}$	

∴ The stationary points are

$(1,2), (1,-2), (-1,2), (-1,-2)$

	(1,2)	(1,-2)	(-1,2)	(-1,-2)
$A = 6x$	$6>0$	$6>0$	$-6<0$	$-6<0$
$B = 0$	0	0	0	0
$AC-B^2 = 36xy$	$72>0$	$-72<0$	$-72<0$	$72>0$
Conclusion	minimum	Saddle	Saddle	maximum

$\therefore$ maximum value of $f(x,y)$ is

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20$$
$$= 38$$

Minimum value of $f(x,y)$ is

$$f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$$
$$= 1 + 8 - 3 - 24 + 20$$
$$= 2$$

(2) Find the extreme values of $f(x,y) = x^3y^2(1-x-y)$

Soln:- [AU JAN 14, R-13]

$$f(x,y) = x^3y^2(1-x-y)$$
$$= x^3y^2 - x^4y^2 - x^3y^3$$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2$$

$$A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$$

$$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$$

To find Stationary points :-

$f_x = 0$

$$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$
$$x^2y^2[3 - 4x - 3y] = 0$$
$$\Rightarrow x=0,\ y=0,\ 4x+3y=3$$
$$4x+3y=3 \longrightarrow (1)$$

$f_y = 0$

$$2x^3y - 2x^4y - 3x^3y^2 = 0$$
$$x^3y[2 - 2x - 3y] = 0$$
$$x=0,\ y=0,\ 2x+3y=2$$
$$2x+3y=2 \longrightarrow (2)$$

$$4x + 3y = 3$$
$$2x + 3y = 2$$
$$\overline{2x = 1}$$

$$\boxed{x = 1/2}$$

Substitute $x = 1/2$ in ②

$$2 \times \frac{1}{2} + 3y = 2$$
$$1 + 3y = 2$$
$$3y = 2 - 1 = 1$$
$$\boxed{y = 1/3}$$

$\therefore$ The stationary points are $(0,0)$ & $(1/2, 1/3)$.

At $(1/2, 1/3)$

$$A = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3$$
$$= 6 \times \frac{1}{2} \times \frac{1}{9} - 12 \times \frac{1}{4} \times \frac{1}{9} - 6 \times \frac{1}{2} \times \frac{1}{27}$$
$$= \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{9} = -1/9.$$

$$B = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^2$$
$$= 6 \times \frac{1}{4} \times \frac{1}{3} - 8 \times \frac{1}{8} \times \frac{1}{3} - 9 \times \frac{1}{4} \times \frac{1}{9}$$
$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = \frac{6-4-3}{12} = -1/12$$

$$C = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3)$$
$$= 2 \times \frac{1}{8} - 2 \times \frac{1}{16} - 6 \times \frac{1}{8} \times \frac{1}{3}$$
$$= \cancel{\frac{1}{4}} - \frac{1}{8} - \cancel{1/4} = -1/8$$

$$AC - B^2 = (-\tfrac{1}{9})(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$$

$A = -1/9 < 0$.

$\therefore f(1/2, 1/3)$ is maximum.

Maximum value of $f(x,y)$ is $f(1/2, 1/3) = (1/2)^3 (1/3)^2 (1 - 1/2 - 1/3)$

$$= 1/8 \times 1/9 \left[\frac{6-3-2}{6}\right] = \frac{1}{72}(\ldots$$
$$= \frac{1}{432}.$$

③ Find the maximum or minimum values of $f(x,y) = 3x^2 - y^2 + x^3$.

Soln:- [AU J-18, R-17]

Given:- $f(x,y) = 3x^2 - y^2 + x^3$

$f_x = 6x + 3x^2$

$f_y = -2y$

$A = f_{xx} = 6 + 6x$

$B = f_{xy} = 0$

$C = f_{yy} = -2$

To find Stationary points:

$f_x = 0$	$f_y = 0$
$6x + 3x^2 = 0$	$-2y = 0$
$3(2x + x^2) = 0$	$y = 0$
$3x(2+x) = 0$	
$x = 0;\quad x = -2$	

∴ The stationary points are $(0,0)$ & $(-2,0)$

	$(0,0)$	$(-2,0)$
$A = 6+6x$	$6 > 0$	$-6 < 0$
$B = 0$	0	0
$C = -2$	-2	-2
$AC - B^2$	$-12 < 0$	$12 > 0$
Conclusion	Saddle	Maximum

∴ Maximum value of $f(x,y)$ is

$f(-2,0) = 3(-2)^2 - 0 + (-2)^3$

$= 3(+4) - 8 = 12 - 8 = 4$

④ Examine the maxima and minima of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Soln:- [AU ~~D/J~~ J16, R-13 / AU M/J 16, ~~R-08~~]

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$f_x = 3x^2 + 3y^2 - 30x + 72$

$f_y = 6xy - 30y$

$A = f_{xx} = 6x - 30$

$B = f_{xy} = 6y$

$C = f_{yy} = 6y - 30$

To find the stationary points.

$f_x = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$

$f_y = 0$

$6xy - 30y = 0$

$6y(x-5) = 0$

$y = 0, \ x = 5$

when $y = 0$

$3x^2 - 30x + 72 = 0$

$3(x^2 - 10x + 24) = 0$

$x^2 - 10x + 24 = 0$

$(x-6)(x-4) = 0$

$x = 6, 4.$

$\therefore$ The stationary points are $(6,0), (4,0)$

	$(6,0)$	$(4,0)$
$A = 6x - 30$	$36 - 30 = 6 > 0$	$24 - 30 = -6 < 0$
$B = 6y$	0	0
$C = 6y - 30$	-30	-30
$AC - B^2$	$-180 < 0$	$180 > 0$
Conclusion	Saddle	maximum

$\therefore$ Maximum value of $f(x,y)$ is

$f(4,0) = (4)^3 + 0 - 15(4)^2 - 0 + 72(4)$

$= 64 - 240 + 288$

$= 328$

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS:-

To find the maximum and minimum values of $f(x,y,z)$ where x,y,z are subject to a constraint equation $g(x,y,z)=0$

We define a function

$$F(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) \longrightarrow (1)$$

where λ is called Lagrange multiplier which is independent of x,y,z.

The necessary conditions for a maximum or minimum are

$$\frac{\partial F}{\partial x} = 0 \to (2), \quad \frac{\partial F}{\partial y} = 0 \to (3) \quad \text{and} \quad \frac{\partial F}{\partial z} = 0 \to (4).$$

PROBLEMS:-

(1) A rectangular box opens at the top is to have a volume of 32cc. Find the dimensions of the box, that requires the least material for its construction. [A/M 17 R-13, N/D 15 R-13, M/J 16 R-13]

Soln:-

Let x,y,z be the length, breadth and height of the box.

$$\text{Surface area} = xy + 2yz + 2xz = f(x,y,z)$$

$$\text{volume} = xyz = 32 \quad = g(x,y,z)$$

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$\Rightarrow F(x,y,z) = xy + 2yz + 2xz + \lambda(xyz - 32) \longrightarrow (1)$$

$$\frac{\partial F}{\partial x} = 0$$

$$y + 2z + \lambda yz = 0$$

$$y + 2z = -\lambda yz$$

$$\frac{y+2z}{yz} = -\lambda$$

$$\frac{1}{z} + \frac{2}{y} = -\lambda \to (2)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow x + 2z + \lambda xz = 0$$

$$x + 2z = -\lambda xz$$

$$\frac{x+2z}{xz} = -\lambda$$

$$\frac{1}{z} + \frac{2}{x} = -\lambda \to (3)$$

$$\frac{\partial F}{\partial z} = 0$$

$$2y + 2x + \lambda xy = 0$$

$$2y + 2x = -\lambda xy$$

$$\frac{2y+2x}{xy} = -\lambda$$

$$\frac{2}{x} + \frac{2}{y} = -\lambda \longrightarrow (4)$$

From (2) & (3)

$$\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$x = y \longrightarrow (5)$$

From (3) & (4)

$$\frac{1}{z} + \cancel{\frac{2}{x}} = \cancel{\frac{2}{x}} + \frac{2}{y}$$

$$\frac{1}{z} = \frac{2}{y}$$

$$y = 2z \longrightarrow (6)$$

From ⑤ & ⑥

$$x = y = 2z$$

Volume : $xyz = 32$

$$(2z)(2z)z = 32$$

$$4z^3 = 32$$

$$z^3 = \frac{32}{4} = 8$$

$$\boxed{z = 2}$$

$\therefore$ $x = 4$; $y = 4$; $z = 2$.

$\therefore$ Dimension of the box are 4, 4, 2.

② Find the dimensions of the rectangular box without top of maximum capacity whose surface area is 108 sq cm.

Soln:-

[Jan 18 R-17]

Let x, y, z be the length, breadth & height of the box.

Surface area : $xy + 2yz + 2zx = 108 = g$

Volume $= xyz = f$

$\therefore$ $F(x,y,z,\lambda) = xyz + \lambda(xy + 2yz + 2zx - 108) \longrightarrow ①$

$F_x = 0$

$$yz + \lambda(y + 2z) = 0$$

$$yz = -\lambda(y + 2z)$$

$$\frac{y + 2z}{yz} = -\frac{1}{\lambda}$$

$$\frac{1}{z} + \frac{2}{y} = -\frac{1}{\lambda} \longrightarrow ②$$

$F_y = 0$

$$xz + \lambda(x + 2z) = 0$$

$$xz = -\lambda(x + 2z)$$

$$\frac{x + 2z}{xz} = -\frac{1}{\lambda}$$

$$\frac{1}{z} + \frac{2}{x} = -1/\lambda \longrightarrow ③$$

$F_z = 0$

$$xy + \lambda(2x + 2y) = 0$$

$$xy = -\lambda(2x + 2y)$$

$$\frac{2x + 2y}{xy} = -\frac{1}{\lambda}$$

$$\frac{2}{y} + \frac{2}{x} = -1/\lambda \longrightarrow ④$$

From ② & ③

$$\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x} \Rightarrow x = y \longrightarrow ⑤$$

From ③ & ④

$$\frac{1}{z} + \cancel{\frac{2}{x}} = \frac{2}{y} + \cancel{\frac{2}{x}}$$

$$\frac{1}{z} = \frac{2}{y} \Rightarrow y = 2z \longrightarrow ⑥$$

From ⑤ & ⑥

$x = y = 2z$

Surface area : $xy + 2yz + 2zx = 108$

$(2z)(2z) + 2(2z)z + 2z(2z) = 108$

$4z^2 + 4z^2 + 4z^2 = 108$

$12z^2 = 108$

$z^2 = \frac{108}{12} = 9$

$\boxed{z = 3}$

$\therefore x = 6 \; ; \; y = 6 \; ; \; z = 3$

$\therefore$ The dimensions are 6, 6, 3.

③ The temperature $u(x,y,z)$ at any point in space is $u = 400xyz^2$. Find the highest temperature on surface of the sphere $x^2+y^2+z^2 =$

[N/D-17, R-13, A/M-18, R-13]

Soln:-

Given:- $u = f = 400xyz^2$

$g = x^2+y^2+z^2-1$

$F(x,y,z,\lambda) = 400xyz^2 + \lambda(x^2+y^2+z^2-1) \longrightarrow ①$

$F_x = 0$

$400yz^2 + \lambda(2x) = 0$

$400yz^2 = -2\lambda x$

$\frac{400yz^2}{2x} = -\lambda$

$\frac{200yz^2}{x} = -\lambda \longrightarrow ②$

$F_y = 0$

$400xz^2 + \lambda(2y) = 0$

$400xz^2 = -\lambda 2y$

$\frac{400xz^2}{2y} = -\lambda$

$\frac{200xz^2}{y} = -\lambda \longrightarrow ③$

$F_z = 0$

$800xyz + \lambda(2z) = 0$

$800xyz = -\lambda 2z$

$\frac{800xyz}{2z} = -\lambda$

$400xy = -\lambda \longrightarrow ④$

From ② & ③

$\frac{200yz^2}{x} = \frac{200xz^2}{y}$

$\frac{y}{x} = \frac{x}{y}$

$y^2 = x^2 \longrightarrow ⑤$

From ③ & ④

$\frac{200xz^2}{y} = 400xy$

$\frac{z^2}{y} = 2y$

$2y^2 = z^2 \longrightarrow ⑥$

From (5) & (6)

$x^2 = y^2 = \frac{1}{2}z^2$

We have $x^2+y^2+z^2 = 1$

$\frac{1}{2}z^2 + \frac{1}{2}z^2 + z^2 = 1$

$\frac{z^2+z^2+2z^2}{2} = 1$

$\frac{4z^2}{2} = 1 \Rightarrow 2z^2 = 1 \Rightarrow z^2 = \frac{1}{2} \Rightarrow z = \pm\frac{1}{\sqrt{2}}$

$\therefore x^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$

& $y^2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} \Rightarrow y^2 = \frac{1}{4} \Rightarrow y = \pm\frac{1}{2}$

$\therefore$ Temperature $u = 400xyz^2$

$= 400\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2$

$= 400 \times \frac{1}{4} \times \frac{1}{2} = 50$

$\therefore$ Maximum temperature is 50.

(4) Find the maximum volume of the largest rectangular parallelopiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$. [AU A/M 2015 R-13, AU N/D 2015]

Soln: Let the vertex of the parallelopiped be (x, y, z)

All other vertices will be $(\pm x, \pm y, \pm z)$

Sides of the solid be $2x, 2y, 2z$

Volume $V = (2x)(2y)(2z) = 8xyz = f$

We have to maximize V subject to the condition

$g(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} - 1 = 0$

$$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} - 1\right) \longrightarrow (1)$$

$F_x = 0$

$$8yz + \frac{2x\lambda}{a^2} = 0$$

$$8yz = -\frac{2x\lambda}{a^2}$$

$$\frac{8yz}{-2\lambda} = \frac{x}{a^2}$$

$$-\frac{4yz}{\lambda} = \frac{x}{a^2}$$

$$-\frac{4xyz}{\lambda} = \frac{x^2}{a^2} \rightarrow (2)$$

$F_y = 0$

$$8xz + \frac{2y\lambda}{b^2} = 0$$

$$8xz = -\frac{2y\lambda}{b^2}$$

$$\frac{8xz}{-2\lambda} = \frac{y}{b^2}$$

$$-\frac{4xz}{\lambda} = \frac{y}{b^2}$$

$$-\frac{4xzy}{\lambda} = \frac{y^2}{b^2} \rightarrow (3)$$

$F_z = 0$

$$8xy + \frac{2z\lambda}{c^2} = 0$$

$$8xy = -\frac{2z\lambda}{c^2}$$

$$\frac{8xy}{-2\lambda} = \frac{z}{c^2}$$

$$-\frac{4xy}{\lambda} = \frac{z}{c^2}$$

$$-\frac{4xyz}{\lambda} = \frac{z^2}{c^2} \rightarrow (4)$$

From (2), (3) & (4)

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Given:- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{c^2} = 1$$

$$\frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly, $y = \frac{b}{\sqrt{3}}$ & $z = \frac{c}{\sqrt{3}}$

$\therefore$ Extremum point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$, maximum point.

$\therefore$ Maximum value is $V = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$

$$= \frac{8abc}{3\sqrt{3}}.$$

(5) Find the maximum Value of $x^m y^n z^p$ when $x+y+z=a$

Soln:-

Let $f = x^m y^n z^p$

$g = x+y+z-a$

$$F(x,y,z,\lambda) = x^m y^n z^p + \lambda(x+y+z-a) \longrightarrow (1)$$

$F_x = 0$

$$mx^{m-1} y^n z^p + \lambda = 0$$
$$mx^{m-1} y^n z^p = -\lambda$$
$$\frac{mx^m y^n z^p}{x} = -\lambda \quad \rightarrow (2)$$

$F_y = 0$

$$nx^m y^{n-1} z^p + \lambda = 0$$
$$nx^m y^{n-1} z^p = -\lambda$$
$$\frac{nx^m y^n z^p}{y} = -\lambda \quad \rightarrow (3)$$

$F_z = 0$

$$px^m y^n z^{p-1} + \lambda = 0$$
$$px^m y^n z^{p-1} = -\lambda$$
$$\frac{px^m y^n z^p}{z} = -\lambda \quad \rightarrow (4)$$

From (2), (3) & (4)

$$\frac{mx^m y^n z^p}{x} = \frac{nx^m y^n z^p}{y} = \frac{px^m y^n z^p}{z}$$

$$\frac{m}{x} = \frac{n}{y} = \frac{p}{z}$$

$$\frac{m}{x} = \frac{n}{y} \Rightarrow my = nx \Rightarrow x = \frac{m}{n}y$$

$$\frac{n}{y} = \frac{p}{z} \Rightarrow nz = py \Rightarrow z = \frac{p}{n}y$$

Given:- $x+y+z = a$

$$\frac{m}{n}y + y + \frac{y}{n}p = a$$

$$my + ny + yp = na$$

$$y(m+n+p) = na$$

$$y = \frac{na}{m+n+p}$$

Also $x = \dfrac{ma}{m+n+p}$ & $z = \dfrac{pa}{m+n+p}$

$\therefore$ The stationary point is $\left(\dfrac{am}{m+n+p}, \dfrac{na}{m+n+p}, \dfrac{pa}{m+n+p}\right)$

$\therefore$ Maximum value of f is

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$= \frac{a^{m+n+p}\, m^m n^n p^p}{(m+n+p)^{m+n+p}}.$$

⑥ Find the minimum values of x^2yz^3 subject to the condition $2x+y+3z=a$ [AU M/J 2007, AU AM 2017 -R-13]

Soln:-

Let $f = x^2yz^3$

$g = 2x+y+3z-a$

$F(x,y,z,\lambda) = x^2yz^3 + \lambda(2x+y+3z-a) \longrightarrow$ ①

$F_x = 0$

$2xyz^3 + 2\lambda = 0$

$2xyz^3 = -2\lambda$

$xyz^3 = -\lambda \longrightarrow$ ②

$F_y = 0$

$x^2z^3 + \lambda = 0$

$x^2z^3 = -\lambda \longrightarrow$ ③

$F_z = 0$

$3x^2yz^2 + 3\lambda = 0$

$3x^2yz^2 = -3\lambda$

$x^2yz^2 = -\lambda \longrightarrow$ ④

From ② & ③

$xyz^3 = x^2z^3$

$y = x \longrightarrow$ ⑤

From ③ & ④

$x^2z^3 = x^2yz^2$

$z = y \longrightarrow$ ⑥

From ⑤ & ⑥

$x = y = z$

Given:- $2x+y+3z = a$

$2z + z + 3z = a$

$6z = a$

$z = \frac{a}{6}$

$\therefore \quad x = y = \frac{a}{6}$

$\therefore$ The stationary point is $\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right)$.

$\therefore$ Minimum value of f is $\left(\frac{a}{6}\right)^2\left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3 = \frac{a^6}{6^6} = \left(\frac{a}{6}\right)^6$.

⑦ Find the shortest and the longest distances from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2=24$ using Lagrange's method of constrained maxima and minima. [AU N/D 2016 R-13]

Soln:-

Let (x,y,z) be any point of the sphere.

Distance of the point (x,y,z) from $(1,2,-1)$ is given by

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

We have to find the maximum and minimum values of d.

$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$ subject to $x^2+y^2+z^2-24=0$

Let $f = (x-1)^2 + (y-2)^2 + (z+1)^2$

$g = x^2+y^2+z^2-24 = 0$

$$F(x,y,z,\lambda) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2+y^2+z^2-24) \longrightarrow (1)$$

$F_x = 0$

$2(x-1) + 2\lambda x = 0$

$x - 1 + \lambda x = 0$

$(1+\lambda)x = 1$

$x = \frac{1}{1+\lambda} \longrightarrow (2)$

$F_y = 0$

$2(y-2) + 2\lambda y = 0$

$y - 2 + \lambda y = 0$

$(1+\lambda)y = 2$

$\frac{y}{2} = \frac{1}{1+\lambda} \longrightarrow (3)$

$F_z = 0$

$2(z+1) + 2\lambda z = 0$

$z + 1 + \lambda z = 0$

$(1+\lambda)z = -1$

$\frac{z}{-1} = \frac{1}{1+\lambda} \longrightarrow (4)$

From (2) & (4)

$x = -z \longrightarrow (5)$

From (3) & (4)

$\frac{y}{2} = -z \Rightarrow y = -2z \longrightarrow (6)$

Given :- $x^2+y^2+z^2 = 24$

$(-z)^2 + (-2z)^2 + z^2 = 24$

$z^2 + 4z^2 + z^2 = 24$

$6z^2 = 24$

$z^2 = 4$

$z = \pm 2$

If $z = 2$, then $x = -2$ and $y = -4$

$\therefore$ The point is $(-2,-4,2)$; $d = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$

If $z = -2$, then $x = 2$ and $y = 4$

$\therefore$ The point is $(2,4,-2)$; $d = \sqrt{(-3)^2+(-6)^2+3^2} = 3\sqrt{6}$

$\therefore$ Shortest and longest distances are $\sqrt{6}$ and $3\sqrt{6}$.

1. A thin closed rectangular box is to have one edge equal to twice the other and constant volume $72 m^3$. Find the least surface area of the box.

[Nov/Dec -2019]

(Problem under Legrange's topic).

Soln:

Let $x, y, 2y$ be the length, breadth and height of the box respectively.

Surface Area $= 2(x)(y) + 2(y)(2y) + 2(x)(2y)$

$= 2xy + 4y^2 + 4xy = 6xy + 4y^2$ —— (A)

Volume : $xyz = 72$

$x(y)(2y) = 72$

$2xy^2 = 72 \Rightarrow xy^2 = 36$ —— (B)

∴ The auxillary Function F be

$F(x,y,z,\lambda) = (6xy + 4y^2) + \lambda(xy^2 - 36)$

$F_x = 6y + \lambda y^2$; $F_y = 8y + 6x + 2\lambda xy$

To find the extremum,

$F_x = 0$

$6y + \lambda y^2 = 0$

$6y = -\lambda y^2$

$\frac{6}{y} = -\lambda$ —— (1)

$F_y = 0$

$6x + 8y + 2\lambda xy = 0$

$6x + 8y = -2\lambda xy$

$-\lambda = \frac{3x + 4y}{xy}$

$$\frac{3}{y} + \frac{4}{x} = -\lambda \quad \text{---(2)}$$

From (1) & (2)

$$\frac{6}{y} = \frac{3}{y} + \frac{4}{x} \Rightarrow \frac{3}{y} = \frac{4}{x}$$

$$3x = 4y \Rightarrow y = \frac{3}{4}x \quad \text{---(3)}$$

(B) $\Rightarrow xy^2 = 36$

$$\Rightarrow x\left(\frac{3}{4}x\right)^2 = \frac{9}{16}x^3 = 36$$

$$\Rightarrow x^3 = 64$$

$$\Rightarrow x = 4.$$

$\therefore$ (3) $\Rightarrow$ $y = \left(\frac{3}{4}\right)(4) = 3$

$\therefore$ f is minimum at $(4, 3)$

$\therefore$ The minimum surface $f = (6)(4)(3) + 4(3)^2$

$$= 108.$$

(8) Verify Eulers Theorem for the function $u = x^2 + y^2 + 2xy$

(Nov/Dec-2019)

2 marks

Soln

Given $u = x^2 + y^2 + 2xy$

$$u(tx, ty) = (tx)^2 + (ty)^2 + 2(tx)(ty)$$

$$= t^2x^2 + t^2y^2 + 2t^2xy.$$

$$= t^2 u(x, y)$$

$\therefore u$ is a homogeneous fnctn of degree 2.

To verify Euler's Theorem we show that

$$x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 2(x^2+y^2+2xy)$$

$$= 2x^2 + 2y^2 + 4xy$$

LHS

$$x(2x+2y) + y(2y+2x) = 2x^2+2y^2+4xy$$

RHS

$$nu = 2x^2+2y^2+4xy.$$

$\therefore$ Euler's Theorem is verified for this function.

(3) Find the Taylor Series expansion of the function $f(x) = \sqrt{1+x^2+y^2}$ in powers of $(x-1)$ and y upto second terms.

[Nov/Dec-2018]

Soln Given $f(x,y) = \sqrt{1+x^2+y^2}$

We have to write Taylor's series in terms of $x-1$ and y.

To expand $f(x,y)$ about the point $(1,0)$

(i.e) $(a,b) = (1,0)$

Taylors series about (a,b)

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$

$f(x,y) = \sqrt{1+x^2+y^2}$ $\qquad (a,b) = (1,0)$

$$f_x = \frac{1}{2\sqrt{1+x^2+y^2}} \times 2x = \frac{x}{\sqrt{1+x^2+y^2}}$$

$$f_y = \frac{1}{2\sqrt{1+x^2+y^2}} \times 2y = \frac{y}{\sqrt{1+x^2+y^2}}.$$

$$f_{xx} = \frac{\sqrt{1+x^2+y^2}\,(1) - x\cdot\frac{1}{2\sqrt{1+x^2+y^2}} \times 2x}{1+x^2+y^2}$$

$$= \frac{1+x^2+y^2-x^2}{(1+x^2+y^2)^{3/2}} = \frac{1+y^2}{(1+x^2+y^2)^{3/2}}.$$

$$f_{xy} = \frac{\sqrt{1+x^2+y^2}\,(0) - x\cdot\frac{1}{2\sqrt{1+x^2+y^2}} \times 2y}{1+x^2+y^2}$$

$$= \frac{-xy}{(1+x^2+y^2)^{3/2}}$$

|||ly

$$f_{yy} = \frac{1+x^2}{(1+x^2+y^2)^{3/2}}$$

At $(1,0)$

$$f(1,0) = \sqrt{2} \quad ; \quad f_x(1,0) = \frac{1}{\sqrt{2}}, \quad f_y(1,0) = 0.$$

$$f_{xx}(1,0) = \frac{1}{2\sqrt{2}} \quad ; \quad f_{yy}(1,0) = \frac{1}{\sqrt{2}}.$$

$$\therefore f(x,y) = \sqrt{1+x^2+y^2} = \sqrt{2} + (x-1)\frac{1}{\sqrt{2}} + \frac{1}{2!}\left[(x-1)^2\frac{1}{2\sqrt{2}} + y^2\cdot\frac{1}{\sqrt{2}}\right]$$

(4) Find the minimum distance from the point (1, 2, 0) to the cone $z^2 = x^2 + y^2$

[Nov/Dec - 2018]

Soln:

Let $P(x, y, z)$ be a point on the cone $z^2 = x^2 + y^2$ and let A be $(1, 2, 0)$

Then $AP = \sqrt{(x-1)^2 + (y-2)^2 + (z-0)^2}$

$= \sqrt{(x-1)^2 + (y-2)^2 + z^2}$

Let $f(x, y, z) = (x-1)^2 + (y-2)^2 + z^2$

A.P is minimum if $f(x, y, z)$ is minimum.

Hence we have to minimize AP subject to $x^2 + y^2 + z^2 = 0$

Let $\phi(x, y, z) = x^2 + y^2 - z^2$.

Let form the auxillary function.

$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

$= (x-1)^2 + (y-2)^2 + z^2 + \lambda(x^2 + y^2 - z^2)$

$F_x = 2(x-1) + 2\lambda z$

$F_y = 2(y-2) + 2\lambda y$

$F_z = 2z - 2\lambda z$

To find the stationary points we solve

$F_x = 0 \qquad F_y = 0 \qquad F_z = 0, \quad \phi = 0$

$F_x = 0 \Rightarrow 2(x-1) + 2\lambda x = 0 \Rightarrow \lambda x = -(x-1)$

$\lambda \cancel{x} = -\frac{(x-1)}{x}$

$F_y = 0 \Rightarrow \cancel{2z - 2\lambda z} \quad 2(y-2) + 2\lambda y = 0 \Rightarrow \lambda = -\frac{(y-2)}{y}$

$F_z = 0 \Rightarrow 2z - 2\lambda z = 0 \Rightarrow \lambda = 1$

$\therefore \frac{(x-1)}{x} = 1 \Rightarrow x = -x+1 \Rightarrow x = \frac{1}{2}.$

And $-\frac{(y-2)}{y} = 1 \Rightarrow \boxed{y=1}$

Subst in $\phi = 0$

$\Rightarrow z^2 = \frac{1}{4} + 1 = \frac{5}{4}$

$\Rightarrow z = \pm\frac{\sqrt{5}}{2}$

$\therefore$ the stationary points are

$P\left(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2}\right), P\left(\frac{1}{2}, 1, -\frac{\sqrt{5}}{2}\right)$

$AP = \sqrt{\left(\frac{1}{2}-1\right)^2 + (1-2)^2 + \frac{5}{4}}$

$= \sqrt{\frac{1}{4} + 1 + \frac{5}{4}} = \sqrt{\frac{6}{4} + 1} = \frac{\sqrt{10}}{2}.$

and $AP' = \frac{\sqrt{10}}{2}.$

$\therefore$ the minimum distance is $\frac{\sqrt{10}}{2}.$

(5) Find the maximum or minimum value of the function $f(x,y) = x^2 + y^2 + 6x + 12.$ [Nov/Dec - 2019]

Soln

The given function is $f = x^2 + y^2 + 6x + 12.$

$f_x = 2x+6, \ f_y = 2y, \ f_{xx} = 2, \ f_{xy} = 0, \ f_{yy} = 2$

To find the stationary points

$f_x = 0 \Rightarrow 2x + 6 = 0 \Rightarrow x = -3$

$f_y = 0 \Rightarrow 2y = 0 \Rightarrow y = 0.$

$\therefore (-3, 0)$ is a stationary point

Now $A = f_{xx} = 2, \quad B = f_{xy} = 0, \quad C = f_{yy} = 2$

$AC - B^2 = 4 > 0$ and $A > 0.$

$\therefore$ the function attain minimum at $(-3, 0)$ and hence the minimum value is

$$f(-3, 0) = (-3)^2 + 0 + 6(-3) + 12 = 3.$$

(b) Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ using Taylor's series upto third degree terms..

Soln Given function $f(x,y) = x^2y^2 + 2x^2y + 3xy^2.$

Taylor series at (a, b)

$$f(x,y) = f(a,b) + \frac{1}{1!}[h f_x(a,b) + k f_y(a,b)]$$
$$+ \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \frac{1}{3!}[h^3 f_{xxx}(a,b) + 3h^2 f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)] + \cdots$$

$$h = x + 2, \quad k = y - 1$$

Function

$f(x,y) = x^2y^2 + 2x^2y + 3xy^2$

$f_x(x,y) = 2x^2 + 4xy + 3y^2$

$f_y(x,y) = 2x^2y + 2x^2 + 6xy$

$f_{xx} = 2y^2 + 4y$

$f_{xy} = 4xy + 4x + 6y$

at $(-2, 1)$

$f(-2, 1) = 6.$

$f_x = -9$

$f_y = 4.$

$f_{xx} = 6$

$f_{xy} = -10$

$f_{yy}(x,y) = 2x^2+6x$ $\qquad$ $f_{yy}(-2,1) = -4.$

$f_{xxx} = 0$ $\qquad$ $f_{xxx} = 0$

$f_{xxy} = 4y+4$ $\qquad$ $f_{xxy} = \cancel{2}\,8$

$f_{xyy} = 4x+6$ $\qquad$ $f_{xyy} = -2$

$f_{yyy} = 0$ $\qquad$ $f_{yyy} = 0$

$$f(x,y) = 6 - 9(x+2) + 4(y-1) + \frac{1}{2!}\Big[6(x+2)^2 - 20(x+2)(y-1) - 4(y-1)^2\Big]$$

$$+ \frac{1}{3!}\Big[24(x+2)^2(y-1) - 6(x+2)(y-1)^2\Big] + \cdots$$

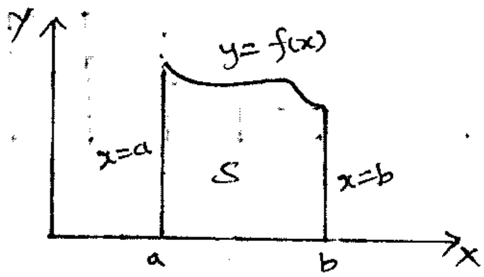
$$= 6 - 9(x+2) + 4(y-1) + \cancel{\tfrac{1}{2}}\Big[3(x+2)^2 - 10(x+2)(y-1) - (y-1)^2$$

$$+ 4(x+2)^2(y-1) - 3(x+2)(y-1)^2 + \cdots$$

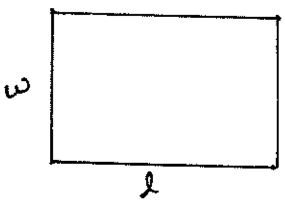
Introduction

In Mathematics, integral assign numbers to functions in a way that can describe displacement, area, volume and open concepts that arise by combining infinitesimal data. Integral calculus plays a vital role in Mathematics, Engineering, Science and Economics.

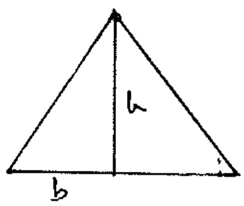
First let us concentrate to solve the area problem. Given a function f which is continuous and non-negative on an interval [a,b], find the areas between the graph of f and the interval [a,b] on the x-axis.



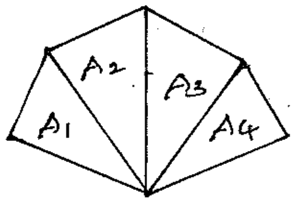
From this diagram, S is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x=a$ and $x=b$ and the x axis.



$A = lw$



$A=\frac{1}{2}bh$



$A = A_1+A_2+A_3+A_4$

For a rectangle, the area is defined as the product of the length and the width. The area of triangle is half of the base times the height. The are of a polygon is found by dividing it into triangles and adding the areas of the triangles. It is not easy to find the area of a region with curved sides. In defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

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We follow a similar idea for areas. We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

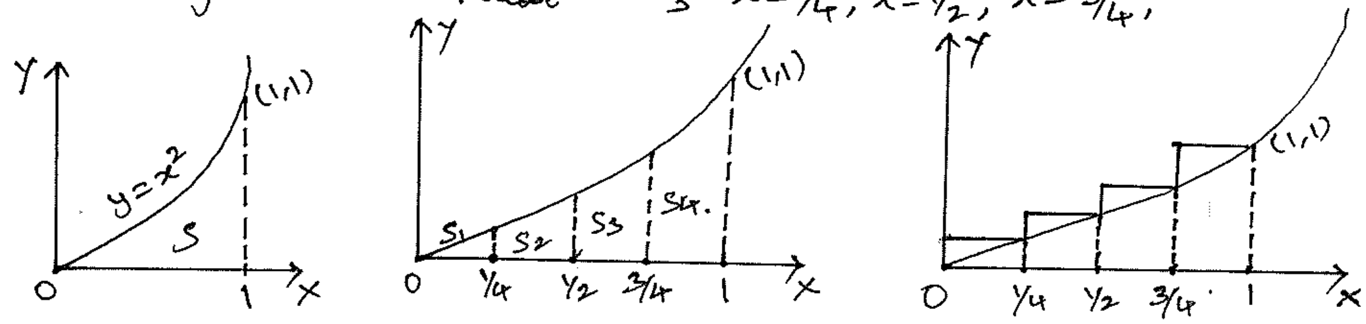
The area problem.

① Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

Solution:

Given that, the area S is between 0 and 1

We divide S into 4 strips S_1, S_2, S_3 and S_4 by drawing the vertical lines $x = 1/4$, $x = 1/2$, $x = 3/4$,



We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip.

The height of these rectangles are the values of the function $f(x) = x^2$ at the right end points of the of the sub-intervals. $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ & $[3/4, 1]$

Each rectangle has width $1/4$ and the height are $(1/4)^2$, $(1/2)^2$, $(3/4)^2$ and 1. If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(3/4\right)^2 + \frac{1}{4}(1)^2 = \frac{15}{32} = 0.46875$$

∴ From the above diagram we see that the area A of S is less than R_4.

$$\therefore A < 0.46875$$

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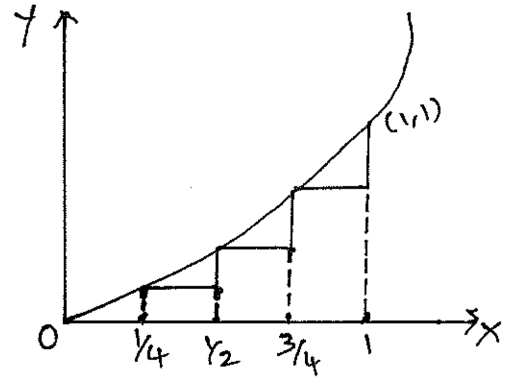
Instead of using the above rectangles, we can use the smaller rectangles from the following graph.



From the graph the heights are the values of f at the left end points of the sub intervals. The sum of the areas of these approximating rectangle is

$$L_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$

$$L_4 = 7/32 = 0.21875$$

We see that the area of S is larger than L_4, so we've lower and upper estimates for A.

$$\therefore 0.21875 < A < 0.46875$$

We should repeat this procedure with a larger number of strips. Now the given region is subdivided into 8 strips of equal width.

$$L_8 = \frac{1}{8}(0)^2 + \frac{1}{8}\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{2}{8}\right)^2 + \frac{1}{8}\left(3/8\right)^2 + 1/8\left(4/8\right)^2 + \frac{1}{8}\left(5/8\right)^2 + \frac{1}{8}\left(6/8\right)^2 + \frac{1}{8}\left(7/8\right)^2$$

$$= \frac{1}{8}\left[\frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4} + \frac{25}{64} + \frac{9}{16} + \frac{49}{64}\right]$$

$$\therefore L_4 = 0.273434$$

$$R_8 = \frac{1}{8}\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{2}{8}\right)^2 + \frac{1}{8}\left(3/8\right)^2 + \frac{1}{8}\left(4/8\right)^2 + \frac{1}{8}\left(5/8\right)^2 + \frac{1}{8}\left(\frac{6}{8}\right)^2 + \frac{1}{8}\left(7/8\right)^2 + \frac{1}{8}(1)^2$$

$$R_8 = 0.3984375$$

$$\therefore 0.2734375 < A < 0.3984375$$

We can obtain better estimates by increasing the no. of strips

n	L_n	R_n
10	0.285000	0.385000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

$\therefore$ A good estimate is obtained by averaging these numbers

$$A = 0.333335$$

② For the region S in $y = x^2$ from 0 to 1, show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$. ie: $\lim_{n\to\infty} R_n = 1/3$.

Solution: Let R_n be the sum of the areas of the n rectangles.

Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, n/n$. ie: the heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \dots, \left(\frac{n}{n}\right)^2$.

Then $$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \dots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n} \cdot \frac{1}{n^2}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right)$$

$$R_n = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{1}{6} n \frac{\left(1+\frac{1}{n}\right)}{n} \cdot n \frac{\left(2+\frac{1}{n}\right)}{n}$$

$$\therefore \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6} \cdot 1 \cdot 2 = 1/3.$$

ie: the sum of the areas of the upper approximating rectangles approaches $1/3$.

Definition:

The area of a region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x\right]$$

We can get the same value for left end points.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \left[f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_{n-1})\Delta x\right]$$

$$\therefore A = \lim_{n\to\infty} \cancel{R_n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

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③ Find the area under the curve $y=x^3$ on the interval $[0,1]$.

Solution:

Dividing $[0,1]$ into n strips of equal length

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

$$x_0 = 0, \quad x_1 = 0+\frac{1}{n} = \frac{1}{n} \quad x_2 = 0+2\cdot\frac{1}{n} = 2/n, \ldots, x_n = 1.$$

If R_n is the right end point approximation using n approximating rectangles, then

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

$$A = \lim_{n\to\infty}\left[\left(\frac{1}{n}\right)^3\frac{1}{n} + \left(\frac{2}{n}\right)^3\frac{1}{n} + \left(\frac{3}{n}\right)^3\frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^3\frac{1}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}(1^3+2^3+\cdots+n^3)$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2$$

$$\therefore A = \frac{1}{4}.$$

Similarly, if L_n is the ~~rig~~ left end point approximation using n approximating rectangles, then.

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

$$= \lim_{n\to\infty}\left[\left(\frac{0}{n}\right)^3\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)^3\left(\frac{1}{n}\right) + \left(\frac{2}{n}\right)^3\left(\frac{1}{n}\right) + \cdots + \left(\frac{n-1}{n}\right)^3\left(\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^4}\left(1^3+2^3+3^3+\cdots+(n-1)^3\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left(\frac{(n-1)n}{2}\right)^2$$

$$= \frac{1}{4}\lim_{n\to\infty} \frac{n^4\left(1-\frac{1}{n}\right)^2}{n^4}$$

$$\therefore A = \frac{1}{4}.$$

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The Definite Integral

The limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{x\to\infty}\left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\right]$$

If f is a function defined for $a \le x \le b$, we devide $[a,b]$ into n sub-intervals of equal width $\Delta x = \frac{b-a}{n}$. Let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the end points of these sub-intervals and let $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ be any sample points in these sub-intervals. So x_i^* lies in the i^{th} sub-interval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i^*)\Delta x.$$

provided that this limit exists and gives the same value for all possible points. If it exists, then f is integrable on $[a,b]$.

Theorem 1

If f is continuous on $[a,b]$ or if f has only a finite number of discontinuities, then f is integrable on (a,b). ie: The definite integral $\int_a^b f(x)\,dx$. exists.

Theorem 2

If f is integrable on $[a,b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Properties of Definite integral

Consider the integral $\int_a^b f(x)dx$.

Let $a<b$. Then $\Delta x = \frac{a-b}{n}$

$\therefore \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

If $a=b$, then $\Delta x=0$, & $\int_a^a f(x)\,dx = 0$

Let us assume that f and g are continuous functions

Then (i) $\int_a^b c\,dx = c(b-a)$, where c is any constant

(ii) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

(iii) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant.

(iv) $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$

(v) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

(vi) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0.$

(vii) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$

(viii) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

---

④ Let A be the area of the region that lies under the graph of $f(x)=e^{-x}$ between $x=0$ and $x=2$. (a) Using right end points, Find an expression for A as a limit. (b) Estimate the area by taking the sample points to be mid-points and using four sub-intervals and then ten sub-intervals.

Solution:

(a) Since $a=0$ and $b=2$, then $\Delta x = \frac{2-0}{n} = 2/n$.

$\therefore x_1 = 2/n,\ x_2 = 4/n,\ x_3 = 6/n, \ldots\ldots,\ x_i = \frac{2i}{n}$ & $x_n = \frac{2n}{n}$

$$\therefore R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}(2/n)$$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left[e^{-2/n} + e^{-4/n} + \cdots + e^{-2n/n}\right]$$

$$A = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} e^{-2i/n}$$

(b) When $n=4$, $\Delta x = 0.5$. The sub intervals are $[0,0.5]$, $[0.5,1.0]$, $[1.0,1.5]$ and $[1.5,2]$.

The mid-points are $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$ and $x_4^* = 1.75$

$$M_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x = f(0.25)\Delta x + f(0.75)\Delta x + f(1.25)\Delta x + f(1.75)\Delta x.$$

$$= e^{-0.25}(0.5) + e^{-0.75}(0.5) + e^{-1.25}(0.5) + e^{-1.75}(0.5)$$

$$= \frac{1}{2}\left(e^{-0.25} + e^{-0.75} + e^{-1.25} + e^{-1.75}\right)$$

$$M_4 = 0.8557$$

When $n=10$ the sub-intervals are $[0,0.2]$, $[0.2,0.4]$, $[0.4,0.6]$, $[0.6,0.8]$, $[0.8,1.0]$, $[1.0,1.2]$, $[1.2,1.4]$, $[1.4,1.6]$, $[1.6,1.8]$, $[1.8,2.0]$

The mid points are 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5, 1.7 and 1.9.

$$A = M_{10} = f(0.1)\Delta x + f(0.3)\Delta x + f(0.5)\Delta x + f(0.7)\Delta x + \cdots + f(1.9)\Delta x$$

$$= 0.2\left[e^{-0.1} + e^{-0.3} + e^{-0.5} + e^{-0.7} + e^{-0.9} + e^{-1.1} + e^{-1.3} + e^{-1.5} + e^{-1.7} + e^{-1.9}\right]$$

$$A = 0.8632$$

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(5) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right end points and $a=0, b=3$ and $n=6$. Also evaluate $\int_0^3 (x^3-6x)\,dx$.

Solution:-

(a) When $n=6$, $\Delta x = \frac{b-a}{n} = 1/2$

$\therefore$ the right end points are $x_1=0.5, x_2=1.0, x_3=1.5, x_4=2.0$, $x_5=2.5$ and $x_6=3.0$.

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$= f(0.5)\Delta x + f(1.0)\Delta x + f(1.5)\Delta x + f(2.0)\Delta x + f(2.5)\Delta x + f(3.0)\Delta x.$$

$$= \frac{1}{2}[-2.875 - 5 - 5.625 - 4 + 0.625 + 9]$$

$$R_6 = -3.9375$$

(b) With n sub-intervals, we've $\Delta x = \frac{b-a}{n} = 3/n$.

$\Rightarrow x_0 = 0,\ x_1 = 3/n,\ x_2 = 6/n,\ x_3 = 9/n$ & in general $x_i = \frac{3i}{n}$

$$\int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(3i/n)\, 3/n.$$

$$= \lim_{n\to\infty} 3/n \sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27$$

$$\left\{\because \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right\}$$

$$\int_0^3 (x^3-6x)\,dx = -6.75$$

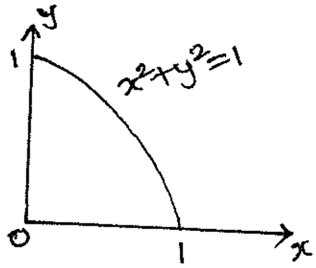
⑥ Evaluate the following integrals by interpreting each in terms of areas. (a) $\int_0^1 \sqrt{1-x^2}\,dx$ (b) $\int_0^3 (x-1)\,dx$

Solution:

(a) Let $f(x)=\sqrt{1-x^2}$. This integral as the area under the curve $y=\sqrt{1-x^2}$, from 0 to 1.

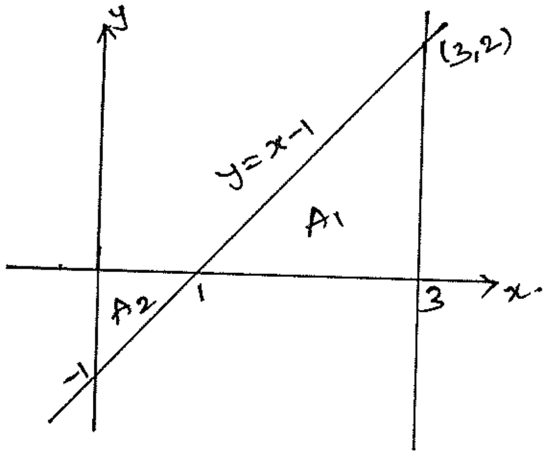
Since $y^2=1-x^2 \Rightarrow x^2+y^2=1$. (ie: a quadratic circle with radius 1.)

$$\therefore \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1) = \pi/4.$$



(b) The graph of $y=1-x$ is the line with slope 1.

We compute the integral as the difference of the areas of the two triangles.



$$\int_0^3 (x-1)\,dx = A_1 - A_2$$

$$= \frac{1}{2}(2\times 2) - \frac{1}{2}(1\times 1)$$

$$\int_0^3 (x-1)\,dx = 1.5$$

⑦ Use mid point rule with $n=5$, to approximate $\int_1^2 \frac{1}{x}\,dx$.

Solution:

Let $a=1, b=2, n=5$

Then $\Delta x = \frac{b-a}{n} = \frac{1}{5}$.

$\therefore$ the end points of the sub-intervals are 1, 1.2, 1.4, 1.6, 1.8 and 2.

Also the mid points are 1.1, 1.3, 1.5, 1.7 & 1.9

$\therefore$ The mid point rule gives

$$\int_1^2 \frac{1}{x}\,dx = \Delta x\,[f(1.1)+f(1.3)+f(1.5)+f(1.7)+f(1.9)]$$

$$= \frac{1}{5}\left[\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right]$$

$$= 0.691908$$

Since $f(x)=\frac{1}{x}>0$ for $1\le x\le 2$ the integral represents an area, and the approximation given by the mid-point rule is the sum of the area of the rectangles.

(8) Prove that (a) $\int_a^b x\,dx = \frac{b^2-a^2}{2}$ & (b) $\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$

Solution:

(a) With n sub-intervals, we have $\Delta x = \frac{b-a}{n}$

$\Rightarrow x_i = a + \frac{(b-a)}{n} i$. To evaluate the integral, we use Riemann sum,

$$\int_a^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\left(a + \frac{(b-a)}{n} i\right)$$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \left[a + \left(\frac{b-a}{n}\right) i\right]$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \left[\sum_{i=1}^{n} a + \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) i\right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right) a \sum_{i=1}^{n} 1 + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[a\left(\frac{b-a}{n}\right) \cdot n + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)\right]$$

$$= ab - a^2 + \frac{1}{2}(a^2 - 2ab + b^2)\cdot 1$$

$$= \frac{1}{2}\left[2ab - 2a^2 + a^2 - 2ab + b^2\right]$$

$$\int_a^b x\,dx = \frac{b^2-a^2}{2}$$

(b) With n sub-intervals, we've $\Delta x = \frac{b-a}{n} \Rightarrow x_i = a + \left(\frac{b-a}{n}\right) i$

$$\therefore \int_a^b x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\left[a + \left(\frac{b-a}{n}\right) i\right]^2$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} \left[a + \left(\frac{b-a}{n}\right) i\right]^2$$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \left[a^2 + 2a\left(\frac{b-a}{n}\right) i + \left(\frac{b-a}{n}\right)^2 i^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{a^2(b-a)}{n} + 2a\left(\frac{b-a}{n}\right)^2 i + \left(\frac{b-a}{n}\right)^3 i^2\right]$$

$$\int_a^b x^2\,dx = \lim_{n\to\infty}\left[\frac{a^2(b-a)}{n}\sum_{i=1}^{n} 1 + 2a\left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^3\sum_{i=1}^{n} i^2\right]$$

$$= a^2(b-a) + 2a(b-a)^2 \lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2}$$

$$+ (b-a)^3 \lim_{n\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3}$$

$$= a^2b - a^3 + ab^2 - 2a^2b - a^3 + \frac{b^3}{3} - ab^2 + a^2b - \frac{a^3}{3}.$$

$$\int_a^b x^2\,dx. = \frac{b^3-a^3}{3}$$

---

(9) Find the Riemann sum for $f(x) = \sin x$, $0 \le x \le \frac{3\pi}{2}$ with six terms, taking the sample points to be right end-points correct to six decimal places. Repeat the problem with mid-points as the sample points.

Solution:

given that $f(x) = \sin x$, $0 \le x \le 3\pi/2$

$\Delta x = \frac{b-a}{2} = \pi/4$. Since we are using right end-points, $x_i^* = x_i$

Now $R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x. = \Delta x\, f(x_1) + \Delta x\, f(x_2) + \cdots + \Delta x\, f(x_6)$

$$= \frac{\pi}{4}\left[f(\pi/4) + f(2\pi/4) + f(3\pi/4) + f(\pi) + f\left(\frac{5\pi}{4}\right) + f\left(\frac{6\pi}{4}\right)\right]$$

$$= \frac{\pi}{4}\left[\sin\pi/4 + \sin\pi/2 + \sin 3\pi/4 + \sin\pi + \sin 5\pi/4 + \sin\left(\frac{3\pi}{2}\right)\right]$$

$$= \frac{\pi}{4}\left[\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} - 1\right]$$

$$= \frac{\pi}{4}\,\frac{1}{\sqrt{2}}$$

$$R_6 = 0.555360$$

Since we are using the mid points $x_i^* = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

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$$M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\Delta x = \Delta x[f(\bar{x}_1)+f(\bar{x}_2)+\cdots+f(\bar{x}_6)]$$

$$= \frac{\pi}{4}\left[f(\pi/8)+f(3\pi/8)+f(5\pi/8)+f(7\pi/8)+f(9\pi/8)+f(11\pi/8)\right]$$

$$= \frac{\pi}{4}\left[\sin\frac{\pi}{8}+\sin\frac{3\pi}{8}+\sin\frac{5\pi}{8}+\sin\frac{7\pi}{8}+\sin\frac{9\pi}{8}+\sin\frac{11\pi}{8}\right]$$

$$= \frac{\pi}{4}[1.306563]$$

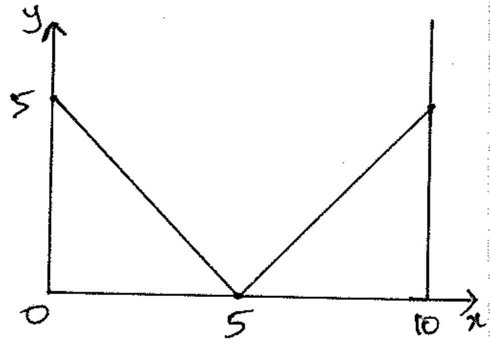
$\therefore M_6 = 1.026172$

(10) Evaluate the integral by interpreting it interms of axes.

$$\int_0^{10} |x-5|\,dx$$

Solution: Let $f(x) = |x-5|$ between 0 & 10.

The value of the integral can be interpreted as the sum of the areas of the two triangles of base length 5 and height 5.



$$\int_0^{10} |x-5|\,dx = 2\cdot\left(\tfrac{1}{2}\right)\cdot 5\cdot 5 = 25.$$

Theorem (1): Fundamental theorem of calculus-part-1.

If f is continuous on $[a,b]$ then the function g is defined by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) = f(x)$.

Theorem (2): Fundamental theorem of calculus-part-2.

If f is continuous on $[a,b]$ then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any anti-derivative of f, (ie) a function such that $F' = f$.

① Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

Sol.

Since $f(t) = \sqrt{1+t^2}$ is continuous, part 1 of the fundamental theorem of calculus gives

$g'(x) = \sqrt{1+x^2}$

② Evaluate the integral $\int_1^3 e^x dx$.

Sol.

Since $f(x) = e^x$ is continuous every where

WKT anti-derivative of $f(x)$ is $F(x) = e^x$.

So part 2 of fundamental theorem gives

$\int_1^3 e^x dx = F(3) - F(1) = e^3 - e^1$.

③ What is wrong with the following calculation.

$\int_{-1}^3 \left(\frac{1}{x^2}\right) dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^3 = -\frac{4}{3}$.

Sol.

Since $f(x) = \frac{1}{x^2} \geq 0$, $\int_a^b f(x)dx \geq 0$ when $f(x) \geq 0$.

Also $\frac{1}{x^2}$ is discontinuous on $[-1,3]$, $f(x) = \frac{1}{x^2}$ has an infinite discontinuity at $x=0$.

$\therefore \int_{-1}^3 \left(\frac{1}{x^2}\right) dx$ does not exists.

④ Find the derivative of $y = \int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt$.

Sol

Let $u = \tan x$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Since $\frac{du}{dx} = \sec^2 x$

Then $\frac{d}{dx}\left\{\int_0^{\tan x} \sqrt{t+\sqrt{t}}\,dt\right\} = \frac{d}{dx}\left\{\int_0^u \sqrt{t+\sqrt{t}}\,dt\right\}$

$= \frac{d}{du}\left\{\int_0^u \sqrt{t+\sqrt{t}}\,dt\right\}\frac{du}{dx} = \sqrt{u+\sqrt{u}}\,\sec^2 x$.

$= \sqrt{\tan x + \sqrt{\tan x}}\,\sec^2 x$.

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Indefinite Integral

In calculus, an indefinite integral of a function $f(x)$ is a differentiable function F whose derivative is equal to the original function $f(x)$. i.e., $F' = f(x)$.

Remark:

From the definite and indefinite integrals, we note that a definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)\,dx$ is a function. The relation between these two integrals is given by part 2 of the fundamental theorem.

Formulae.

1. $\int c f(x)\,dx = c\int f(x)\,dx$, c is constant.
2. $\int k\,dx = kx + c$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
4. $\int \frac{1}{x}\,dx = \log|x| + c$
5. $\int e^x\,dx = e^x + c$
6. $\int a^x\,dx = a^x \frac{1}{\log a} + c$
7. $\int \sin x\,dx = -\cos x + c$
8. $\int \cos x\,dx = \sin x + c$
9. $\int \sec^2 x\,dx = \tan x + c$
10. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
11. $\int \sec x \tan x\,dx = \sec x + c$.
12. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$
13. $\int \frac{dx}{x^2+1} = \tan^{-1} x \text{ (or) } -\cot^{-1}(x) + c$
14. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c \text{ or } -\cos^{-1}(x) + c$
15. $\int \sinh x\,dx = \cosh x + c$
16. $\int \cosh x\,dx = \sinh x + c$
17. $\int \operatorname{sech}^2 x\,dx = \tanh x + c$
18. $\int \operatorname{cosech}^2 x\,dx = -\coth x + c$
19. $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + c$
20. $\int \operatorname{cosech} x \coth x\,dx = -\operatorname{cosech} x + c$
21. $\int \frac{dx}{\sqrt{x(x^2-1)}} = \sec^{-1}(x) + c \text{ or } \operatorname{cosec}^{-1}(x) + c$
22. $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x) + c$
23. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c$
24. $\int \frac{dx}{x^2-1} = \tanh^{-1} x + c \text{ (or) } \coth^{-1}(x) + c$.

DEPARTMENT OF SCIENCE AND HUMANITIES

① Evaluate $\int(10x^4-2\sec^2x)\,dx$.

Sol Let $I=\int(10x^4-2\sec^2x)\,dx$.

$$=10\frac{x^5}{5}-2\tan x+c.$$

$$\therefore I=2x^5-2\tan x+c.$$

② $\int\frac{\cos\theta}{\sin^2\theta}\,d\theta=?$

Sol: Let $I=\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$.

$$I=\int\frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\,d\theta.$$

$$I=\int\cot\theta\cdot\operatorname{cosec}\theta\,d\theta$$

$$\therefore I=-\operatorname{cosec}\theta+c.$$

③ Evaluate $\int\left[\sqrt{x^3}+\sqrt[3]{x^2}\right]dx$

Sol. Let $I=\int\left[\sqrt{x^3}+\sqrt[3]{x^2}\right]dx$.

Then $I=\int(x^{3/2}+x^{2/3})\,dx$.

$$I=\frac{x^{5/2}}{5/2}+\frac{x^{5/3}}{5/3}+c.$$

$$=\frac{1}{5}\left[2x^{5/2}+3x^{5/3}\right]+c.$$

④ $\int\frac{\sin2x}{\sin x}\,dx.=?$

Sol

$$\int\frac{\sin2x}{\sin x}\,dx=\int\frac{2\sin x\cos x}{\sin x}\,dx.$$

$$=\int2\cos x\,dx.$$

$$=2\sin x+c.$$

⑤ $\int\left[(x^2+1)+\frac{1}{x^2+1}\right]dx=?$

Sol $\int\left[x^2+1+\frac{1}{x^2+1}\right]dx.$

$$=\frac{x^3}{3}+x+\tan^{-1}x+c.$$

⑥ Evaluate $\int_1^2\left(\frac{1}{x^2}-\frac{4}{x^3}\right)dx$.

Sol

$$I=\int_1^2\left(\frac{1}{x^2}-\frac{4}{x^3}\right)dx=\int_1^2(x^{-2}-4x^{-3})\,dx$$

$$I=\left[\frac{x^{-2+1}}{-2+1}-4\frac{x^{-3+1}}{-3+1}\right]_1^2$$

$$\therefore I=\left[\frac{x^{-1}}{-1}+\frac{4}{3}x^{-2}\right]_1^2=-\frac{1}{2}+\frac{1}{2}1-2$$

$$\therefore I=-1.$$

⑦ Evaluate $\int_0^1(5x-5^x)\,dx$.

Sol $I=\int_0^1(5x-5^x)\,dx$.

$$I=5\left[\frac{x^2}{2}\right]_0^1-\left[\frac{5^x}{\log5}\right]_0^1$$

$$=\frac{5}{2}[1-0]-\frac{1}{\log5}[5^1-5^0]$$

$$I=\frac{5}{2}-\frac{4}{\log5}$$

⑧ Evaluate $\int_{\pi/4}^{\pi/3}\operatorname{cosec}^2\theta\,d\theta$.

Sol

$$\int_{\pi/4}^{\pi/3}\operatorname{cosec}^2\theta\,d\theta=\left[-\cot\theta\right]_{\pi/4}^{\pi/3}$$

$$=-\left[\cot\pi/3-\cot\pi/4\right]$$

$$=\cot\pi/4-\cot\pi/3.$$

① Find $\int \frac{dx}{\sin^2 x \cos^2 x}$

Sol

$$\int \frac{1}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x * \cos^2 x} dx.$$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx.$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx.$$

$$= \tan x - \cot x + c.$$

② $\int \sqrt{1+\sin 2x} \, dx.$

Sol $\int \sqrt{1+\sin 2x} \, dx.$

$$= \int \sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x} \, dx$$

$$= \int \sqrt{(\cos x + \sin x)^2} \, dx$$

$$= \int (\cos x + \sin x) \, dx.$$

$$= \sin x - \cos x + c.$$

③ $\int \frac{1}{1-\cos x} dx.$

Sol

$$\int \frac{1}{1-\cos x} dx$$

$$= \int \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} dx$$

$$= \int \frac{1+\cos x}{1^2 - \cos^2 x} dx.$$

$$= \int \frac{1+\cos x}{(1-\cos x)(1+\cos x)} dx.$$

$$= \int \frac{1+\cos x}{\sin^2 x} dx.$$

$$= \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx.$$

$$= \int \operatorname{cosec}^2 x \, dx + \int \cot x \operatorname{cosec} x \, dx$$

$$= -\cot x - \operatorname{cosec} x + c.$$

Properties of Definite Integrals

We assume that f and g are continuous functions

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

3. $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ $\quad a<b<c.$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

6. (i) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx.$ iff $f(2a-x) = f(x)$

(ii) $\int_0^{2a} f(x)\,dx = 0$ iff $f(2a-x) = -f(x)$

7 (i) If $f(x)$ is an even fun.. Then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

(ii) If $f(x)$ is an odd fun... then $\int_{-a}^{a} f(x)\,dx = 0.$

CHENNAI INSTITUTE OF TECHNOLOGY DEPARTMENT OF SCIENCE AND HUMANITIES

① Evaluate $\int_{-1}^{1}(2-|x|)\,dx$.

Sol Let $I=\int_{-1}^{1}(2-|x|)\,dx$

Then $I=\int_{-1}^{1}2\,dx-\int_{-1}^{1}|x|\,dx$.

$=4-2\int_{0}^{1}x\,dx$ {∵ $|x|$ is even}

$=4-2\left[\frac{x^2}{2}\right]_0^1 \Rightarrow I=3$

② $\int_0^{\pi/2}\frac{1}{1+\tan x}\,dx$.

Let $I=\int_0^{\pi/2}\frac{1}{1+\tan x}\,dx$.

$I=\int_0^{\pi/2}\frac{\cos x}{\sin x+\cos x}\,dx$ —①

{∵ $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$}

∴ $I=\int_0^{\pi/2}\frac{\cos(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}\,dx$

$\Rightarrow I=\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx$ —②

①+② ⇒

$2I=\int_0^{\pi/2}\frac{\sin x+\cos x}{\sin x+\cos x}\,dx$.

$2I=\int_0^{\pi/2}dx \Rightarrow 2I=[x]_0^{\pi/2}$

∴ $I=\pi/4$.

③ $\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$.

Sol Let $I=\int_0^{\pi}\frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$ —①

Then we've $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$.

∴ $I=\int_0^{\pi}\frac{e^{\cos(\pi-x)}}{e^{\cos(\pi-x)}+e^{-\cos(\pi-x)}}\,dx$

$I=\int_0^{\pi}\frac{e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}\,dx$ —②

①+② ⇒ $2I=\int_0^{\pi}\frac{e^{\cos x}+e^{-\cos x}}{e^{-\cos x}+e^{\cos x}}\,dx$

$\Rightarrow 2I=\int_0^{\pi}dx \Rightarrow 2I=[x]_0^{\pi}=\pi$

$\Rightarrow I=\pi/2$

④ $\int_{-2}^{2}|x+1|\,dx$.

Sol Since $|x+1|=\begin{cases}-(x+1), & -2<x<-1\\ (x+1), & -1<x<2\end{cases}$

∴ $I=\int_{-1}^{1}|x+1|\,dx=-\int_{-2}^{-1}(x+1)\,dx+\int_{-1}^{2}(x+1)\,d$

∴ $I=-\left[\frac{x^2}{2}+x\right]_{-2}^{-1}+\left[\frac{x^2}{2}+x\right]_{-1}^{2}$

$=-\left[\frac{-1}{2}-0\right]+\left[4-\left(\frac{-1}{2}\right)\right]$

$I=5$

(5) Evaluate $\int_{-\pi/2}^{\pi/2} \sin^{199}x \, dx$.

Sol

Let $I = \int_{-\pi/2}^{\pi/2} \sin^{199}x \, dx$

Let $f(x) = \sin^{199}x$

Then $f(-x) = \sin^{199}(-x)$

$= -\sin^{199}x$.

$= -f(x)$

$\therefore f(x)$ is an ~~even~~ odd fun.

$\therefore \int_{-\pi/2}^{\pi/2} \sin^{199}x \, dx = 0$.

$\left\{ \because \int_{-a}^{a} f(x)\,dx = 0, \text{ where } f(x) \text{ is odd} \right\}$

Integration by Substitution rule.

Type I (A).

$\int [f(x)]^n f'(x)\,dx$ (or)

$\int \phi[f(x)] f'(x)\,dx$. —(1)

Substitute $u = f(x)$,

$du = f'(x)dx$

(1) $\Rightarrow \int u^n \, du$ or $\int \phi(u)\,du$ and then proceed.

(1) Solve $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$.

Sol

Let $I = \int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

$\left\{ \because \int f'(x)\, \phi[f(x)]\,dx \right\}$

$\therefore u = 2 - 1/x \qquad du = \frac{1}{x^2}dx$

$\therefore I = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + c$

Hence $I = \frac{2}{3}\left(2-\frac{1}{x}\right)^{3/2} + c$.

(2) Evaluate $\int x^2/\sqrt{x+5}\,dx$.

Sol let $u = \sqrt{x+5}$

$du = \frac{1}{2\sqrt{x+5}}dx \Rightarrow 2du = \frac{dx}{\sqrt{x+5}}$.

Now $u^2 = x+5 \Rightarrow x^2 = (u^2-5)^2$

$\therefore x^2 = u^4 - 10u^2 + 25$

$\therefore I = \int (u^4 - 10u^2 + 25)\, 2du$

$= 2\left(\frac{u^5}{5} - \frac{10u^3}{3} + 25u\right) + c$

$= \frac{2}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + 50(x+5)^{1/2} + c$

(3) Evaluate $\int \frac{1}{(3x-4)^{3/2}}\,dx$.

Sol Let $I = \int \frac{1}{(3x-4)^{3/2}}\,dx$.

Let $u = 3x-4 \qquad du = 3dx$.

$\Rightarrow \frac{du}{3} = dx$.

$\therefore I = \int \frac{du/3}{u^{3/2}} = \frac{-2}{3}\frac{1}{\sqrt{u}} + c$

$\Rightarrow I = \frac{-2}{3}\frac{1}{\sqrt{3x-4}} + c$.

Type 1 (B) [logarithmic fun.]

① Evaluate $\int_1^e \frac{\log x}{x}\,dx$.

Sol

Let $I = \int_1^e \frac{\log x}{x}\,dx$.

Let $u = \log x \quad du = \frac{dx}{x}$.

x	1	e
u	0	1

$\therefore I = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

② Evaluate $\int \frac{\sec^2(\log x)}{x}\,dx$.

Sol

Let $I = \int \frac{\sec^2(\log x)}{x}\,dx$.

Let $u = \log x \quad du = \frac{dx}{x}$.

$\therefore I = \int \sec^2 u\,du = \tan u + c$

$\therefore I = \tan(\log x) + c$.

Type 1 (C) [Exponential funs.]

① Evaluate $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$.

Sol

Let $I = \int_1^2 \frac{e^{1/x}}{x^2}\,dx$.

Let $u = e^{1/x}; \; du = e^{1/x}\left(\frac{-1}{x^2}\right)dx$

$\Rightarrow -du = \frac{e^{1/x}}{x^2}\,dx$

x	1	2
u	e	$e^{1/2}$

$\therefore I = \int_e^{e^{1/2}} (-du) = -[u]_e^{e^{1/2}} = e - \sqrt{e}$

② Evaluate $\int e^{\cos x} \sin x\,dx$.

Sol

Let $I = \int e^{\cos x} \sin x\,dx$

$u = e^{\cos x} \quad du = e^{\cos x}(-\sin x)dx$

$\therefore I = \int (-du) = -u + c = -e^{\cos x} + c$.

③ Evaluate $\int (\log a)^x\,dx$.

Sol Let $I = \int (\log a)^x\,dx$.

Then

$I = \int e^{\log[\log a]^x}\,dx$.

$= \int e^{x[\log(\log a)]}\,dx$.

$= \frac{e^{[\log(\log a)]x}}{\log(\log a)} + c$.

$\therefore I = \frac{(\log a)^x}{\log(\log a)} + c$.

Type 1 (D) [Trigonometric funs.]

① Evaluate $\int_0^{\pi/2} \cos x \sin(\sin x)\,dx$

Sol

Let $I = \int_0^{\pi/2} \cos x \sin(\sin x)\,dx$

put $u = \sin x \quad du = \cos x\,dx$

x	0	$\pi/2$
u	0	1

$\therefore I = \int_0^1 \sin u\,du = [-\cos u]_0^1$

$\therefore I = 1 - \cos 1$

CHENNAI INSTITUTE OF TECHNOLOGY DEPARTMENT OF SCIENCE AND HUMANITIES

② $\int \sec x\,dx = ?$

Sol Let $I = \int \sec x\,dx$

~~Let~~ $I = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\,dx$

$\therefore I = \int \frac{1}{u}\,du = \log u + c.$

$\Rightarrow I = \log(\sec x + \tan x) + c.$

③ Evaluate $\int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right]dx$

Let $I = \int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right]dx$

Let $u = \tan^{-1}x \quad du = \frac{dx}{1+x^2}$

$\Rightarrow \tan u = x$

$1+x+x^2 = \tan u + \sec^2 u$

$\therefore I = \int e^u(\tan u + \sec^2 u)\,du$

put $t = e^u \tan u$

$dt = [e^u \sec^2 u + \tan u\, e^u]\,du$

$\therefore I = \int dt = t + c$

$\Rightarrow I = x\,e^{\tan^{-1}x} + c.$

TECHNIQUES OF INTEGRATION

Integration by parts method

$\int u\,dv = uv - \int v\,du.$

① Solve $\int_0^1 \frac{y}{e^{2y}}\,dy$

Let $I = \int_0^1 y\,e^{-2y}\,dy$

Let $u = y \qquad dv = e^{-2y}\,dy$

$du = dy \qquad v = e^{-2y}/-2$

$\therefore I = y\left(\frac{e^{-2y}}{-2}\right)\Big|_0^1 + \int_0^1 \frac{e^{-2y}}{2}\,dy$

$= \frac{-1}{2}e^{-2} + \frac{1}{2}\left[\frac{e^{-2y}}{-2}\right]_0^1$

$= \frac{-1}{2}e^{-2} - \frac{1}{4}[e^{-2} - 1]$

$\Rightarrow I = \frac{1}{4} - \frac{3}{4}e^{-2}$

② Evaluate $\int \left(\frac{\log x}{x}\right)^2 dx.$

Sol

Let $I = \int (\log x)^2/x^2\,dx.$

~~Let~~ $u = (\log x)^2 \qquad dv = \frac{1}{x^2}dx.$

$du = 2\log x\,(1/x)\,dx \qquad v = \frac{-1}{x}.$

$\therefore I = \frac{-1}{x}(\log x)^2 + 2\int(\log x)\left(\frac{1}{x^2}\right)dx$ — ①

Consider $\int(\log x)\left(\frac{1}{x^2}\right)dx$

Let $u = \log x \qquad dv = dx/x^2$

$du = dx/x \qquad v = -1/x.$

$\therefore \int(\log x)\left(\frac{1}{x^2}\right)dx$

$= \frac{-1}{x}\log x + \int \frac{1}{x^2}\,dx$

$= \frac{-1}{x}\log x - \frac{1}{x} + c.$

$\therefore I = \frac{-1}{x}(\log x)^2 - 2\frac{1}{x}\log x - 2\left(\frac{1}{x}\right) + 2c$

DEPARTMENT OF SCIENCE AND HUMANITIES

③ Evaluate $\int \frac{x}{1+\sin x}\,dx$.

Sol

Let $I = \int \frac{x}{1+\sin x}\,dx$

Then $I = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$.

$= \int \frac{(x - x\sin x)}{\cos^2 x}\,dx$

$I = \int (x\sec^2 x - x\sec x\tan x)\,dx$ — ①

Take $\int x\sec^2 x\,dx$.

Let $u = x \qquad dv = \sec^2 x\,dx$.

$du = dx \qquad v = \int \sec^2 x\,dx$

$\therefore \int x\sec^2 x\,dx$

$= x\tan x - \int \tan x\,dx$

$= x\tan x - \log(\sec x)$

Take $\int x\sec x\tan x\,dx$.

$u = x, \qquad dv = \sec x\tan x\,dx$

$du = dx \qquad v = \sec x$

$\therefore \int x\sec x\tan x\,dx$

$= x\sec x - \log(\sec x + \tan x)$

$\therefore I = \int \frac{x}{1+\sin x}\,dx$

$= x\tan x - \log(\sec x)$

$- x\sec x + \log(\sec x + \tan x) + c$

④ Evaluate $\int_0^1 \tan^{-1}x\,dx$

Let $I = \int_0^1 \tan^{-1}x\,dx$.

$u = \tan^{-1}x \qquad dv = dx$.

$du = dx/1+x^2 \qquad v = x$.

$\therefore I = x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2}$

$= \frac{\pi}{4} - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\log 2$.

⑤ Evaluate $\int_0^{1/2} \cos^{-1}x\,dx$.

Sol

Let $I = \int_0^{1/2} \cos^{-1}x\,dx$.

$u = \cos^{-1}x \qquad dv = dx$:

$du = \frac{-dx}{\sqrt{1-x^2}} \qquad v = x$.

$\therefore I = x\cos^{-1}(x)\Big|_0^{1/2} + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$

$= \frac{1}{2}\left(\frac{\pi}{3}\right) + \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}}$ — ①

Let $t = 1-x^2 \qquad dt = -2x\,dx$

$\Rightarrow -\frac{dt}{2} = x\,dx$.

x	0	1/2
t	1	3/4

$\therefore \int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} = \int_1^{3/4} \frac{-dt/2}{\sqrt{t}}$

$= \frac{-1}{2}\left[2\sqrt{t}\right]_0^{3/4} = \frac{-\sqrt{3}}{2} + 1$

$\therefore I = \frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$

⑥ Evaluate $\int e^{ax}\cos bx\,dx$.

Sol

Let $I = \int e^{ax}\cos bx\,dx$

$u = \cos bx \qquad dv = e^{ax}\,dx$

$du = -\sin bx\,(b) \qquad v = \frac{e^{ax}}{a}$

$\therefore I = \frac{e^{ax}}{a}\cos bx + \int \frac{e^{ax}}{a}\sin bx\,(b)\,dx$

$= \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx \quad \text{--- (1)}$

Take $\int e^{ax}\sin bx\,dx$

$u = \sin bx \qquad dv = e^{ax}\,dx$

$du = b\cos bx\,dx \qquad v = \frac{e^{ax}}{a}$

$\therefore \int e^{ax}\sin bx\,dx$

$= \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a}\,b\cos bx\,dx$

$= \frac{e^{ax}}{a}\sin bx - \frac{b}{a} I$

$\therefore (1) \Rightarrow I = \frac{e^{ax}}{a}\cos bx \cancel{+\frac{b}{a}\left(\frac{b}{a}\right)I}$

$+\frac{b}{a}\frac{e^{ax}}{a}\sin bx - \frac{b^2}{a^2} I$

$\Rightarrow I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a}\cos bx + b\frac{e^{ax}}{a^2}\sin bx.$

$\Rightarrow I\left(\frac{a^2+b^2}{a^2}\right) = \frac{e^{ax}}{a^2}\left[a\cos bx + b\sin bx\right]$

$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + c.$

⑦ Evaluate $\int e^{ax}\sin bx\,dx$.

⑧ Evaluate $\int e^{-ax}\cos bx\,dx$.

⑨ Evaluate $\int e^{-ax}\sin bx\,dx$.

⑩ Find the reduction formula for $\int \sin^n x\,dx$ and also find $\int_0^{\pi/2} \sin^n x\,dx$, $\int_0^{\pi/2} \sin^5 x\,dx$

Sol Let $I_n = \int \sin^n x\,dx$.

Then $I_n = \int \sin^{n-1}x\,\sin x\,dx$.

$u = \sin^{n-1}x; \qquad dv = \sin x\,dx$

$du = (n-1)\sin^{n-2}x\cos x\,dx$

$v = -\cos x$

$\therefore I_n = -\cos x\sin^{n-1}x + (n-1)\int \cos x(\sin^{n-2}x\cos x\,dx)$

$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x\cos^2 x\,dx$

$= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2}x(1-\sin^2 x)\,dx.$

$= -\cos x\sin^{n-1}x. + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$

$\therefore I_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$.

$\Rightarrow I_n = \frac{-1}{n} \cos x \sin^{n-1}x + \frac{n-1}{n} I_{n-2}$.

Now $I_n = \int_0^{\pi/2} \sin^n x\, dx$.

$= \frac{-1}{n} \cos x \sin^{n-1}x \Big|_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$.

$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$.

$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$.

In general,

$$\int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} I_0 \text{ (n is even)} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} I_1 \text{ (n is odd)} \end{cases} \quad (1)$$

Since $I_n = \int_0^{\pi/2} \sin^n x\, dx$.

$I_0 = \int_0^{\pi/2} dx = \pi/2$

$I_1 = \int_0^{\pi/2} \sin x\, dx = -\cos x \Big|_0^{\pi/2}$

$\therefore I_1 = \cancel{\pi/2}\ 1$

$\therefore (1) \Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx$.

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, \text{ n is even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, \text{ n is odd.} \end{cases}$$

Also $\int_0^{\pi/2} \sin^5 x\, dx$.

$= \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = 8/15$

(11) Reduction formula for $\int \cos^n x\, dx$ and hence find $\int_0^{\pi/2} \cos^n x\, dx$, $\int_0^{\pi/2} \cos^9 x\, dx$

Sol Let $I_n = \int \cos^n x\, dx$

Then $I_n = \int \cos^{n-1}x \cos x\, dx$.

$u = \cos^{n-1}x \qquad dv = \cos x\, dx$

$du = (n-1)\cos^{n-2}x\,(-\sin x)\,dx$

$v = \sin x$.

$\therefore I_n = \sin x \cos^{n-1}x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx$.

$= \sin x \cos^{n-1}x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\, dx$.

$= \sin x \cos^{n-1}x + (n-1)\int (\cos^{n-2}x\, \cancel{dx} + \cos^n x\, \cancel{dx})dx$

$= \sin x \cos^{n-1}x + (n-1) I_{n-2} + (n-1) I_n$

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$\Rightarrow nI_n = \sin x \cos^{n-1}x + (n-1) I_{n-2}$

$\therefore I_n = \frac{1}{n} \sin x \cos^{n-1}x + \frac{n-1}{n} I_{n-2}$

Now $I_n = \int_0^{\pi/2} \cos^n x \, dx$.

$= \frac{1}{n} \sin x \cos^{n-1}x \Big|_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$; $I_{n-2} = \frac{n-3}{n-2} I_{n-4}$

In general.

$$I_n = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{1}{2} I_0 & (n \text{ is even}) \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3} I_1 & (n \text{ is odd}) \end{cases} \quad \text{— (c)}$$

$I_0 = \int_0^{\pi/2} \cos^0 x \, dx = \pi/2$

$I_1 = \int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2}$

$\therefore I_1 = 1$

$$\text{(c)} \Rightarrow I_n = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots \frac{2}{3} \cdot 1, & n \text{ is odd} \end{cases}$$

also $\int_0^{\pi/2} \cos^9 x \, dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

$\therefore \int_0^{\pi/2} \cos^9 x \, dx = \frac{128}{315}$

(12) Evaluate $\int \sec^n x \, dx$

Let $I_n = \int \sec^n x \, dx$. — (1)

Then $I_n = \int \sec^{n-2}x \sec^2 x \, dx$.

$\therefore I_n = \int \sec^{n-2}x \; d(\tan x)$.

$u = \sec^{n-2}x \qquad dv = d(\tan x)$

$du = (n-2) \sec^{n-3}x (\sec x \tan x) dx$

$\& \; V = \tan x$.

$\therefore I_n = \tan x \sec^{n-2}x - \int (n-2) \sec^{n-2}x \tan^2 x \, dx$.

$I_n = \tan x \sec^{n-2}x - (n-2) \int \sec^{n-2}x (-1+\sec^2 x) dx$

$I_n = \tan x \sec^{n-2}x - (n-2)\left[\int \sec^n x \, dx - \int \sec^{n-2}x \, dx\right]$

$I_n = \tan x \sec^{n-2}x - (n-2) I_n + (n-2) I_{n-2}$.

$\Rightarrow I_n + (n-2) I_n = \tan x \sec^{n-2}x + (n-2) I_{n-2}$

$\Rightarrow (n-1) I_n = \tan x \sec^{n-2}x + (n-2) I_{n-2}$

$\Rightarrow I_n = \frac{1}{n-1} \tan x \sec^{n-2}x + \frac{n-2}{n-1} I_{n-2}$, $n \geq 2$.

$I_0 = \int dx = x + c$ (from (1))

$I_1 = \int \sec x \, dx = \log(\sec x + \tan x) + c$

(13) Evaluate $\int (\log x)^n dx$. also find $\int (\log x)^2 dx$.

Sol Let $I_n = \int (\log x)^n dx$.

$u = (\log x)^n \qquad dv = dx$.

$du = n(\log x)^{n-1} \frac{1}{x} dx \qquad v = x$.

$\therefore I_n = x(\log x)^n - \int x\, n(\log x)^{n-1} \frac{1}{x} dx$

$= x(\log x)^n - n\int (\log x)^{n-1} dx$.

$\therefore I_n = x(\log x)^n - n I_{n-1}$

$I_0 = \int dx = x + c$

To find $\int (\log x)^2 dx$, put $n=2$ in I_n.

$\therefore I_2 = \int (\log x)^2 dx$

$= x(\log x)^2 - 2I_1$

$= x(\log x)^2 - 2I_1 \qquad \text{(c)}$

But $I_1 = x\log x - \cancel{1} I_0$

ie: $I_1 = x\log x - \cancel{1}x \cancel{+c}$

$\therefore \text{(c)} \Rightarrow I_2 = x(\log x)^2 - 2(x\log x - x) + c$

$I_2 = x(\log x)^2 - 2x\log x + 2x + c$.

(14) Find the reduction formula for $\int x^n \sin mx\, dx$.

Sol Let $I_n = \int x^n \sin mx\, dx$.

$= \int x^n d\left(-\frac{\cos mx}{m}\right)$

$= -x^n \frac{\cos mx}{m} + \int \frac{\cos mx}{m} d(x^n)$

$= -x^n \frac{\cos mx}{m} + \frac{1}{m}\int \cos mx\,(n x^{n-1}) dx$.

$= -x^n \frac{\cos mx}{m} + \frac{n}{m}\int x^{n-1} d\left(\frac{\sin mx}{m}\right)$

$= -x^n \frac{\cos mx}{m} + \frac{n}{m}\left[x^{n-1}\frac{\sin mx}{m} - \int \frac{\sin mx}{m}(n-1)x^{n-2} dx\right]$

$= -x^n \frac{\cos mx}{m} + \frac{n}{m^2} x^{n-1}\sin mx - \frac{n(n-1)}{m^2}\int x^{n-2}\sin mx\, dx$.

$\therefore I_n = -x^n \frac{\cos mx}{m} + \frac{n}{m^2} x^{n-1}\sin mx - \frac{(n-1)n}{m^2} I_{n-2}$.

(14) Evaluate $\int x^n e^x dx$

Let $u = x^n \qquad dv = e^x dx$

$du = nx^{n-1}dx \qquad v = e^x$

$\therefore \int x^n e^x dx = x^n e^x - \int e^x n x^{n-1} dx$

$\therefore I_n = x^n e^x - n I_{n-1}$

Problems based on trigonometric integrals

① Evaluate $\int \sin^6 x \cos^3 x\, dx$

Sol: Let $I = \int \sin^6 x \cos^3 x\, dx$.

Then $I = \int \sin^6 x \cos^2 x \cos x\, dx$.

$= \int \sin^6 x (1-\sin^2 x) \cos x\, dx$

put $u = \sin x$ $\quad du = \cos x\, dx$

$I = \int u^6(1-u^2)\, du = \int (u^6 - u^8)\, du$

$\therefore I = \frac{u^7}{7} - \frac{u^9}{9} + c$.

$\therefore I = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + c$

② Solve $I = \int_0^{\pi/3} \tan^5 x \sec^4 x\, dx$

Sol:

$I = \int_0^{\pi/3} \tan^4 x \sec^3 x (\sec x \tan x)\, dx$

$= \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x (\sec x \tan x)\, dx$.

let $u = \sec x$ $\quad du = \sec x \tan x\, dx$

x	0	$\pi/3$
u	1	2

$\therefore I = \int_1^2 (u^2-1)^2 u^3\, du$

$= \int_1^2 (u^4 - 2u^2 + 1) u^3\, du$

$= \int_1^2 (u^3 - 2u^5 + u^7)\, du$.

$= \left[\frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8}\right]_1^2 = \frac{117}{8}$.

Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$	$1-\sin^2\theta = \cos^2\theta$.
$\sqrt{a^2+x^2}$	$x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$	$1+\tan^2\theta = \sec^2\theta$.
$\sqrt{x^2-a^2}$	$x = a\sec\theta$, $0 \le \theta < \pi/2$, $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2 x$.

① Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\, dx$.

Sol

$I = \int \frac{\sqrt{9-x^2}}{x^2}\, dx$.

put $x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \pi/2$

$dx = 3\cos\theta\, d\theta$

$\therefore I = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} (3\cos x)\, d\theta$.

$= \int \frac{3\sqrt{1-\sin^2\theta}}{9\sin^2\theta} (3\cos\theta)\, d\theta$.

$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int (\operatorname{cosec}^2\theta - 1)\, d\theta$

$= -\cot\theta - \theta + c$.

$\therefore I = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$.

② Evaluate $I = \int \frac{1}{\sqrt{a^2-x^2}}$

$x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \pi/2$

$dx = a\cos\theta\, d\theta$.

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$$I = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \, a\cos\theta \, d\theta$$

$$= \int \frac{1}{a\cos\theta} \, a\cos\theta \, d\theta$$

$$\therefore I = \int d\theta = \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

③ Let $I = \int \frac{1}{x^2\sqrt{x^2-1}} \, dx, \; x > 1$

put $x = \sec\theta$

$dx = \sec\theta \tan\theta$

$$\therefore I = \int \frac{1}{\sec^2\theta \tan\theta} \sec\theta \tan\theta \, d\theta$$

$$= \int \cos\theta \, d\theta$$

$$= \sin\theta + c$$

$$= \frac{\sqrt{x^2-1}}{x} + c$$

Partial Fraction method.

① Solve $\int \frac{dx}{(x+1)(x+2)}$

Sol

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$\Rightarrow A(x+2) + B(x+1) = 1$

put $x = -2$, $-B = 1 \Rightarrow \boxed{B = -1}$

put $x = -1$, $\boxed{A = 1}$

$$\therefore \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log\left(\frac{x+1}{x+2}\right) + c$$

② Solve $\int \frac{x^2+1}{(x-3)(x-2)^2} \, dx$

Sol

$$\frac{x^2+1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \quad (1)$$

$A(x-2)^2 + B(x-2)(x-3) + C(x-3) = x^2+1.$

put $x = 3$, $\boxed{A = 10}$

put $x = 2$, $\boxed{C = -5}$

equating coefft of x^2 on both sides. $1 = A + B$

$\therefore \boxed{B = -9}$

$$\therefore (1) \Rightarrow \frac{x^2+1}{(x-3)(x-2)^2} = \frac{10}{x-3} - \frac{9}{x-2} - \frac{5}{(x-2)^2}$$

$$\therefore \int \frac{x^2+1}{(x-3)(x-2)^2} \, dx = 10\int \frac{dx}{x-3} - 9\int \frac{dx}{x-2} - 5\int \frac{dx}{(x-2)^2}$$

$$I = 10\log(x-3) - 9\log(x-2) + 5\left(\frac{1}{x-2}\right) + c$$

③ Solve $\int \frac{10 \, dx}{(x-1)(x^2+9)}$

Sol

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$\Rightarrow A(x^2+9) + (Bx+C)(x-1) = 10$

put $x = 1$, $\boxed{A = 1}$

put $x = 0$, $9A - C = 10$

$\Rightarrow \boxed{C = -1}$

Equating the coefficient of x^2, we get, $A+B=0$
$\Rightarrow \boxed{B=-1}$

$$\therefore \frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{(-x-1)}{x^2+9}$$

$$\text{Now} \int \frac{10\,dx}{(x-1)(x^2+9)} = \int \frac{dx}{x-1} - \int \frac{x\,dx}{x^2+9} - \int \frac{dx}{x^2+9}$$

$$\therefore I = \log(x-1) - \frac{1}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c.$$

(4) Integrate the following.

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$$

Sol Since deg (Nr) > deg (Dr), it is improper

So, $f(x) = q(x) + \frac{r(x)}{p(x)}$

i.e.:

$$\begin{array}{r|l} & \quad x+1 \\ \hline x^3-x^2-x+1 & x^4-0x^3-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ \hline & \quad x^3-x^2+3x+1 \\ & \quad x^3-x^2-x+1 \\ (-) & \hline & \qquad 4x. \end{array}$$

$$\therefore \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1}. \quad \text{— (1)}$$

$$\text{Now } \frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)}$$

$$= \frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\Rightarrow 4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

put $x=1$, $\boxed{B=2}$

put $x=-1$, $\boxed{C=-1}$

equating the coefft. of x^2,

$A+C=0 \Rightarrow \boxed{A=1}$

$$\therefore (1) \Rightarrow \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$$

$$= (x+1) + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$\therefore I = \int (x+1)\,dx + \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x+1}$$

$$I = \frac{x^2}{2} + x + \log(x-1) + 2\,\frac{(-1)}{(x-1)} - \log(x+1) + c$$

$$\therefore I = \frac{x^2}{2} + x - \frac{2}{x-1} + \log\left(\frac{x-1}{x+1}\right) + c$$

The given integral is of the form $\int \frac{dx}{ax^2+bx+c}$

① Evaluate $\int \frac{dx}{3x^2-4x-5}$

Sol

Let $I = \int \frac{dx}{3x^2-4x-5}$

Then $I = \frac{1}{3}\int \frac{dx}{x^2-\frac{4}{3}x-\frac{5}{3}}$

Now

$$x^2-\frac{4}{3}x-\frac{5}{3}$$
$$= x^2-\frac{4}{3}x+\left(\frac{2}{3}\right)^2-\left(\frac{2}{3}\right)^2-\frac{5}{3}$$
$$= \left(x-\frac{2}{3}\right)^2-\frac{19}{9}$$
$$= \left(x-\frac{2}{3}\right)^2-\left(\frac{\sqrt{19}}{3}\right)^2$$

$$\therefore I = \frac{1}{3}\int \frac{dx}{\left(x-\frac{2}{3}\right)^2-\left(\frac{\sqrt{19}}{3}\right)^2}$$

$$\left\{\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c\right\}$$

$$\therefore I = \frac{1}{3}\frac{1}{2\left(\frac{\sqrt{19}}{3}\right)}\log\left[\frac{\left(x-\frac{2}{3}\right)-\frac{\sqrt{19}}{3}}{\left(x-\frac{2}{3}\right)+\frac{\sqrt{19}}{3}}\right]+c$$

$$\Rightarrow I = \frac{1}{2\sqrt{19}}\log\left[\frac{3x-2-\sqrt{19}}{3x-2+\sqrt{19}}\right]+c.$$

The given integral is of the form $\int \frac{px+q}{ax^2+bx+c}$

$$Nr = A\frac{d}{dx}(Dr)+B.$$

i.e. $\int \frac{px+q}{ax^2+bx+c} = \int \frac{A\frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c} + \int \frac{B}{ax^2+bx+c}dx.$

① $\int \frac{2x+3}{x^2+2x+5}dx = ?$

Sol Let $2x+3 = A\frac{d}{dx}(x^2+2x+5)+B.$

$$\Rightarrow 2x+3 = A(2x+2)+B \quad \text{—①}$$

Equating the coefficients of x we get

$2A = 2 \Rightarrow \boxed{A=1}$ & $\cancel{B=3}$

$2A+B = 3 \Rightarrow \boxed{B=1}$

$$\therefore 2x+3 = (2x+2)+1.$$

$$\therefore I = \int \frac{2x+3}{x^2+2x+5}dx.$$
$$= \int \frac{(2x+2)+1}{x^2+2x+5}dx$$
$$= \int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{dx}{x^2+2x+5}$$
$$= \log(x^2+2x+5) + \int \frac{dx}{x^2+2x+5} \quad \text{—②}$$

Take $\int \frac{dx}{x^2+2x+5}$

$$\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{x^2+2x+1^2-1^2+5}$$

$$= \int \frac{dx}{(x+1)^2+4}.$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c.$$

(2) $\Rightarrow I = \log(x^2+2x+5)$
$+\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$

The given integral is of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$$

(1) Solve $\int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx$

Sol

$$2x-1 = A\frac{d}{dx}(x^2+5x+6)+B$$

$$\Rightarrow 2x-1 = (2x+5)A+B$$

Equating the coefficients of x we get

$$2A = 2 \Rightarrow \boxed{A=1}$$

~~put~~ Also, $-1 = 5A+B$

$$\Rightarrow -1 = 5+B \Rightarrow \boxed{B=-6}$$

$$\therefore I = \int \frac{2x-1}{\sqrt{x^2+5x+6}}\,dx$$

$$= \int \frac{(2x+5)}{\sqrt{x^2+5x+6}}\,dx - 6\int \frac{dx}{\sqrt{x^2+5x+6}}$$

$$\therefore I = 2\sqrt{x^2+5x+6}$$
$$-6\int \frac{dx}{\left(x+\frac{5}{2}\right)^2+6-\frac{25}{4}}$$

$$= 2\sqrt{x^2+5x+6}$$
$$-6\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= 2\sqrt{x^2+5x+6}$$
$$-6\log\left[\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right]+c$$

The given integral is of the form $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

put $px+q = A\,d(ax^2+bx+c)+B$

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx.$$

$$= A\int \sqrt{ax^2+bx+c}\;d(ax^2+bx+c)$$
$$+B\int \sqrt{ax^2+bx+c}\,dx$$

$$= A\sqrt{u}\,du + B\sqrt{ax^2+bx+c}\,dx$$

(1) Solve $\int (3x-2)\sqrt{x^2+x+1}\,dx$

Sol

$$3x-2 = A\frac{d}{dx}(x^2+x+1)+B$$

$$3x-2 = A(2x+1)+B$$

Equating the coefficients of x, we get

$$2A = 3 \Rightarrow \boxed{A=\frac{3}{2}}$$

also, $A+B=-2 \Rightarrow \boxed{B=\frac{-7}{2}}$

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$$\therefore I = \int (3x-2)\sqrt{x^2+x+1}\, dx.$$

$$= \frac{3}{2}\int \sqrt{x^2+x+1}\; d(x^2+x+1) - \frac{7}{2}\int \sqrt{x^2+x+1}\; dx$$

$$= \frac{3}{2}\, \frac{(x^2+x+1)^{3/2}}{3/2} - \frac{7}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\; dx.$$

$$= (x^2+x+1)^{3/2} - \frac{7}{2}\left[\left(\frac{2x+1}{4}\right)\sqrt{x^2+x+1} + \frac{3}{8}\log\left[x+\frac{1}{2}+\sqrt{x^2+x+1}\right]\right] + c$$

Improper Integrals

In a regular (proper) definite integral $\int_a^b f(x)\,dx$ it is assumed that the limits of integration are finite and that the integrand $f(x)$ is continuous for every value of x in the interval $a \le x \le b$.

If atleast one of these conditions is violated, then the integral is known as an improper integral. (Singular or indefinite integral)

Defn. of an improper integral (Type I)

① If $\int_a^t f(x)\,dx$ exists for every number $t \ge a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx.$$

② If $\int_t^b f(x)\,dx$ exists for every number $t \le b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists. (as a finite no.)

③ If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, if the corresponding limit exists and divergent if the limit does not exists.

③ If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

'a' is any real number.

Defn. of improper Integral

Type:2

① If f is continuous on $[a,b)$ and is discontinuous at b, then $\int_a^b f(x)\,dx$

$$= \lim_{t\to b^-} \int_a^t f(x)\,dx$$ If this limit exists.

② If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

If this limit exist

The improper integral $\int_a^b f(x)\,dx$ is convergent if the corresponding limit exists and divergent if the limit does not exist.

③ If f has a discontinuity at c, where $a<c<b$ and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

① Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

Sol given $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx.$

$$= \lim_{t\to\infty} [\log x]_1^t$$

$$= \lim_{t\to\infty} [\log t - \log 1].$$

$$= \lim_{t\to\infty} \log t = \infty \text{ (not finite)}$$

$\therefore \int_1^\infty \frac{1}{x}\,dx$ is divergent.

② Determine whether the integral $\int_1^\infty \frac{\log x}{x^2}\,dx$ is convergent or divergent.

Sol Take $\int \frac{\log x}{x^2}\,dx$

$u = \log x \qquad dv = \frac{1}{x^2}\,dx$

$du = \frac{dx}{x} \qquad \Rightarrow v = -1/x$

$\therefore \int \log x/x^2 \,dx$

$$= \frac{-1}{x}\log x + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\log x - \frac{1}{x}$$

$$= \frac{-1}{x}[\log x + 1]$$

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$$\therefore \int_1^{\infty} \frac{\log x}{x^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{\log x}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{-1}{x}\left(\log x+1\right)\right]_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{t}\left(\log t+1\right) + \log 1+1\right]$$

$$= \lim_{t\to\infty} \left(\frac{-\log t}{t}\right) = \left(\frac{\infty}{\infty}\text{ form}\right)$$

Apply L'Hospital rule,

$$\cancel{\therefore I = \lim_{t\to\infty}\left[\frac{\log t}{t}\right]}$$

$$I = \lim_{t\to\infty}\left(\frac{-1/t}{1}\right) = 0$$

$$\therefore I = \int_1^{\infty} \frac{\log x}{x^2}\,dx = 1 \text{ (finite)}$$

$$\therefore \int_1^{\infty} \frac{\log x}{x^2} \text{ is Convergent.}$$

(3) Evaluate $\int_{-\infty}^{0} x e^x\,dx$

Sol $I = \lim_{t\to-\infty} \int_t^0 x e^x\,dx.$

$$= \lim_{t\to-\infty} x\, d(e^x)$$

$$= \lim_{t\to-\infty} \left\{\left[x e^x\right]_t^0 - \int_t^0 e^x\,dx\right\}$$

$$= \lim_{t\to-\infty}\left[-te^t - (1-e^t)\right]$$

$$= \lim_{t\to-\infty}\left[-te^t - 1 + e^t\right] = -1 \text{ (finite)}$$

$$\therefore \int_{-\infty}^{0} x e^x\,dx \text{ is Convergent.}$$

(4) For what values of p in the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$. Convergent?

Sol

If $p \neq 1$, $\lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx$

$$= \lim_{t\to\infty} \int_1^t x^{-p}\,dx.$$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right]$$

$$= \lim_{t\to\infty} \frac{1}{p-1}\left[1 - \frac{1}{t^{p-1}}\right]$$

$$= \begin{cases} \frac{1}{p-1}, & p>1, \text{ Converges} \\ \infty, & p\le 1, \text{ diverges} \end{cases}$$

(5) $\int_3^{\infty} \frac{1}{(x-2)^{3/2}}\,dx = ?$

Sol Take $\int \frac{1}{(x-2)^{3/2}}\,dx$ —(1)

put $u = x-2$: $du = dx$

$$(1) \Rightarrow \int \frac{dx}{(x-2)^{3/2}} = \int \frac{du}{u^{3/2}}$$

$$= \int u^{-3/2}\,du$$

$$= \frac{u^{-3/2+1}}{\frac{-3}{2}+1}$$

$$= \frac{u^{-1/2}}{-1/2} = \frac{-2}{\sqrt{u}} = \frac{-2}{\sqrt{x-2}}$$

$$\therefore \int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty}\int_3^t \frac{1}{(x-2)^{3/2}}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x-2}}\right]_3^t$$

$$= \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t-2}} - \left(\frac{-2}{\sqrt{1}}\right)\right]$$

$$= \lim_{t\to\infty} \frac{-2}{\sqrt{t-2}} + 2$$

$$= 2 \text{ (finite)}$$

$\therefore \int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$ is Convergent

⑥ Evaluate $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$.

Sol at $x=3$ $f(x)$ is discontinuity

$$\therefore \int_2^3 \frac{dx}{\sqrt{3-x}} = \lim_{t\to 3^-}\int_2^t \frac{dx}{\sqrt{3-x}}$$

$$= \lim_{t\to 3^-}\left[-2\sqrt{3-x}\right]_2^t$$

$$= \lim_{t\to 3^-}\left[-2\sqrt{3-t}+2\right]$$

$$= 2 \text{ (finite)}$$

$\therefore \int_2^3 \frac{dx}{\sqrt{3-x}}$ is ~~div~~ convergent

⑦ Evaluate $\int_0^3 \frac{1}{\sqrt{x}}\,dx$.

Sol Let $f(x) = \frac{1}{\sqrt{x}}$.

at $x=0$, $f(x)$ is discontinuous

$$\therefore \int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^3 x^{-1/2}\,dx$$

$$= \lim_{t\to 0^+}\left[\frac{x^{1/2}}{1/2}\right]_t^3$$

$$= \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^3$$

$$= \lim_{t\to 0^+}\left[2\sqrt{3}-2\sqrt{t}\right]$$

$$= 2\sqrt{3} \text{ (finite)}$$

$\therefore \int_0^3 \frac{1}{\sqrt{x}}$ is Convergent.

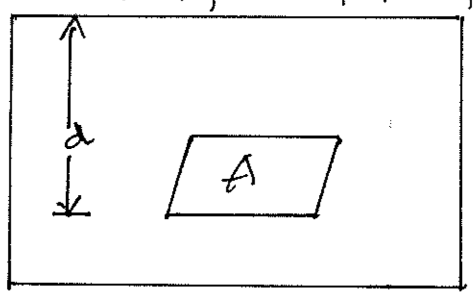
Applications of Integral calculus.

Hydrostatic force and force.

Sea diverse realize that water pressure increases as they dive deeper into the sea. This significant feature happens because the weight of the water above them increases.

Let us assume that a thin horizontal plate with area A square meters is submerged in a fluid of density 'ρ' kilograms per cubic meter at a depth 'd' meters below the surface of the fluid.

Surface of the fluid.



The fluid directly above the plate has volume $V = Ad$.

Then its mass is $m = \rho V = \rho Ad$.

The force exerted by the fluid on the plate is

$$F = mg = \rho g A d.$$

where 'g' is the acceleration due to gravity

$\therefore$ The pressure on the plate is

$$P = \frac{F}{A} = \rho g d.$$

The SI unit for measuring pressure is Newton per square meter is called pascal.

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We know that the density of water is given by $\rho = 1000 kg/m^3$

The pressure at the bottom of a swimming pool 2 meter 2 meter deep is given by

$$p = \rho g d = 1000 kg/m^3 \times 9.8 m/s^2 \times 2 \text{ meter}.$$

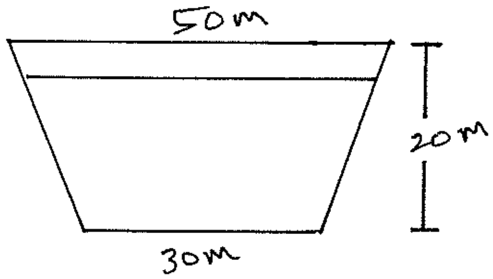
$$\therefore p = 19,600 pa = 19.6 kpa$$

The principle of fluid pressure is that, at any point in a liquid the pressure is the same in all directions. Therefore, the divers has the same pressure on nose and both ears.

Hence the pressure in any direction at a depth "d" in a fluid with mass density "ρ" is given by $p = \rho g d = \delta d$.

This relation helps to determine the hydrostatic force against a vertical plate or wall or dam in a fluid.

① A dam has the shape of the trapezoid as shown in figure.



The height is 20m and the width is 50m at the top and 30m at the bottom. Find the force on the dam due hydrostatic pressure if the water level is 4m from the top of the dam.

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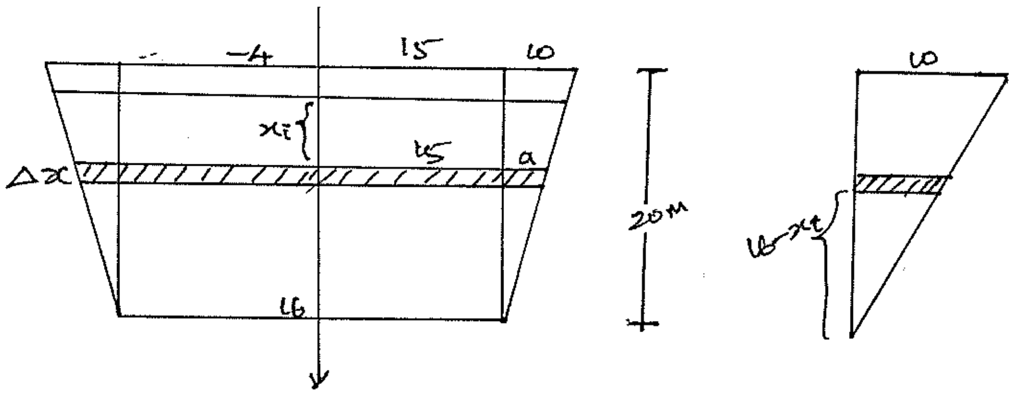
Sol: Let us consider a vertical x axis with origin at the surface of the water and directed downward.

Then the depth of the water is 16 m.

$\therefore$ the interval $[0,16]$ can be divided into subintervals of equal length with end points x_i, $x_i \in [x_{i-1}, x_i]$

Let us consider the i^{th} horizontal strip of the dam is approximated by a rectangle with height Δx and width w_i

$$\therefore \frac{a}{16-x_i} = \frac{10}{20} \Rightarrow a = \frac{16-x_i}{2} = 8 - \frac{x_i}{2}$$



$$w_i = 2(15+a) = 2\left(15+8-\frac{x_i}{2}\right)$$

$$\therefore w_i = 46 - x_i$$

If A_i is the area of the i^{th} strip, then

$$A_i = w_i \Delta x = (46 - x_i)\,\Delta x$$

If Δx is very small, then the pressure P_i on the ~~ith~~ i^{th} strip is almost constant.

$$\therefore P = \rho g d = \delta d.$$

$$P_i = 1000 g x_i$$

Now the hydrostatic force F_i is

$$F_i = P_i A_i = 1000 g x_i (46 - x_i)\Delta x.$$

As $n \to \infty$,

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 1000 g x_i (46 - x_i)\Delta x.$$

$$= \int_0^{16} 1000 g x (46 - x)\, dx$$

$$= 1000 g \int_0^{16} (46x - x^2)\, dx$$

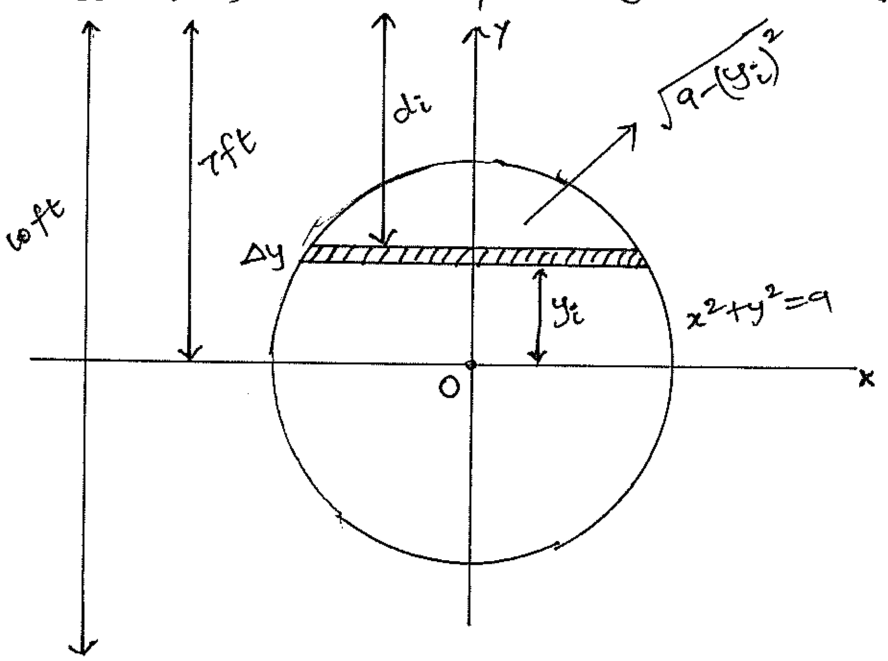
$$= 1000(9.8)\left[\frac{46x^2}{2} - \frac{x^3}{3}\right]_0^{16}$$

$$= 9800\left[23(16)^2 - \frac{1}{3}(16)^3\right]$$

$$= 4.43 \times 10^7 \text{ N}.$$

② Find the hydrostatic force on one end of a cylindrical drum with radius 3 feet if the drum is submerged in water 10 feet deep.

Sol The circle has a simple equation $x^2 + y^2 = 9$. —①



we divide the circular region into horizontal strips of equal width.

From (i), the length of the i^{th} strip is given by $2\sqrt{9-(y_i)^2}$

$\therefore$ Its area (strip) is $A_i = 2\sqrt{9-(y_i)^2}\,\Delta y$.

The pressure on the strip is $\delta d_i = 62.7(7-y_i)$

$\therefore$ the force on the strip is

$$\delta d_i A_i = 62.7(7-y_i)\,2\sqrt{9-(y_i)^2}\,\Delta y$$

$\therefore$ The total force is

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} 62.7(7-y_i)\,2\sqrt{9-(y_i)^2}\,\Delta y$$

$$= 125\int_{-3}^{3} (7-y)\sqrt{9-y^2}\,dy$$

$$= 125 \times 7 \times \int_{-3}^{3} \sqrt{9-y^2}\,dy - 125\int_{-3}^{3} y\sqrt{9-y^2}\,dy$$

$$= 875\int_{-3}^{3}\sqrt{9-y^2}\,dy - 0$$

$$= 875\int_{-3}^{3}\sqrt{3^2-y^2}\,dy \qquad \left\{\because \int\sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}(x/a)\right\}$$

$$= 875\left[\frac{1}{2}y\sqrt{9-y^2} + \frac{1}{2}3^2\sin^{-1}(y/3)\right]_{-3}^{3}$$

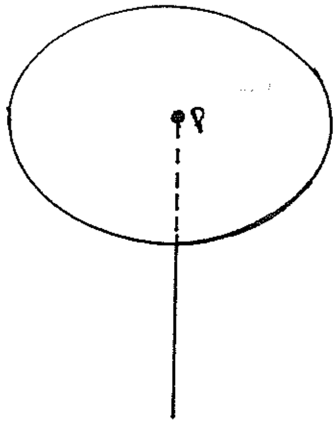
$$= 875\left[\left(\frac{3}{2}(0) + \frac{9}{2}\sin^{-1}(1)\right) - \left(-\frac{3}{2}(0) - \frac{9}{2}\sin^{-1}(1)\right)\right]$$

$$= 875\left[9\sin^{-1}(1)\right]$$

$$F = 12,375 \text{ lb}$$

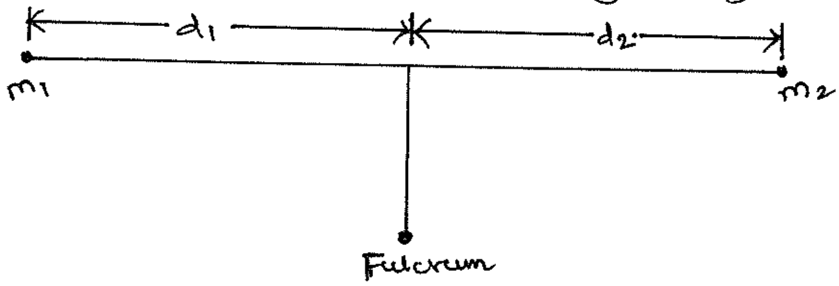
Moments and Centre of Mass.

Let us consider the diagram.



From the diagram, to find the point 'p' on which a thin plate of any given shape balances horizontally, is called the centre of mass or centre of gravity of the plate.

Ex



In the above diagram, the two masses m_1 and m_2 are fixed to a rod of negligible mass on opposite sides of a fulcrum and at distances d_1 and d_2 from the fulcrum.

The rod will be a balanced one, if

$$m_1 d_1 = m_2 d_2 \quad \text{—— (1)}$$

Let us assume the rod lies along the x axis with m_1 at the point x_1 and m_2 at the point x_2 and the centre of mass at the point $\bar{x}$.

$$\therefore \; d_1 = \bar{x} - x_1 \quad \& \quad d_2 = x_2 - \bar{x}$$

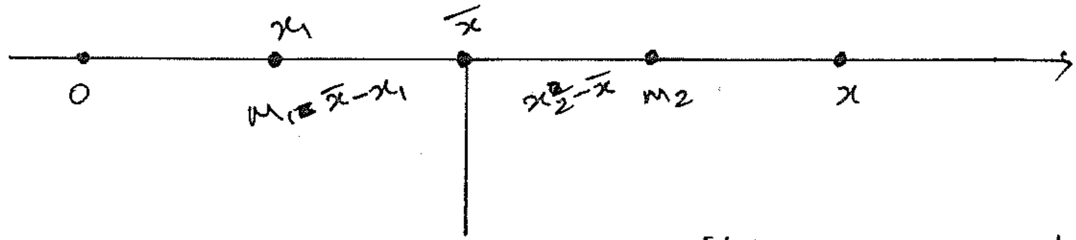
$$(1) \Rightarrow m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$\Rightarrow (m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2$$

$$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \text{—— (2)}$$

where m_1x_1 & m_2x_2 are moments of m_1 & m_2

and $m = m_1 + m_2$



Suppose we've 'n' particles with corresponding masses $m_1, m_2, m_3, \ldots, m_n$ located at the points $x_1, x_2, \ldots, x_n$ on the x axis, then the centre of mass of the entire system is

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

where $m = \sum m_i$ is the total mass of the system and the sum of the individual moments

$$M = \sum_{i=1}^{n} m_i x_i$$ is called the moment of the system about the origin.

$$\therefore M = m\bar{x}.$$

Let us consider a system of n particles with masses $m_1, m_2, m_3, \ldots, m_n$ located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane.

$\therefore$ the moment of the system about y-axis & x axis is given by $M_y = \sum_{i=1}^{n} m_i x_i$, $M_x = \sum_{i=1}^{n} m_i y_i$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency to rotate about the x-axis.

$\therefore (\bar{x}, \bar{y})$ of the centre of mass are given by

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

Where $m = \sum_{i=1}^{n} m_i$ is the total mass.

Since $m\bar{x} = M_y$ & $m\bar{y} = M_x$, the centre of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m will have the same moments.

Centroid and Symmetry

Consider a flat plate which is called as a lamina with uniform density ρ that occupies a region R of the plane. We wish to locate the centre of mass of the plate which is called as the centroid of the region R.

If R is symmetric about a line l, then the centroid of R lies on l is called principle of symmetry. Hence the centroid of a rectangle is its centre.

Moments should be defined so that if the entire mass of a region is concentrated at the centre of mass, then its moments remains unchanged. The moments of the union of two non-overlapping regions should be the sum of the moments of the individual regions.

The moment of the region R above y-axis is given by

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \, \bar{x}_i \, f(\bar{x}_i) \Delta x = \rho \int_a^b x f(x) \, dx$$

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The moment of the region R about the x-axis is

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \, \frac{1}{2}\left[f(\bar{x}_i)\right]^2 \Delta x = \rho \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

For system of particles, the centre of mass of the plate is defined as $m\bar{x} = M_y$ & $m\bar{y} = M_x$.

The mass of the plate is the product of its density and area.

$$m = \rho A = \rho \int_a^b f(x)\, dx.$$

Then

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\, dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b x f(x)\, dx}{\int_a^b f(x)\, dx}.$$

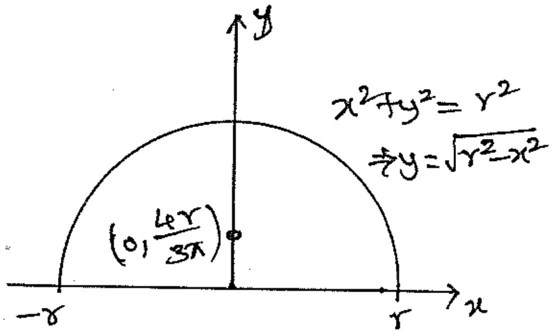
$$\bar{y} = \frac{M_x}{m} = \frac{\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx}{\int_a^b f(x)\, dx}.$$

$\therefore$ The centre of mass of the plate or the centroid of R is located at the point $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{A}\int_a^b x f(x)\, dx$

and

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx.$$

① Find the centre of mass of a semicircular plate of radius r.



Sol

Equation of the circle is $x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2$

$\therefore y = f(x) = \sqrt{r^2-x^2}$ and

$a=-r$, $b=r$.

Since the centre of mass must lie on the y-axis, $\bar{x}=0$

$\therefore$ Area $A = \frac{1}{2}\pi r^2$

Now $\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}[f(x)]^2 dx = \frac{1}{\frac{1}{2}\pi r^2}\int_{-r}^{r} \frac{1}{2}\left[\sqrt{r^2-x^2}\right]^2 dx$

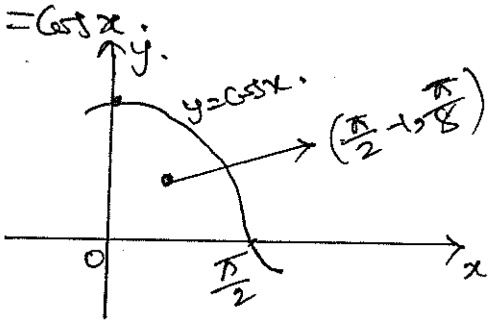
$\therefore \bar{y} = \frac{2}{\pi r^2}\int_0^r (r^2-x^2)\,dx. = \frac{2}{\pi r^2}\left[r^2x - \frac{x^3}{3}\right]_0^r$

$\bar{y} = \frac{2}{\pi r^2}\left[r^3 - \frac{r^3}{3}\right] \Rightarrow \bar{y} = \frac{4r}{3\pi}$

$\therefore (\bar{x},\bar{y}) : (0, \frac{4r}{3\pi})$

---

② Find the centroid of the region bounded by the curves $x=0$, $x=\pi/2$, $y=0$ and $y=\cos x$.



Sol

The area of the region

$A = \int_0^{\pi/2} \cos x\,dx.$

$= \sin x\Big]_0^{\pi/2} \Rightarrow \boxed{A=1}$

We know that the centroid of the region is given by $(\bar{x},\bar{y})$, where

$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$ & $\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2 dx.$

Now $\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx = \int_0^{\pi/2} x\cos x\,dx.$

$u = x \qquad u' = 1 \qquad u'' = 0.$

$v = \cos x \qquad v_1 = \sin x \qquad v_2 = -\cos x.$

$\therefore \bar{x} = \left[x\sin x + \cos x\right]_0^{\pi/2} = \left(\frac{\pi}{2} + 0\right) - (0 - 1)$

$\therefore \bar{x} = \frac{\pi}{2} - 1$

$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx. = \frac{1}{2}\int_a^{\pi/2} \cos^2 x\,dx$

$\bar{y} = \frac{1}{4}\int_0^{\pi/2} (1+\cos 2x)\,dx = \frac{1}{4}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$\bar{y} = \frac{\pi}{8}$

$\therefore$ the centroid is $(\bar{x}, \bar{y}) : \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$

---

③ Note

If the region R lies between $y = f(x)$ and $y = g(x)$ where $f(x) \geq g(x)$ then the centroid of R is $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$ and

$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[[f(x)]^2 - [g(x)]^2\right]dx.$

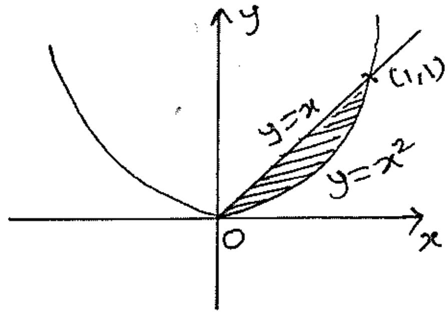
---

③ Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.

Sol

let $f(x) = x$ & $g(x) = x^2$

$a = 0$ & $b = 1$



Now Area $A = \int_0^1 (x - x^2)\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \Rightarrow \boxed{A = \frac{1}{6}}$

We know that $\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx.$

$$\bar{x} = \frac{1}{(1/6)}\int_0^1 x(x - x^2)\,dx$$

$$\bar{x} = 6\int_0^1 (x^2 - x^3)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$\boxed{\bar{x} = \frac{1}{2}}$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[[f(x)]^2 - [g(x)]^2\right]dx.$$

$$= \frac{1}{1/6}\int_0^1 \frac{1}{2}\left[x^2 - x^4\right]dx$$

$$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$\boxed{\bar{y} = \frac{2}{5}}.$$

$\therefore$ the centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$

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Unit - 5

Multiple Integrals

Double integrals - change of order of integration - Double integrals in Polar co-ordinates - Area enclosed by plane curves - Triple integrals - volume of Solids change of variables in double and triple integrals - Applications: Moments and centre of mass, moment of inertia.

Double integrals in cartesian co-ordinates:

Type: 1 (Limits are constants)

1) Evaluate $\int_0^a \int_0^b xy(x-y)\,dx\,dy$

Answer:

$$\int_0^a \int_0^b xy(x-y)\,dx\,dy = \int_0^a \int_0^b (x^2y - xy^2)\,dx\,dy$$

$$= \int_0^a \left(\frac{x^3y}{3} - \frac{x^2}{2}y^2\right)_0^b dy$$

$$= \int_0^a \left(\frac{b^3y}{3} - \frac{b^2y^2}{2}\right) dy$$

$$= \int_0^a b^2\left[\frac{by}{3} - \frac{y^2}{2}\right] dy$$

$$= b^2\left\{\frac{by^2}{6} - \frac{y^3}{6}\right\}_0^a$$

$$= b^2\left\{\frac{a^2b}{6} - \frac{a^3}{6}\right\} = \frac{a^2b^2}{6}(b-a)$$

2) Evaluate $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$

Answer: $\int_0^1 \int_1^2 x(x+y)\,dy\,dx = \int_0^1 \int_1^2 (x^2+xy)\,dy\,dx$

$$= \int_0^1 \left(x^2y + \frac{xy^2}{2}\right)_1^2 dx$$

$$= \int_0^1 (2x^2+2x) - \left(x^2+\frac{x}{2}\right) dx$$

$$= \int_0^1 \left(x^2+\frac{3x}{2}\right) dx$$

$$= \left(\frac{x^3}{3}+\frac{3x^2}{4}\right)_0^1 = \left(\frac{1}{3}+\frac{3}{4}\right) - (0+0)$$

$$= \frac{13}{12}$$

3) Evaluate $\int_2^a \int_2^b \frac{dx\,dy}{xy}$

Ans:

$$\int_2^a \int_2^b \frac{dx\,dy}{xy} = \int_2^a \frac{dx}{x} \int_2^b \frac{dy}{y}$$

$$= (\log x)_2^a \, (\log y)_2^b$$

$$= (\log a - \log 2)\,(\log b - \log 2)$$

$$= \log\frac{a}{2} \log\frac{b}{2}$$

4) Evaluate $\int_0^3 \int_0^2 e^{x+y}\, dy\, dx$

Ans:

$$\int_0^3 \int_0^2 e^{x+y}\, dy\, dx = \int_0^3 e^x dx \int_0^2 e^y dy$$

$$= (e^x)_0^3 \, (e^y)_0^2$$

$$= (e^3 - e^0)\,(e^2 - e^0) = (e^3-1)(e^2-1)$$

5) Evaluate $\int_1^2 \int_1^3 xy^2\, dx\, dy$

Ans:

$$\int_1^2 \int_1^3 xy^2\, dx\, dy = \int_1^2 x\, dx \int_1^3 y^2 dy$$

$$= \left(\frac{x^2}{2}\right)_1^2 \left(\frac{y^3}{3}\right)_1^3$$

$$= \left(\frac{4}{2}-\frac{1}{2}\right)\left(\frac{27}{3}-\frac{1}{3}\right) = \frac{3}{2}\cdot\frac{26}{3} = 13.$$

Type-II (Limits are variables)

1) Evaluate $\int_0^1 \int_0^{1-x} y\, dy\, dx$

Ans:

$$\int_0^1 \int_0^{1-x} y\, dy\, dx = \int_0^1 \int_0^{1-x} y\, dx\, dy \quad \text{(correct form)}$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^2}{2}\, dx = \frac{1}{2}\left\{\frac{(1-x)^3}{-3}\right\}_0^1$$

$$= -\frac{1}{6}(0-1) = 1/6$$

2) Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)\, dx\, dy$

Ans:

$$\int_0^1 \int_x^{\sqrt{x}} xy(x+y)\, dx\, dy = \int_0^1 \int_x^{\sqrt{x}} (x^2y + xy^2)\, dy\, dx \quad \text{(correct form)}$$

$$= \int_0^1 \int_x^{\sqrt{x}} (x^2y + xy^2)\, dy\, dx$$

$$= \int_0^1 \left.\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\frac{x^2(\sqrt{x})^2}{2} - \frac{x(\sqrt{x})^3}{3}\right] - \left[\frac{x^2 \cdot x^2}{2} + \frac{x \cdot x^3}{3}\right] dx$$

$$= \int_0^1 \frac{x^3}{2} - \frac{x^{5/2}}{3} - \frac{x^4}{2} - \frac{x^4}{3}\, dx$$

$$= \left.\frac{x^4}{8} + \frac{x^{7/2}}{3(7/2)} - \frac{x^5}{10} - \frac{x^5}{15}\right]_0^1$$

$$= \frac{1}{8} + \frac{2}{21} - \frac{1}{10} - \frac{1}{15} = 3/56$$

3) Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$

N/D '18

Ans: $\int_0^4 \int_0^{x^2} e^{y/x}\,dy\,dx$ (correct form)

$= \int_0^4 \left\{\frac{e^{y/x}}{1/x}\right\}_0^{x^2} dx$

$= \int_0^4 x\{e^{x^2/x} - e^0\}\,dx$

$= \int_0^4 x(e^x - 1)\,dx$

$= \int_0^4 (xe^x - x)\,dx$

$= \left\{x e^x - e^x - \frac{x^2}{2}\right\}_0^4$

$= (4e^4 - e^4 - 8) - (-1) = 3e^4 - 7$

4) Evaluate $\int_0^1 \int_y^{y^2+1} x^2 y\,dx\,dy$

Ans:

$\int_0^1 \int_y^{y^2+1} x^2 y\,dx\,dy = \int_0^1 y\left(\frac{x^3}{3}\right)_y^{y^2+1} dy$

$= \frac{1}{3}\int_0^1 y\{(y^2+1)^3 - y^3\}\,dy$

$= \frac{1}{3}\int_0^1 y\{y^6 + 1 + 3y^4 + 3y^2 - y^3\}\,dy$

$= \frac{1}{3}\left\{\frac{y^8}{8} + \frac{y^2}{2} + \frac{3y^6}{6} + \frac{3y^4}{4} - \frac{y^5}{5}\right\}_0^1$

$= \frac{1}{3}\left\{\frac{1}{8} + \frac{1}{2} + \frac{1}{2} + \frac{3}{4} - \frac{1}{5} - 0\right\}$

$= \frac{1}{3}\left\{\frac{1}{8} + 1 + \frac{3}{4} - \frac{1}{5}\right\}$

$= \frac{1}{3}\left(\frac{67}{40}\right)$

$= \frac{67}{120}$

Sketch the region of integration:

1) Sketch the region of integration $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx\,dy$

Answer:

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx\,dy = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dy\,dx \quad \text{(correct form)}$$

$$y : \sqrt{ax-x^2} \longrightarrow y : \sqrt{a^2-x^2}$$

$$y = \sqrt{ax-x^2} \longrightarrow y = \sqrt{a^2-x^2}$$

$$y^2 = ax-x^2 \longrightarrow y^2 = a^2-x^2$$

$$x^2+y^2-ax=0 \longrightarrow y^2+x^2=a^2$$

$$x^2+y^2-ax+\frac{a^2}{4}-\frac{a^2}{4}=0 \longrightarrow x^2+y^2=a^2$$

$$\left(x-\frac{a}{2}\right)^2+(y-0)^2=\left(\frac{a}{2}\right)^2 \longrightarrow x^2+y^2=a^2$$

Circle, $C=(a/2, 0)$ & radi $= a/2$ $\longrightarrow$ Circle, $C=(0,0)$ & radi $= a$

Rough

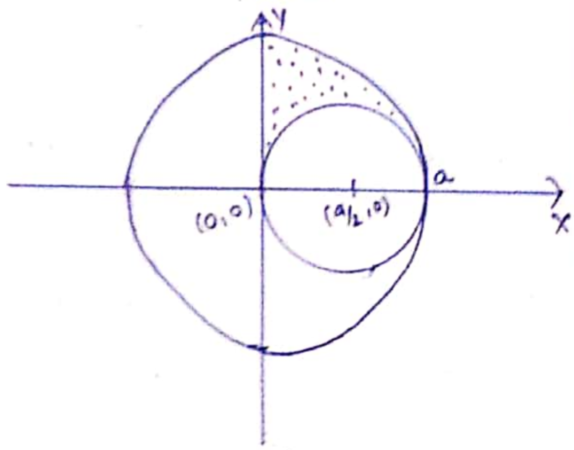
$2b = -a$

$b = -a/2$

$b^2 = \frac{a^2}{4}$

Also $x : 0 \longrightarrow a$

$x=0 \longrightarrow x=a$



2) Sketch the region of integration for $\int_0^1 \int_0^x f(x,y)\,dy\,dx$

M/D'15
M/J 16
MD 18

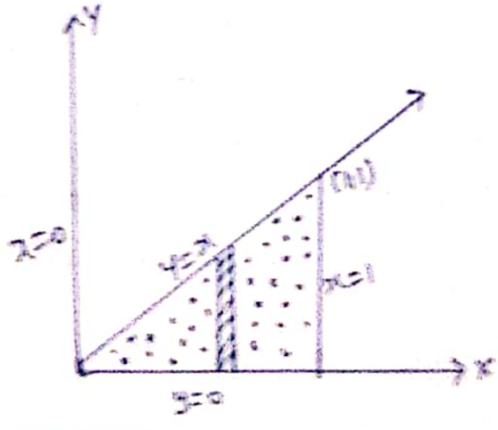
Ans:

Given $y : 0 \rightarrow x$

$y=0 \rightarrow y=x$

Also $x : 0 \rightarrow 1$

$x=0 \rightarrow x=1$



3) Sketch the region $\int_0^a \int_0^{\sqrt{a^2-x^2}} dx\,dy$

<u>Ans:</u>

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx \quad \text{[Correct form]}$$

Given $y: 0 \longrightarrow \sqrt{a^2-x^2}$

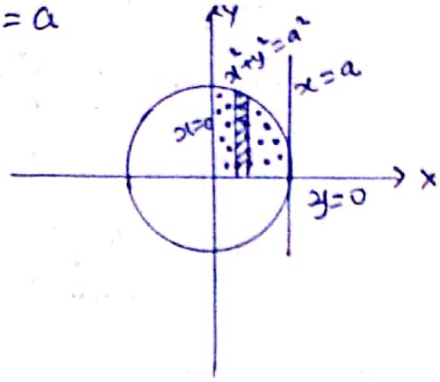
$y=0 \longrightarrow y=\sqrt{a^2-x^2}$

$y^2=a^2-x^2$

$x^2+y^2=a^2$ circle, $C=(0,0)$ & rad $=a$

Also $x: 0 \rightarrow a$

$x=0 \longrightarrow x=a$



Change of order of integration:

when we change the order of integration the limits are also changed. The following Points ~~p~~ are very useful when the change of order of integration.

1) If the limits of the inner integral is a function of x. (or function of y) then the first integration should be wrt y (or wrt x).

2) Draw the region of integration by using the given limits.

3) If the integration is first wrt x keeping y as a constant then consider the horizontal strip and vice versa.

4) Then change into vertical strip, to find the limits of x & y.

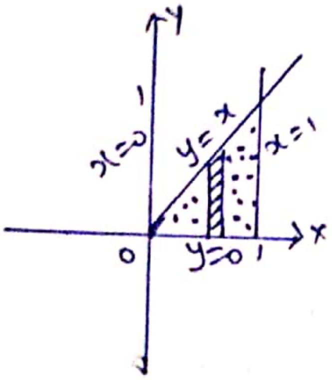
1) change the order of integration in $\int_0^1 \int_0^x f(x,y)\, dy\, dx$

Ans: Given $y: 0 \to x$

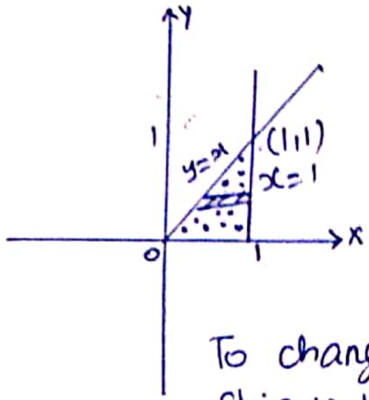
$y=0 \to y=x$

Also $x: 0 \to 1$

$x=0 \to x=1$



Consider the strip is $\parallel^{el}$ to Y axis.



To change the strip is $\parallel^{el}$ to x axis

$x=y \to x=1$

& $y=0 \to y=1$

$$\therefore \int_0^1 \int_0^x f(x,y)\, dy\, dx = \int_0^1 \int_y^1 f(x,y)\, dx\, dy$$

2) change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ & hence evaluate it.

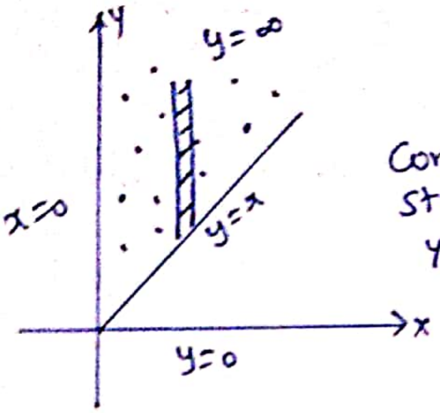
Ans: $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ (correct form)

Given $y: x \to \infty$

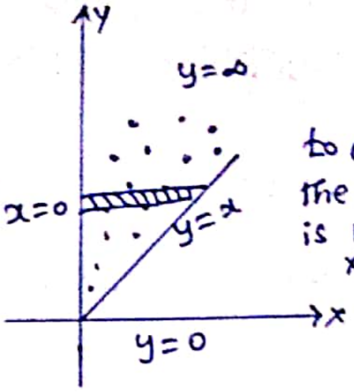
$y=x \to y=\infty$

Also $x: 0 \to \infty$

$x=0 \to x=\infty$



Consider the strip is ||el to y axis



to change the strip is ||el to x axis

Hence $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$

$$= \int_0^\infty \frac{e^{-y}}{y}(x)_0^y\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}y\,dy = \int_0^\infty e^{-y}\,dy$$

$$= -1[e^{-\infty} - e^{0}]$$

$$= -1(0-1) = 1$$

3) change the order of integration & hence evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$.

Jan' 2014
M/J 2016
A/M 2018

Ans

Given $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

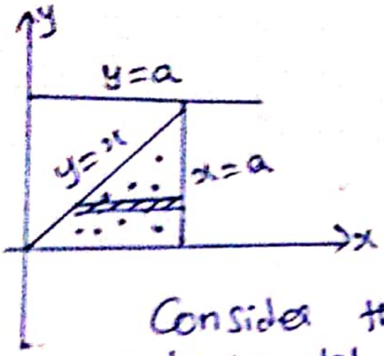
The region is bounded by $x=y$, $x=a$, $y=0$ & $y=a$

$x: y \to a$

$x=y \to x=a$

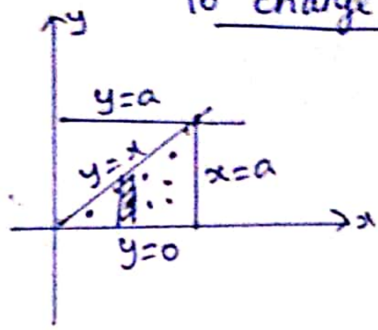
& $y: 0 \to a$

$y=0 \to y=a$



Consider the horizontal strip

To change the order



To change the strip is ∥el to y axis.

$$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy = \int_0^a \int_0^x \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_0^a x \left(\frac{1}{x} \tan^{-1} y/x\right)_0^x dx$$

$$= \int_0^a \left(\tan^{-1} y/x\right)_0^x dx$$

$$= \int_0^a \tan^{-1}(1) - \tan^{-1}(0)\, dx$$

$$= \int_0^a \left(\frac{\pi}{4} - 0\right) dx = \frac{\pi}{4}(x)_0^a = \frac{a\pi}{4}$$

4) change the order of integration, hence evaluate it

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$$

Given $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$

$y: x^2/4a \longrightarrow 2\sqrt{ax}$

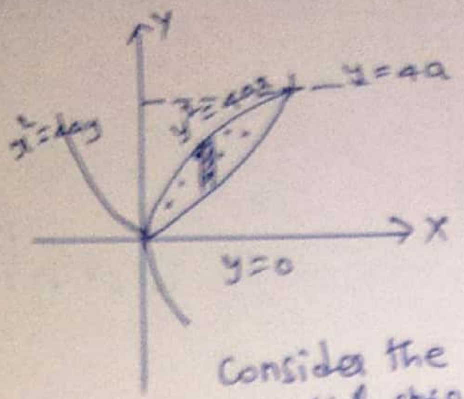
$y = x^2/4a \longrightarrow y = 2\sqrt{ax}$

$x^2 = 4ay \longrightarrow y^2 = 4ax$ (Parabola's)

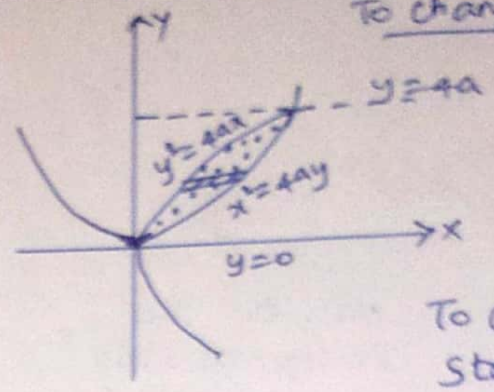
Also $x: 0 \longrightarrow 4a$

$x = 0 \longrightarrow x = 4a$

To change the order



Consider the vertical strip.



To change the strip into horizontal

$x: y^2/4a \longrightarrow 2\sqrt{ay}$

$y: 0 \longrightarrow 4a$

$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a} y\left(\frac{x^2}{2}\right)_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^{4a} y\left\{(2\sqrt{ay})^2 - \left(\frac{y^2}{4a}\right)^2\right\} dy$$

$$= \frac{1}{2}\int_0^{4a} y\left\{4ay - \frac{y^4}{16a^2}\right\} dy$$

$$= \frac{1}{2}\int_0^{4a} 4ay^2 - \frac{y^5}{16a^2}\, dy$$

$$= \frac{1}{2}\left\{4a\frac{y^3}{3} - \frac{y^6}{6\times 16a^2}\right\}_0^{4a}$$

$$= \frac{1}{2}\left\{(4a)^4\right\}\left\{\frac{1}{3} - \frac{(4a)^2}{96a^2}\right\}$$

$$= \frac{256a^4}{2}\left(\frac{1}{3} - \frac{1}{6}\right)$$

$$= 128a^4\left(1/6\right) = \frac{64}{3}a^4$$

5 Change the order of integration & hence evaluate

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dx\,dy$$

Ans: Given $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ (correct form)

$y: x^2/a \to 2a-x$

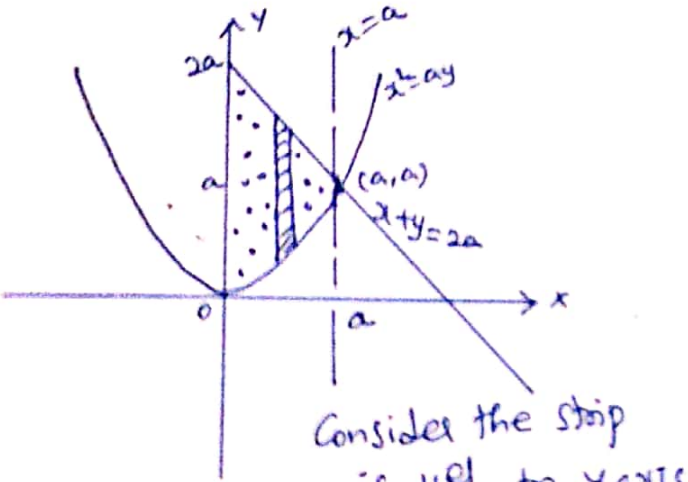
$y = x^2/a \to y = 2a - x$

$x^2 = ay \to y = 2a - x$

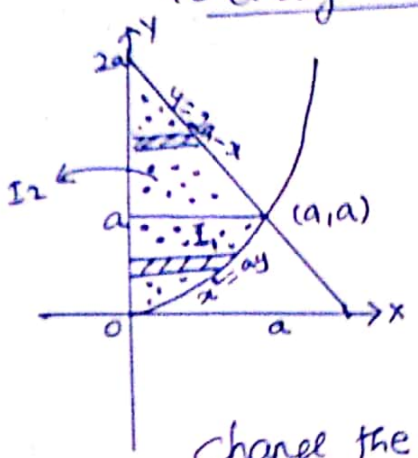
Also $x: 0 \to a$

$x = 0 \to x = a$

To change the order



Consider the strip is ||el to Y axis



Change the strip ||el to x axis.

Consider I_1

$x: 0 \to \sqrt{ay}$

& $y: 0 \to a$

$$\therefore I_1 = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^a y\left\{\frac{x^2}{2}\right\}_0^{\sqrt{ay}} dy$$

$$= \int_0^a y\,\frac{(\sqrt{ay})^2}{2}\,dy = \frac{1}{2}\int_0^a y(ay)\,dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{a}{2}\left(\frac{a^3}{3}\right) = \frac{a^4}{6}$$

$$I_2 = \int_a^{2a}\int_0^{2a-y} xy\,dx\,dy$$

$$= \int_a^{2a} y\left\{\frac{x^2}{2}\right\}_0^{2a-y} dy$$

$$= \frac{1}{2}\int_a^{2a} y\{4a^2 - 4ay + y^2\}\,dy$$

$$= \frac{1}{2}\left\{4\frac{a^2y^2}{2} - \frac{4ay^3}{3} + \frac{y^4}{4}\right\}_a^{2a}$$

$$= \frac{1}{2}\left\{\left[2a^2(2a)^2 - \frac{4a}{3}(2a)^3 + \frac{(2a)^4}{4}\right] - \left[2a^2(a)^2 - \frac{4a(a)^3}{3} + \frac{a^4}{4}\right]\right\}$$

$$= \frac{a^4}{2}\left\{8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right\}$$

$$= \frac{a^4}{2}\left\{10 - \frac{28}{3} - \frac{1}{4}\right\} = \frac{5a^4}{24}$$

$$\therefore I = I_1 + I_2 = \frac{a^4}{6} + \frac{5a^4}{24} = a^4\left\{\frac{1}{6} + \frac{5}{24}\right\}$$

$$= \frac{9a^4}{24} = \frac{3a^4}{8}$$

Double integration in Polar co-ordinates

Type: 1 (Limits are constants).

1) Evaluate $\int_0^{\pi/2}\int_0^2 r\,dr\,d\theta$

Ans $\int_0^{\pi/2}\int_0^2 r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_0^2 d\theta$

$= \frac{1}{2}\int_0^{\pi/2} (4-0)\, d\theta$

$= \frac{4}{2} \{\theta\}_0^{\pi/2}$

$= 2\{\pi/2\} = \pi$

2) Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r}{(r^2+a^2)^2}\, dr\, d\theta$

Formula

$\int [f(x)]^n\, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$

$I = \int_0^{\pi/2}\int_0^{\infty} r(r^2+a^2)^{-2}\, dr\, d\theta$

$= \int_0^{\pi/2}\int_0^{\infty} \frac{1}{2}\, 2r(r^2+a^2)^{-2}\, dr\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{(r^2+a^2)^{-2+1}}{-2+1}\right]_0^{\infty} d\theta$

$= \frac{-1}{2}\int_0^{\pi/2} \left[\frac{1}{r^2+a^2}\right]_0^{\infty} d\theta$

$= -\frac{1}{2}\int_0^{\pi/2} \left(0-\frac{1}{a^2}\right) d\theta$

$= \frac{1}{2a^2}(\theta)_0^{\pi/2} = \frac{1}{2a^2}\,\pi/2 = \pi/4a^2$

Type: 2 (Limits are variables)

1) Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\, dr\, d\theta$

Ans

Given $\int_0^{\pi}\int_0^{\sin\theta} r\, dr\, d\theta = \int_0^{\pi}\left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{4}\left\{\theta - \frac{\sin 2\theta}{2}\right\}_0^{\pi} = \frac{1}{4}\{(\pi-0)-(0-0)\} = \pi/4$

2) Evaluate $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$

Answer:

$$I = \int_0^{\pi/2} \left\{\frac{r^2}{2}\right\}_0^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{4}\left\{\theta - \frac{\sin 2\theta}{2}\right\}_0^{\pi/2}$$

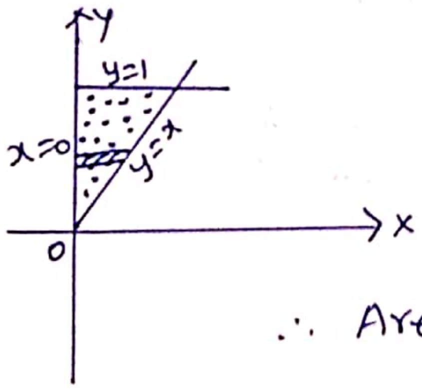
$$= \frac{1}{4}\left\{(\pi/2 - 0) - (0-0)\right\}$$

$$= \pi/8$$

Area Enclosed by plane curves:

Formula $A = \iint dx\,dy$

1) Find the area bounded by the lines $x=0, y=1$ & $y=x$



Consider the horizontal strip

$x: 0 \to y$

& $y: 0 \to 1$

$$\therefore \text{Area} = \iint_R dx\,dy$$

$$= \int_0^1\int_0^y dx\,dy$$

$$= \int_0^1 \{x\}_0^y\,dy$$

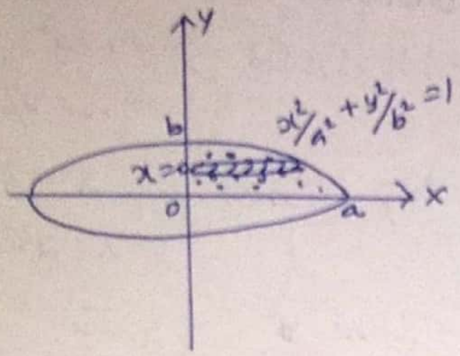
$$= \int_0^1 y\,dy$$

$$= \left.\frac{y^2}{2}\right]_0^1 = 1/2$$

2) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

N/D 2018
A/M 2020

<u>Ans:</u>



Consider the horizontal strip
Strip starts from ellipse Y axis
$\therefore$ Lower lt of $x=0$
Strip ends at ellipse

$$\frac{x^2}{a^2}+y^2/b^2=1$$

$$\Rightarrow \frac{x^2}{a^2}=1-y^2/b^2 \qquad \therefore x:0\to\frac{a}{b}\sqrt{b^2-y^2}$$

$$\Rightarrow x=a/b\sqrt{b^2-y^2}$$

Also $y:0\to b$

$\therefore$ Area = 4 × Area in the I quadrant

$$=4\iint_R dx\,dy$$

$$=4\int_0^b\int_0^{a/b\sqrt{b^2-y^2}} dx\,dy = 4\int_0^b (x)_0^{\frac{a}{b}\sqrt{b^2-y^2}}dy$$

$$=4\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\,dy$$

$$=\frac{4a}{b}\left\{y/2\sqrt{b^2-y^2}+\frac{b^2}{2}\sin^{-1}y/b\right\}_0^b$$

$$=\frac{4a}{b}\left\{\frac{b^2}{2}\sin^{-1}(1)-\frac{b^2}{2}\sin^{-1}(0)\right\}$$

$$=\frac{4a}{b}\,\frac{b^2}{2}\,\pi/2 = \pi ab$$

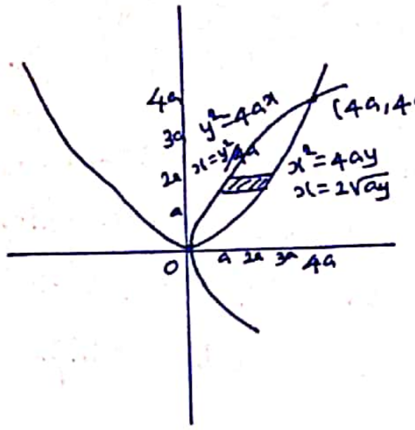
3) show that the area b/w the parabolas $y^2=4ax$ & $x^2=4ay$ is $\frac{16a^2}{3}$

<u>Ans</u> Given $y^2=4ax$

x	-2a	-a	0	a	~~2a~~	3a	4a
y	–	–	0	2a	~~[illegible]~~	–	4a

$y=x^2/4a$

x	-2a	-a	0	a	2a	3a	4a
y	–	–	0	–	–	–	4a



Consider the horizontal strip.

Strip starts from $y^2 = 4ax$

$\Rightarrow x = y^2/4a$

Strip ends at $x^2 = 4ay$

$x = 2\sqrt{ay}$

$\therefore x : \dfrac{y^2}{4a} \rightarrow 2\sqrt{ay}$

& $y : 0 \rightarrow 4a$

$$\therefore \text{Area} = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a} \{x\}_{y^2/4a}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a} \left\{2\sqrt{ay} - y^2/4a\right\} dy$$

$$= 2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{4a\times 3}\Big\}_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\,(4a)^{3/2} - \frac{(4a)^3}{12a}$$

$$= \frac{4\sqrt{a}\;4a(4a)^{1/2}}{3} - \frac{(4a)(4a)(4a)}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

4) Using double integration, find the area enclosed by the curves $y = 2x^2$ & $y^2 = 4x$

Ans Find the intersection points

$y = 2x^2 \rightarrow (1)$

$y^2 = 4x \rightarrow (2)$

From (1) & (2)

$4x = (2x^2)^2$

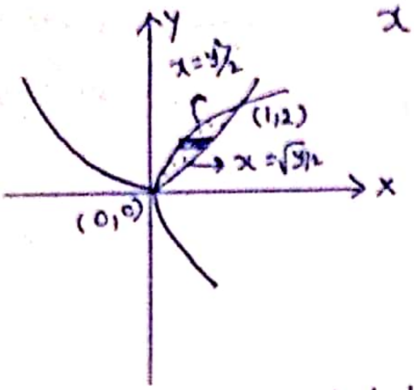
$4x^4 - 4x = 0$

$x\{x^3 - 1\} = 0$

$x = 0, x = 1$

when $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 2$



Consider the horizontal strip.

Here

$x : y^2/2 \to \sqrt{y/2}$

$y : 0 \to 2$

$\therefore$ Area $= \iint dx\,dy$

$$= \int_0^2 \int_{y^2/4}^{\sqrt{y/2}} dx\,dy = \int_0^2 (x)_{y^2/4}^{\sqrt{y/2}} dy$$

$$= \int_0^2 \frac{\sqrt{y}}{\sqrt{2}} - \frac{y^2}{4}\,dy$$

$$= \left. \frac{y^{3/2}}{\sqrt{2}\cdot 3/2} - \frac{y^3}{3\times 4} \right\}_0^2$$

$$= \frac{2^{3/2}}{\sqrt{2}\ 3/2} - \frac{2^3}{3\times 4}$$

$$= \frac{2\sqrt{2}\times 2}{3\sqrt{2}} - \frac{8}{3\times 4}$$

$$= \frac{4}{3} - \frac{2}{3}$$

$$= \frac{2}{3}$$

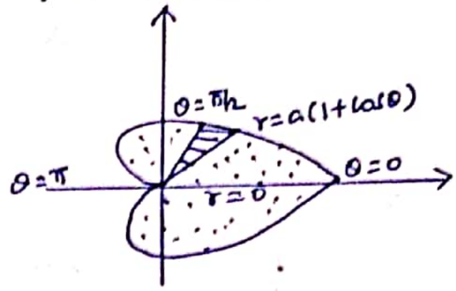
Type : 2 (Polar co-ordinates).

$$\text{Area} = \int_{\theta_1}^{\theta_2} \int_{r = f_1(\theta)}^{r = f_2(\theta)} r\,dr\,d\theta$$

1) Find the area of the cardiod $r = a(1+\cos\theta)$

Ans: $r = a(1+\cos\theta)$

It is symmetrical about the initial line.



$\theta : 0 \to \pi$

$r : 0 \to a(1+\cos\theta)$

$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi} \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2}{2}\int_0^{\pi}\left[a(1+\cos\theta)\right]^2 d\theta$$

$$= a^2\int_0^{\pi}\{1+\cos^2\theta+2\cos\theta\}\,d\theta$$

$$= a^2\int_0^{\pi}\left\{1+\frac{1+\cos 2\theta}{2}+2\cos\theta\right\}d\theta$$

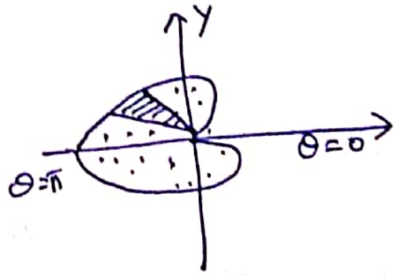
$$= \frac{a^2}{2}\int_0^{\pi}\{3+\cos 2\theta+4\cos\theta\}\,d\theta$$

$$= \frac{a^2}{2}\left\{3\theta+\frac{\sin 2\theta}{2}+4\sin\theta\right\}_0^{\pi}$$

$$= \frac{a^2}{2}\left\{\left(3\pi+\frac{\sin 2\pi}{2}+4\sin\pi\right)-(0+\sin 0+4\sin 0)\right\}$$

$$= \frac{a^2}{2}(3\pi) = \frac{3}{2}\pi a^2$$

2) Find the area of the cardiod $r = a(1-\cos\theta)$



$$\text{Area} = 2\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta$$

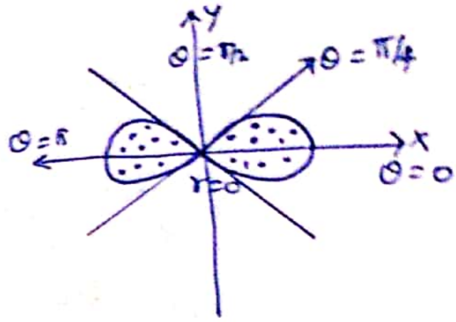
$$= \frac{3}{2}\pi a^2$$

3) Find the area of the lemniscate $r^2 = a^2 \cos 2\theta$ by double integration.

Ans Given $r^2 = a^2 \cos\theta \Rightarrow r = a\sqrt{\cos 2\theta}$

$\theta : 0 \to \pi/4$

$r : 0 \to r\, a\sqrt{\cos 2\theta}$



$$\text{Area} = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos\theta}} d\theta$$

$$= 4/2 \int_0^{\pi/4} (a\sqrt{\cos 2\theta})^2\, d\theta$$

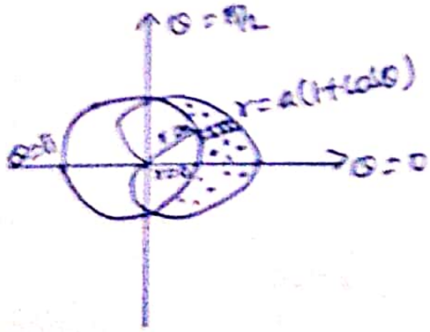
$$= 2\int_0^{\pi/4} a^2 \cos 2\theta\, d\theta$$

$$= 2\left\{a^2 \frac{\sin 2\theta}{2}\right\}_0^{\pi/4}$$

$$= \frac{2a^2}{2}\{1-0\} = a^2$$

4) Find the area that lies inside the cardiod $r = a(1+\cos\theta)$ & outside the circle $r = a$ by double integration.

Ans Given $r = a(1+\cos\theta)$, $r = a$



$\theta : 0 \to \pi/2$

$r : a \to a(1+\cos\theta)$

$$\text{Area} = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{r^2}{2}\right)_a^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2} (a^2(1+\cos\theta)^2 - a^2)\, d\theta$$

$$= a^2 \int_0^{\pi/2} \{1 + \cos^2\theta + 2\cos\theta - 1\}\, d\theta$$

$$= a^2 \int_0^{\pi/2} \left\{\frac{1+\cos 2\theta}{2} + 2\cos\theta\right\} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (1 + \cos 2\theta + 4\cos\theta)\, d\theta$$

$$= \frac{a^2}{2}\left\{\theta + \frac{\sin 2\theta}{2} + 4\sin\theta\right\}_0^{\pi/2}$$

$$= \frac{a^2}{2}\left\{\pi/2 + \frac{\sin\pi}{2} + 4\sin\pi/2 \quad - (0+0+0)\right\}$$

$$= \frac{a^2}{2}\{\pi/2 + 4\}$$

$$= \frac{a^2}{4}\{\pi + 8\}$$

5) Find the area that lies inside the cardiod $r = 2(1+\cos\theta)$ & outside the circle $r = 2$ by double integration.

Ans Same as problem (4), Replace 'a' by 2

we get area $= \pi + 8$

Triple Integrals

1) Evaluate $\int_0^a \int_0^b \int_0^c xyz \, dz\, dy\, dx$

$$= (x^2/2)_0^a \; (y^2/2)_0^b \; (z^2/2)_0^c$$

$$= \frac{a^2b^2c^2}{8}$$

2) Evaluate $\int_0^a\int_0^b\int_0^c e^{x+y+z}\,dz\,dy\,dx$

Ans
$$I = \int_0^a e^x\,dx \int_0^b e^y\,dy \int_0^c e^z\,dz$$
$$= \left(e^x\right)_0^a \left(e^y\right)_0^b \left(e^z\right)_0^c$$
$$= (e^a-1)(e^b-1)(e^c-1)$$

3) Evaluate $\int_0^1\int_0^2\int_0^3 xyz\,dz\,dy\,dx$

Ans
$$I = \int_0^1 x\,dx \int_0^2 y\,dy \int_0^3 z\,dz$$
$$= \left(\frac{x^2}{2}\right)_0^1 \left(\frac{y^2}{2}\right)_0^2 \left(\frac{z^2}{2}\right)_0^3$$
$$= \frac{1}{2}\cdot\frac{2^2}{2}\cdot\frac{3^2}{2}$$
$$= 36/8 = 9/2$$

4) Evaluate $\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$

$$I = \int_0^a\int_0^b \left(\frac{x^3}{3}+y^2x+xz^2\right)_0^c dy\,dz$$
$$= \int_0^a\int_0^b \frac{c^3}{3}+cy^2+cz^2\;dy\,dz$$
$$= \int_0^a \left(\frac{c^3y}{3}+\frac{cy^3}{3}+cz^2y\right)_0^b dz$$
$$= \left\{\frac{c^3bz}{3}+\frac{cb^3}{3}z+\frac{cbz^3}{3}\right\}_0^a$$
$$= \frac{c^3ba}{3}+\frac{cb^3a}{3}+\frac{cba^3}{3}$$
$$= \frac{abc}{3}\left\{a^2+b^2+c^2\right\}$$

5) Evaluate $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$

$$I = \int_0^{\log a}\int_0^{x} e^{x}e^{y}\,\left(e^{z}\right)_0^{x+y}\,dy\,dx$$

$$= \int_0^{\log a}\int_0^{x} e^{x}e^{y}\,\left(e^{x+y}-1\right)dy\,dx$$

$$= \int_0^{\log a}\int_0^{x} \left(e^{2x}e^{2y}-e^{x}e^{y}\right)dy\,dx$$

$$= \int_0^{\log a} \left\{e^{2x}\frac{e^{2y}}{2} - e^{x}e^{y}\right\}_0^{x} dx$$

$$= \int_0^{\log a} \left[\frac{e^{4x}}{2}-e^{2x}\right]-\left[\frac{e^{2x}}{2}-e^{x}\right] dx$$

$$= \int_0^{\log a} \frac{e^{4x}}{2}-\frac{3e^{2x}}{2}+e^{x}\,dx$$

$$= \left.\frac{e^{4x}}{8}-\frac{3e^{2x}}{4}+e^{x}\right)_0^{\log a}$$

$$= \frac{e^{4\log a}}{8}-\frac{3e^{2\log a}}{4}+e^{\log a}-\left(\frac{1}{8}-\frac{3}{4}+1\right)$$

$$= \left(\frac{a^4}{8}-\frac{3a^2}{4}+a\right)-\left(\frac{1-6+8}{8}\right)$$

$$= \frac{a^4-6a^2+8a}{8}-\left(3/8\right)$$

$$= \frac{1}{8}\left(a^4-6a^2+8a-3\right)$$

Volume of solids

$$\text{Volume} = \iiint_R dx\,dy\,dz$$

1) Find the volume of the sphere $x^2+y^2+z^2=a^2$ without transformation.

Ans. Given $x^2+y^2+z^2=a^2$

Volume of the sphere = $8\times$ volume in the I octant

$$= 8\iiint dx\,dy\,dz$$

Here $z: 0 \longrightarrow \sqrt{a^2-x^2-y^2}$

$y: 0 \longrightarrow \sqrt{a^2-x^2}$

$x: 0 \longrightarrow a$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \left(\frac{a^2-x^2}{2}\right)\sin^{-1}\left\{\frac{y}{\sqrt{a^2-x^2}}\right\}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a 0 + \frac{a^2-x^2}{2}\sin^{-1}(1) + 0\;dx$$

$$= 8\int_0^a \frac{a^2-x^2}{2}\cdot\frac{\pi}{2}\,dx$$

$$= 2\pi\left\{a^2x - x^3/3\right\}_0^a$$

$$= 2\pi\left\{a^3 - a^3/3\right\}$$

$$= \frac{4}{3}\pi a^3$$

2) Find the Volume of the tetrahedron bounded by the planes $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the co-ordinates planes (A.U)

Ans

The region bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $x=0, y=0$ & $z=0$.

Limits

$x: 0 \to a$

$y: 0 \to b(1-x/a)$

$z: 0 \to c\left[1-\frac{x}{a}-\frac{y}{b}\right]$

$$\therefore \text{Volume} = \iiint dz\,dy\,dx$$

$$= \int_0^a \int_0^{b[1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-y/b)} dz\,dy\,dx$$

$$\downarrow z$$

$$= \int_0^a \int_0^{b(1-x/a)} c\left(1-\frac{x}{a}-y/b\right) dy\,dx$$

$$= c\int_0^a \left\{y-\frac{xy}{a}-\frac{y^2}{2b}\right\}_0^{b(1-x/a)} dx$$

$$= c\int_0^a \left\{b\left(1-\frac{x}{a}\right)-\frac{xb(1-x/a)}{a}-\frac{b^2(1-x/a)^2}{2b}\right\} dx$$

$$= c\int_0^a b\left(1-\frac{x}{a}\right)\left\{\left(1-\frac{x}{a}\right)-\frac{(1-x/a)}{2}\right\} dx$$

$$= bc\int_0^a (1-x/a)^2\,dx$$

$$= \frac{bc}{2}\left\{\frac{(1-x/a)^3}{3(-1/a)}\right\}_0^a$$

$$= \frac{bc}{2}(-a)(0-1) = \frac{abc}{6}$$

3) Find the Volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the I octant

Ans:

Volume of the ellipsoid in the I octant is bounded by the planes $x=0, y=0, z=0$ & the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Limits

$x: 0 \longrightarrow a$

$y: 0 \longrightarrow b\sqrt{1-x^2/a^2}$

$z: 0 \longrightarrow c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$\therefore$ Volume in I octant

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} (z)_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\, dx$$

$$= c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{b}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}\; dy\, dx$$

$$= \frac{c}{b}\int_0^a \left\{\frac{y}{2}\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)-y^2} + \frac{b^2(1-x^2/a^2)}{2}\sin^{-1}\left\{\frac{y}{b\sqrt{1-\frac{x^2}{a^2}}}\right\}\right\}_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{c}{2b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)\pi/2\; dx$$

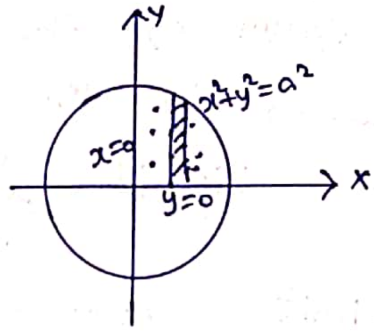
$$= \frac{\pi c}{4b}\, b^2\left\{x-\frac{x^3}{3a^2}\right\}_0^a = \frac{\pi bc}{4}\,\frac{2a^3}{3a^2}$$

$$= \frac{\pi abc}{6}$$

Change of variables

1) change into polar co-ordinates & hence evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$



Ans $y: 0 \rightarrow \sqrt{a^2-x^2}$

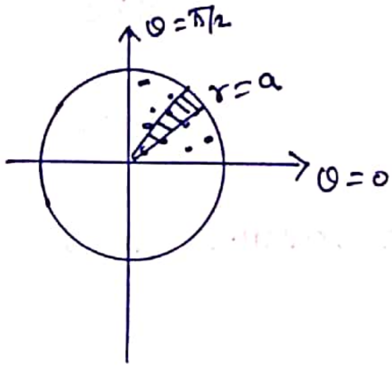
$y=0 \rightarrow y=\sqrt{a^2-x^2}$

$x^2+y^2=a^2$

Circle with $c=(0,0)$

& radi = a

Also $x: 0 \rightarrow a$



put $x=r\cos\theta$, $y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\therefore I = \int_0^{\pi/2}\int_0^a \sqrt{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^2\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^a d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} a^3\, d\theta$$

$$= \frac{a^3}{3}(\theta)_0^{\pi/2}$$

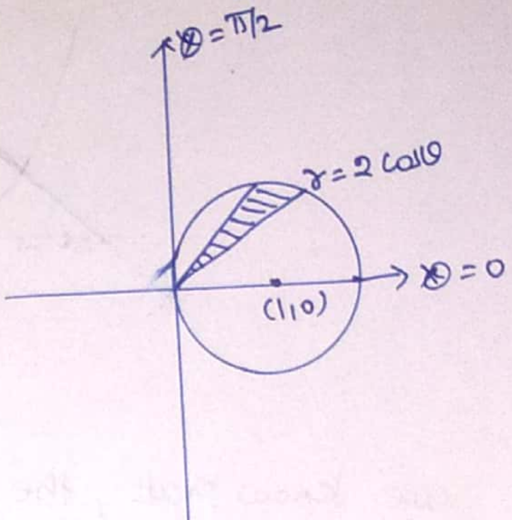
$$= \frac{a^3}{3}(\pi/2) = \frac{a^3\pi}{6}$$

2) By changing into plar co-ordinates, evaluate

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dx\, dy$$

Ans

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dy\, dx \quad \text{(correct form)}$$



$y: 0 \longrightarrow \sqrt{2x-x^2}$

$y=0 \longrightarrow y=\sqrt{2x-x^2}$

$y^2 = 2x - x^2$

$x^2+y^2-2x=0$

circle with

$c = (1,0)$

& radi $=1$

Also $x: 0 \longrightarrow 2$

$x=0 \longrightarrow x=2$

Take $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\therefore I = \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta\, (r)_0^{2\cos\theta}\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta\, (2\cos\theta)\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta$$

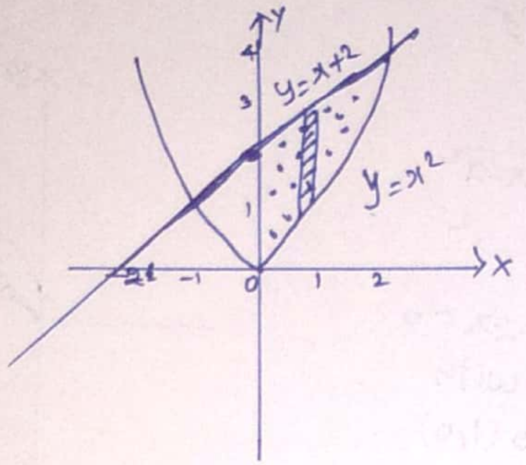
$$= \frac{2}{2}\left\{\theta + \frac{\sin 2\theta}{2}\right\}_0^{\pi/2}$$

$$= \pi/2$$

Applications:

1) Find the mass and centre of mass of the lamina that occupies the region D and has the density function ρ. D is bounded by $y = x^2$ & $y = x+2$, $\rho(x,y) = kx$

Ans:



we know that, the mass of homogeneous lamina is given by

$$m = \iint_R \rho(x,y)\, dA$$

$$m = \int_{-1}^{2} \int_{x^2}^{x+2} kx\, dy\, dx$$

$$= \int_{-1}^{2} kx\,(x - x^2 + 2)\, dx$$

$$= k\left(\frac{x^3}{3} - \frac{x^4}{4} + \frac{2x^2}{2}\right)_{-1}^{2}$$

$$= k\left(\frac{8}{3} - \frac{16}{4} + 4\right) - \left(-\frac{1}{3} - \frac{1}{4} + 1\right)$$

$$= k\left(\frac{32}{12} - \frac{15}{12}\right) = k\left(\frac{27}{12}\right) = \frac{9}{4}k$$

$$\bar{x} = \frac{1}{m} \iint_R x\, \rho(x,y)\, dA$$

$$= \frac{1}{\left(\frac{9}{4}k\right)} \int_{-1}^{2} \int_{x^2}^{x+2} kx^2\, dy\, dx$$

$$= 7/5$$

$$\bar{y} = \frac{1}{m} \iint_R y\,\rho(x,y)\,dA$$

$$= \frac{1}{\left(\frac{9}{4}k\right)} \int_{-1}^{2} \int_{x^2}^{x+2} k\,xy\,dy\,dx$$

$$= \frac{2}{9} \int_{-1}^{2} \left(x^3+4x+4x^2-x^5\right) dx$$

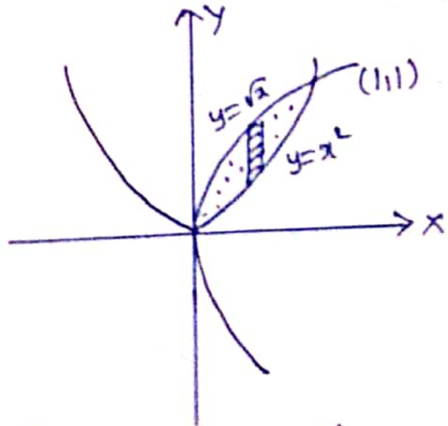
$$= \frac{2}{9} \left\{ 12 - \frac{6}{8} \right)$$

$$= 5/2$$

$\therefore$ Centre of gravity of the lamina is $\left(\frac{7}{5}, \frac{5}{2}\right)$

2) Find the mass and centre of mass of the lamina that occupies the region D and has the given density function ρ. D is bounded by the parabolas $y = x^2$, $x = y^2$; $\rho(x,y) = \sqrt{x}$.

Ans



$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{x}\,dy\,dx = \int_0^1 \sqrt{x}\,(y)_{x^2}^{\sqrt{x}}\,dx$$

$$= \int_0^1 \sqrt{x}\left(\sqrt{x} - x^2\right) dx$$

$$= \int_0^1 x - x^{5/2}\,dx$$

$$= \left.\frac{x^2}{2} - \frac{2}{7}x^{7/2}\right)_0^1 = 3/14$$

$$\bar{x} = \frac{14}{3}\int_0^1 \int_{x^2}^{\sqrt{x}} x\sqrt{x}\, dy\, dx$$

$$= \frac{14}{3}\int_0^1 x^{3/2}\left(x^{1/2} - x^2\right) dx$$

$$= \frac{14}{3}\int_0^1 \left(x^2 - x^{7/2}\right) dx$$

$$= \frac{14}{3}\left\{\frac{x^3}{3} - \frac{x^{9/2}}{9/2}\right\}_0^1 = \frac{14}{3}\left(\frac{1}{3} - \frac{2}{9}\right) = \frac{14}{27}$$

$$\bar{y} = \frac{14}{3}\int_0^1 \int_{x^2}^{\sqrt{x}} y\sqrt{x}\, dy\, dx$$

$$= \frac{14}{3}\int_0^1 \sqrt{x}\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= \frac{14}{3}\int_0^1 \sqrt{x}\left\{\frac{x}{2} - \frac{x^4}{2}\right\} dx$$

$$= \frac{7}{3}\int_0^1 x^{3/2} - x^{9/2}\, dx$$

$$= \frac{7}{3}\left\{\frac{x^{5/2}}{5/2} - \frac{x^{11/2}}{11/2}\right\}_0^1 = \frac{28}{55}$$

$\therefore$ Mass of the region is $m = \frac{3}{14}$

Centre of mass $= \left(\frac{14}{27}, \frac{28}{55}\right)$